7.3 Functional Integral for the Free Theory

As a first example, let us compute $T[J]$ or $Z[J]$ for the free Klein-Gordon theory. We start with the action

$$S_0 = \frac{1}{2}\int d^Dx \left((\partial_\mu \varphi)(\partial^\mu \varphi) - m^2 \varphi^2 \right)$$

$$= -\frac{1}{2}\left(\varphi , (\Box + m^2)\varphi \right)$$

$$= -\frac{1}{2}\left(\varphi , K\varphi \right), \qquad (7.51)$$

which is a quadratic form involving the Klein-Gordon operator $K = \Box + m^2$.

Consider the generating functional

$$Z_0[J] = \int \mathcal{D}\varphi \; e^{iS_0 - i(\varphi, J)} = \int \mathcal{D}\varphi \; e^{-\frac{i}{2}(\varphi, K\varphi) - i(\varphi, J)} \qquad (7.52)$$

In the presence of the source $J(x)$, the equation of motion

$$0 = \frac{\delta S_J[\varphi]}{\delta \varphi(x)} = -K\varphi(x) - J(x) = -(\partial^2 + m^2)\varphi(x) - J(x) \qquad (7.53)$$

is given by the Feynman propagator (for a causal solution),

$$\varphi_0(x) = \int d^Dy \; \Delta_F(x-y) J(y) \equiv (\Delta_F J)(x), \qquad (7.54)$$

since $K\Delta_F = -\mathbb{1}$.

In Eq. (7.52), we however have to integrate over all field configurations ϕ and not just consider the classical solution. Let us parametrize,

$$\phi(x) = \phi_0(x) + \varphi(x), \tag{7.55}$$

such that

$$S_J[\phi] = -\frac{1}{2}\big(\varphi_0 + \varphi, \underbrace{K(\varphi_0+\varphi)}_{=-J+K\varphi}\big) - (\varphi_0+\varphi, J)$$

$$= \frac{1}{2}(\varphi_0+\varphi, J) - \frac{1}{2}(\varphi_0+\varphi, K\varphi) - (\varphi_0+\varphi, J)$$

$$\boxed{\begin{aligned}(\varphi_0, K\varphi) &= \int d^Dx\, \varphi_0(x)\,(\partial^2+m^2)\,\varphi(x) \\ &\overset{\text{i.b.p.}}{=} \int d^Dx\, \big[(\partial^2+m^2)\varphi_0(x)\big]\,\varphi(x) \\ &= -\int d^Dx\, J(x)\,\varphi(x) = -(J,\varphi)\end{aligned}}$$

$$= -\frac{1}{2}(\varphi_0 + \cancel{\varphi}, J) + \frac{1}{2}\underbrace{(J,\varphi)}_{\cancel{\equiv(\varphi,J)}} - \frac{1}{2}(\varphi, K\varphi)$$

$$= -\frac{1}{2}(\varphi_0, J) - \frac{1}{2}(\varphi, K\varphi)$$

$$\overset{(7.54)}{=} -\frac{1}{2}(\Delta_F J, J) - \frac{1}{2}(\varphi, K\varphi)$$

$$= -\frac{1}{2}(J, \Delta_F J) - \frac{1}{2}(\varphi, K\varphi). \tag{7.56}$$

Hence, the generating functional factorizes,

$$Z_0[J] = \int \mathcal{D}\varphi \, e^{iS_J[\varphi]}$$

$$= e^{-\frac{i}{2}(J,\Delta_F J)} \int \mathcal{D}(\varphi_0+\varphi)\, e^{-\frac{i}{2}(\varphi,K\varphi)}$$

$$= e^{-\frac{i}{2}(J,\Delta_F J)} \int \mathcal{D}\varphi \, e^{-\frac{i}{2}(\varphi,K\varphi)}, \qquad (7.57)$$

where we have made use of the fact that the measure is translation-invariant in function space $\mathcal{D}(\varphi_0+\varphi) = \mathcal{D}\varphi$

(this is obvious in the lattice formulation, as the measure at each lattice site can be shifted by a constant (which can differ from site to site) : $d(\varphi_{0\vec{x}} + \varphi_{\vec{x}}) = d\varphi_{\vec{x}}$)

From (7.57), we learn

$$Z_0[0] = \int \mathcal{D}\varphi \, e^{-\frac{i}{2}(\varphi,K\varphi)}, \qquad (7.58)$$

such that the generating functional for correlation functions is given by

$$T_0[J] = \frac{Z_0[J]}{Z_0[0]} = e^{-\frac{i}{2}(J,\Delta_F J)}$$

$$\equiv e^{-\frac{i}{2}\int d^Dx \int d^Dy \, J(x)\,\Delta_F(x-y)\,J(y)} \qquad (7.59)$$

for the free theory.

