1. Scalar field theories

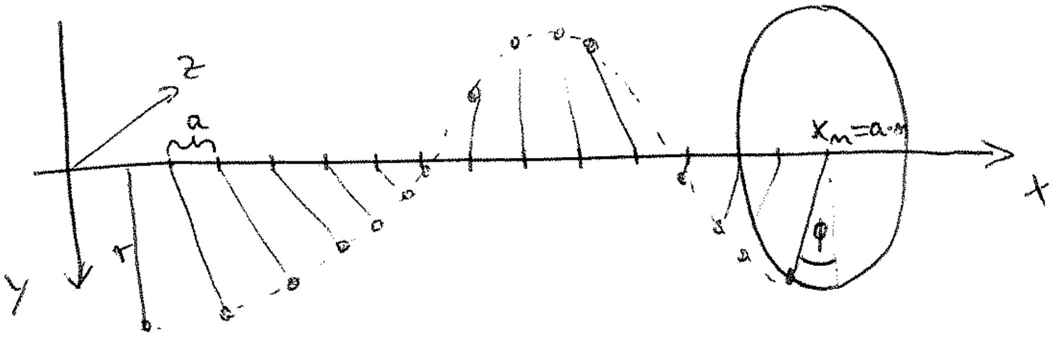
For simplicity, we start with fields that transform as scalars under Lorentz transformations.

Even though most of the fields in particle physics are not scalars (except for the Higgs field), such theories have a wide range of applications from condensed-matter physics to cosmology.

1.1 Classical sine-Gordon theory

We start with a model theory that is already indicative for a rich class of structures of "vacuum" physics.

Let us motivate the model by considering a linear transversal chain in a gravitational field:



which is charactrized by the following energies:

$$\text{kinetic energy:} \quad \frac{1}{2} m r^2 \left[\frac{\partial \varphi(x_n, t)}{\partial t} \right]^2$$

$$\text{interaction energy:} \quad \frac{1}{2} \kappa \left[\varphi(x_{n+1}, t) - \varphi(x_n, t) \right]^2 \qquad (1.1)$$

$$\text{grav. potential} \quad m g r \left[1 - \cos \varphi(x_n, t) \right]$$

We consider the chain in the continuum limit: $a \to 0$ and for $r = 1$

$$m/a \to \rho \to \frac{1}{\lambda} \quad \text{(mass density)}$$

$$\kappa a \to c \xrightarrow{!} \frac{1}{\lambda} \quad \text{(velocity)}$$

$$\frac{mgr}{a} \xrightarrow{\text{new name}} \frac{\mu^2}{\lambda^2} \qquad (1.2)$$

Rescaling the angle $\varphi \to \lambda \varphi$ and with $\sum_n a \to \int dx$,

we obtain the Lagrange function

$$L = \int dx \left[\frac{1}{2} (\partial_t \varphi)^2 - \frac{1}{2} (\partial_x \varphi)^2 - \frac{\mu^2}{\lambda^2} (1 - \cos \lambda \varphi) \right] \quad (1.3)$$

Choosing the metric $\eta_{\cdot} = (-,+)$, we can write the Lagrange density in relativistically covariant form

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - U(\phi) \qquad (1.4)$$

with $$U(\phi) = \frac{\mu^2}{\lambda^2} (1 - \cos \lambda \phi) = \frac{1}{2}\mu^2\phi^2 - \frac{1}{4!}\mu^2\lambda^2\phi^4 + \ldots \qquad (1.5)$$

This defines a 1+1 dimensional field theory.
The corresponding equation of motion (EoM) reads

$$\partial_\mu \partial^\mu \phi = \frac{\mu^2}{\lambda} \sin \lambda \phi . \qquad (1.6)$$

Its resemblance to the Klein-Gordon equation is the reason for the somewhat sophomoric name of the present model.
The sine-Gordon model has an obvious translation symmetry:

$$\phi \to \phi + \frac{2\pi}{\lambda} \qquad (1.7)$$

The corresponding Hamiltonian ($\hat{=}$ energy) is

$$H = \int_{-\infty}^{\infty} dx \left[\frac{1}{2} \dot{\phi}^2 + \frac{1}{2} (\partial_x \phi)^2 + U(\phi) \right] \tag{1.8}$$

The ground state, i.e. the state of minimum energy obviously is

$$\phi_0 = 0 \mod \frac{2\pi}{\lambda} . \tag{1.9}$$

So in the sense of the introduction, the vacuum physics seems to be trivial.