This is indeed directly related to Wick's theorem:

$$T\left[e^{-i\int J\phi}\right] = N\left[e^{-i\int J\phi}\right] e^{-\frac{i}{2}\iint J\Delta_F J} \, . \qquad (7.60)$$

Taking the vacuum expectation value of (7.60), we get

$$\langle 0|\underline{\underline{T\left[e^{-i\int J\phi}\right]}}|0\rangle = \underbrace{\langle 0|N\left[e^{-i\int J\phi}\right]|0\rangle}_{=1} \, e^{-\frac{i}{2}(J,\Delta_F J)} \equiv \underline{\underline{T_0[J]}}, \qquad (7.61)$$

which agrees with the definition $T[J] = \langle\Omega|T\left[e^{-i\int J\phi_H}\right]|\Omega\rangle$, since we have $|\Omega\rangle \Rightarrow |0\rangle$ for the free theory.

Hence all correlation functions can be computed in the free theory:

$$G^{(2)}(x_1,x_2) = i^2 \left.\frac{\delta T_0[J]}{\delta J(x_1)\,\delta J(x_2)}\right|_{J=0} = i\Delta_F(x_1-x_2) \qquad (7.62)$$

$$G^{(2n+1)}(x_1,\dots,x_{2n+1}) = 0 \qquad (7.63)$$

(since an odd number of derivatives of $T_0[J]$ is always $\sim J$)

$$G^{(2n)}(x_1,\dots,x_{2n}) = F[i\Delta_F] \qquad (7.64)$$

The RHS is a function of $i\Delta_F$ which can even be written in closed form. Here we simply note that it can be written as a product of Δ_F. In other words, all higher correlation function factorize in products of the 2-point function in the free theory.

All diagrams here are disconnected. E.g.

$$
\begin{aligned}
G^{(4)}_{(x_1,\dots,x_4)} = -\big(& \Delta_F(x_1-x_2)\,\Delta_F(x_3-x_4) \\
& + \Delta_F(x_1-x_3)\,\Delta_F(x_2-x_4) \\
& + \Delta_F(x_1-x_4)\,\Delta_F(x_2-x_3)\big)
\end{aligned}
$$

$=$  (7.65)

8.4 Functional integrals for interacting theories

In the general interacting case, exact solutions for the generating functional are difficult to obtain – and mostly still unknown for most of the interesting theories. Still, the methods used above are also useful in that case.

The essential pat of the computation was nothing but a completion of the square in the action. In our formal notation, we can repeat the computation in a few lines:

$$Z_0[J] = \int \mathcal{D}\phi\; e^{\frac{i}{2}(\phi,K\phi) - i(J,\phi)} \tag{7.66}$$

$$= \int \mathcal{D}\phi\; e^{\frac{i}{2}\left((\phi - J\frac{1}{K},\, K(\underbrace{\phi - J\frac{1}{K}}_{=:\varphi})\right)}\; e^{-\frac{i}{2}(J,\frac{1}{K}J)}$$

$\frac{1}{K} = \Delta_F$

$$= e^{-\frac{i}{2}(J,\Delta_F J)} \underbrace{\int \mathcal{D}\varphi\; e^{\frac{i}{2}(\varphi,K\varphi)}}_{=Z[0]}$$

$$= Z[0]\, e^{-\frac{i}{2}(J,\Delta_F J)} \tag{7.67}$$

Here, we have used that K is a symmetric operator,

$$(f, Kg) = \int d^D x\, f(x)\, K\, g(x) = \int d^D x\, f(x)\, (\partial^2 + m^2)\, g(x)$$
$$\overset{i.b.p}{=} \int d^D x\, g(x)\, (\partial^2 + m^2)\, f(x) = (g, Kf). \qquad (7.68)$$

Let us now consider φ^4 theory

$$S = \int d^D x \left(\frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - \frac{1}{2} \varphi^2 m^2 - \frac{\lambda}{4!} \varphi^4 \right) \qquad (7.69)$$

with the generating functional

$$Z[J] = \int \mathcal{D}\varphi\, e^{iS - i\int J\varphi}. \qquad (7.70)$$

Decomposing the action into a quadratic part and the interactions

$$S = S_0 + S_I, \quad S_0 = -\frac{1}{2} (\varphi, K\varphi),$$
$$S_I = -\int d^D x\, \frac{\lambda}{4!} \varphi^4, \qquad (7.71)$$

we can formally expand the functional integral in powers of S_I,

$$Z[J] = \int \mathcal{D}\varphi\, e^{iS_0 - i\int J\varphi}\, e^{iS_I}$$
$$= \int \mathcal{D}\varphi \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\int d^D x\, \frac{\lambda}{4!} \varphi^4(x) \right)^n e^{iS_0 - i\int J\varphi}. \qquad (7.72a)$$

Using functional differentiation, we get

$$Z[J] = \int \mathcal{D}\varphi \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \left(\int d^D x \, \frac{\lambda}{4!} \, i^4 \frac{\delta^4}{\delta J(x)^4} \right)^n e^{iS_0 - i\int J\varphi}$$

$$= \int \mathcal{D}\varphi \; e^{iS_I[i\frac{\delta}{\delta J}]} \; e^{iS_0[\varphi] - i\int J\varphi}$$

$$= e^{iS_I[i\frac{\delta}{\delta J}]} \int \mathcal{D}\varphi \; e^{iS_0[\varphi] - i\int J\varphi}$$

$$= e^{iS_I[i\frac{\delta}{\delta J}]} \; Z_0[J]$$

$$= Z_0[0] \; e^{iS_I[i\frac{\delta}{\delta J}]} \; e^{-\frac{i}{2}\int J \Delta J} \; . \qquad (7.72b)$$

In this way, we have rewritten the functional integral representation of the generating functional into a functional differential representation. This representation is straight-forwardly useful for perturbation theory, as an expansion in λ corresponds to an expansion in S_I.

The pattern arising from functional differentiation is identical to that of Wick's theorem. The functional differentiations replace the T-ordered products of Heisenberg / interaction picture operators. This functional differentiation method is particularly easy to implement in Computer algebra systems in order to generate diagrams to arbitrary order. E.g. all diagrams with m vertices contributing to $G^{(n)}$ are generated from

$$G^{(n)}_{(x_1 \ldots x_n)} = i^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \frac{i^m}{m!} S_I^m\left[i \frac{\delta}{\delta J}\right] e^{-\frac{i}{2}\int J \Delta_F J} \Big|_{J=0} \quad (7.73)$$

All combinatorial factors are automatically included and follow from the differentiation.

8.5 Functional integral for fermionic theories

There is a rigorous path integral derivation also for fermionic degrees of freedom (using "coherent states" and Grassmannian integration). The important point is that fermionic field operators $\hat{\psi}(x)$ do not simply reduce to complex numbers, once the operator nature is gone. In order to still keep the statistical nature of fermions, $\hat{\psi}(x)$ must reduce to a Grassmann, i.e. anti-commuting, number (c.f. lectures on Particles and Fields).

Abbreviating the details, we will here construct the functional integral for fermionic fields by analogy to the (complex) bosonic case, taking care of the anticommuting nature of the classical fields.

Let us start from the free theory for Dirac fermions

$$S_0 = \int d^4x \; \bar{\Psi} \, (i\not{\partial} - m) \Psi \, . \qquad (7.74)$$

Treating Ψ and $\bar{\Psi}$ as independent, as before, we need source terms for both in the functional integral

$$Z_0[\eta, \bar{\eta}] = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \; e^{\, i S_0[\Psi, \bar{\Psi}] - i \int \bar{\eta} \Psi + i \int \bar{\Psi} \eta} \qquad (7.75)$$

Here, η and $\bar{\eta}$ are both Dirac-spinor- as well as Grassmann-valued. The relative sign between the source terms is just a convention such that we can introduce functional derivatives WRT $\eta, \bar{\eta}$ which act to the right in a standard manner:

$$i \frac{\delta}{\delta \bar{\eta}(x)} \, e^{-i\int \bar{\eta} \Psi} = \Psi(x) \; e^{-i\int \bar{\eta} \Psi}$$

$$i \frac{\delta}{\delta \eta(x)} \, e^{i\int \bar{\Psi} \eta} = - \underbrace{\left(\frac{\delta}{\delta \eta(x)} \int \bar{\Psi} \eta \right)}_{} e^{i\int \bar{\Psi}\eta} = \bar{\Psi}(x) \, e^{i\int \bar{\Psi}\eta}$$

$$= - \int \bar{\Psi} \frac{\delta \eta}{\delta \eta(x)} = - \bar{\Psi}(x)$$

As for the scalar case, the free theory is solvable

$$Z_0[\eta, \bar{\eta}] = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \; e^{\, i(\bar{\Psi}, K_D \Psi) - i(\bar{\eta}, \Psi) + i(\bar{\Psi}, \eta)}$$

$$\Rightarrow Z_0[\eta,\bar\eta] = \int \mathcal{D}\psi\mathcal{D}\bar\psi\; e^{i\left(\underbrace{(\bar\psi - \bar\eta\frac{1}{K_D}}_{=\tilde{\bar\psi}},\, K_D\underbrace{(\psi + \frac{1}{K_D}\eta)}_{=:\tilde\psi})\right)}\; e^{i(\bar\eta,\frac{1}{K_D}\eta)}$$

$$\text{where } K_D = i\partial\!\!\!/ - m\,,\quad K_D\cdot S_F = i\,\mathbb{1} \tag{7.76}$$