Nevertheless, let us study whether further configurations can exist which have at least finite energy.

(These configurations are also called semi-classical, as they may contribute dominantly to the Euclidean functional integral if they minimize the action.)

With regard to (1.8), finite energy requires that

$$\dot{\phi} \to 0, \ \partial_x \phi \to 0, \ U(\phi) \to 0 \quad \text{as } |x| \to \infty \tag{1.10}$$

sufficiently fast.

This implies that the field is static and constant at infinity:

$$\varphi_{\pm\infty} = \lim_{x \to \pm\infty} \varphi(x,t) \qquad (1.11)$$

and must be at a global minimum of the potential $U(\varphi_{\pm\infty}) = 0$. The trivial vacuum satisfies these conditions obviously,

$$\varphi_0 = \frac{2\pi}{\lambda} p, \quad p \in \mathbb{Z} \quad \Rightarrow \quad H = 0 \qquad (1.12)$$

However, we may find nontrivial vacua if $\varphi_{-\infty}$ and $\varphi_{+\infty}$ lie in different minima of the potential:

$$\varphi_{-\infty} = \frac{2\pi}{\lambda} n \quad , \quad \varphi_{+\infty} = \frac{2\pi}{\lambda} m \quad , \quad m \neq n \qquad (1.13)$$

Of course, the corresponding energy will not vanish, as $U(\varphi(x,t))$ can't vanish everywhere. Also $\dot{\varphi}$ and $\partial_x \varphi \neq 0$ at finite values of x, such that

$$H = E_{n,m} > 0 \qquad (1.14)$$

The explicit solutions satisfying (1.13) representing local minima of the Hamiltonian can be worked out explicitly (these are the so-called solitons). In this introductory section, we will simply concentrate on some qualitative aspects.

First, it is obvious that "smooth" deformations of φ with boundary conditions (1.13) with parameter λ

$$H_\lambda = H + \delta H_\lambda \quad , \; |\delta H_\lambda| < \infty \tag{1.15}$$

cannot lead to a change of n or m as this would require to invest an ∞-amount of energy. Hence, configurations with $m \neq n$ are also referred to as the "non-perturbative" sector, as they cannot be reached by perturbatively deforming the trivial vacuum.

This energy barrier inspires to introduce a conserved charge:

$$Q := \varphi_{+\infty} - \varphi_{-\infty} = \int_{-\infty}^{\infty} dx \, \partial_x \varphi = \frac{2\pi}{\lambda}(m-n) \tag{1.16}$$

which is conserved under smooth deformations as mentioned above. The conservation does not depend on the EoM (as would be for a Noether charge) but relies on the boundary conditions + energy barriers. It is commonly referred to as a

"topological charge"

Q can be related to a topological current

$$J^{\mu} := \varepsilon^{\mu\nu} \partial_{\nu} \phi \qquad (1.17)$$

such that $$Q = \int d\sigma_{\mu} J^{\mu} \qquad (1.18)$$

with $d\sigma_{\mu}$ time-like normal to a space-like surface

J^{μ} trivially satisfies a continuity equation

$$\partial_{\mu} J^{\mu} = \varepsilon^{\mu\nu} \partial_{\mu} \partial_{\nu} \phi = 0 \qquad (1.19)$$

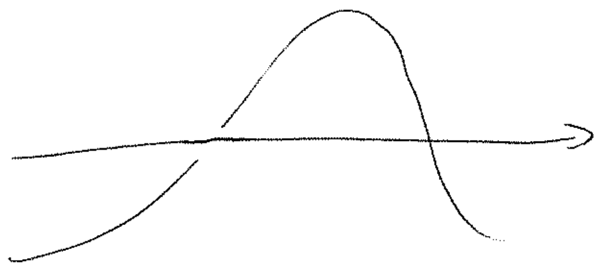
(without the need to use the EoM.)

In the analog system of a linear chain in a gravitational field,

Q is related to the number of windings

$m - n = 1$

$Q = \frac{2\pi}{\lambda}$



$m - n = -3$

$Q = -\frac{2\pi}{\lambda} \cdot 3$