Hence

$$Z_0[\eta,\bar\eta] = Z_0[0,0]\; e^{i(\bar\eta,\frac{1}{K_D}\eta)} = Z_0[0,0]\; e^{(\bar\eta, S_F\eta)}; \tag{7.77}$$

$$\text{where } Z_0[0,0] = \int \mathcal{D}\tilde\psi\,\mathcal{D}\tilde{\bar\psi}\; e^{i(\tilde{\bar\psi},K_D\tilde\psi)}\,.$$

The 2-point function of the free theory is thus given by

$$\begin{aligned}
G_0^{(2)}(x,y) &= \langle 0|T[\psi(x)\,\bar\psi(y)]|0\rangle \\
&= i\frac{\delta}{\delta\bar\eta(x)}\, i\frac{\delta}{\delta\eta(y)}\; T[\eta,\bar\eta]\Big|_{\eta,\bar\eta=0} \\
&= \frac{1}{Z_0[0,0]}\int \mathcal{D}\psi\mathcal{D}\bar\psi\; \psi(x)\,\bar\psi(y)\; e^{iS_0[\psi,\bar\psi]} \\
&\overset{(7.77)}{=} S_F(x,y)\,,
\end{aligned} \tag{7.78}$$

i.e. the Dirac-(Feynman)-Propagator occurring in the fermionic Feynman rules. Also in the free fermionic theory, all higher correlation functions factorize in terms of 2-point correlators.

As an example for a fermionic theory with interactions, let us study the Yukawa theory with action

$$S = \int d^4x \left\{ \bar\psi\,(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - g\bar\psi\psi\phi \right\} \tag{7.79}$$

The generating functional is

$$Z[J,\eta,\bar\eta] = \int \mathcal{D}\psi\,\mathcal{D}\bar\psi\,\mathcal{D}\phi\; e^{iS[\phi,\psi,\bar\psi] - i\int J\phi - i\int\bar\eta\psi + i\int\bar\psi\eta} \tag{7.80}$$

As in the scalar case, we can write the interaction term in terms of functional derivatives

$$S_I = -\int d^4x\; g\bar\psi\psi\phi \;\rightarrow\; -i^3\int d^4x\, g\,\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta J}\,; \tag{7.81}$$

this straightforwardly implies that we can write the generating functional as

$$Z[J,\eta,\bar\eta] = Z[0,0,0]\; e^{-\int d^4x\, g\frac{\delta}{\delta\eta}\frac{\delta}{\delta\bar\eta}\frac{\delta}{\delta J}}\; e^{-\frac{i}{2}(J,\Delta_F J)}\; e^{\bar\eta S_F\eta}. \tag{7.82}$$

An expansion in g and carrying out the functional derivatives reproduce the Feynman rules order by order for the correlation functions

7.6 Functional integral for the electromagnetic field

The classical theory of Maxwell's electrodynamical field is given by the action / Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= -\frac{1}{2} \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A_\mu \right) \tag{7.83}$$

$$= -\frac{1}{2} A^\mu \left(-\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu \right) A^\nu \quad + \text{total derivatives} ,$$

yielding the classical equations of motion

$$0 = \underbrace{\frac{\partial \mathcal{L}}{\partial A_\mu}}_{=0} - \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu A_\mu)} = -\partial_\nu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) = -\partial_\nu F^{\nu\mu} \tag{7.84}$$

which is Maxwell's equation for the field strength tensor in vacuum.

A straightforward attempt to quantize the photon field in direct analogy to the scalar field leads to an elementary problem:

The Maxwell operator $K_{M\,\mu\nu} = -\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu$, corresponding to the kinetic term of A_ν, has a zero eigenvalue:

$$K_{M\,\mu\nu} A^\nu = 0 \qquad \text{for} \quad A^\nu = \partial^\nu \varphi \;, \; \varphi \text{ arbitrary} , \tag{7.85}$$

since $\left(-\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu \right) \partial^\nu \varphi = \left(-\partial^2 \partial_\mu + \partial_\mu \partial^2 \right) \varphi = 0$.

Hence, we cannot straightforwardly define a propagator

$$D \sim \frac{1}{K_M}$$

The construction of such an inverse of V_M is a prerequisite for constructing the functional integral for the free photon field. (This is similar to a Gaußian integral $\int dx \; e^{-kx^2}$, which does not exist for $k=0$.)

There is, of course, a reason for the occurrence of such a zero eigenvalue: the gauge field A^ν contains redundant information, i.e. not all components are independent degrees of freedom. Maxwell's theory is invariant under gauge transformations

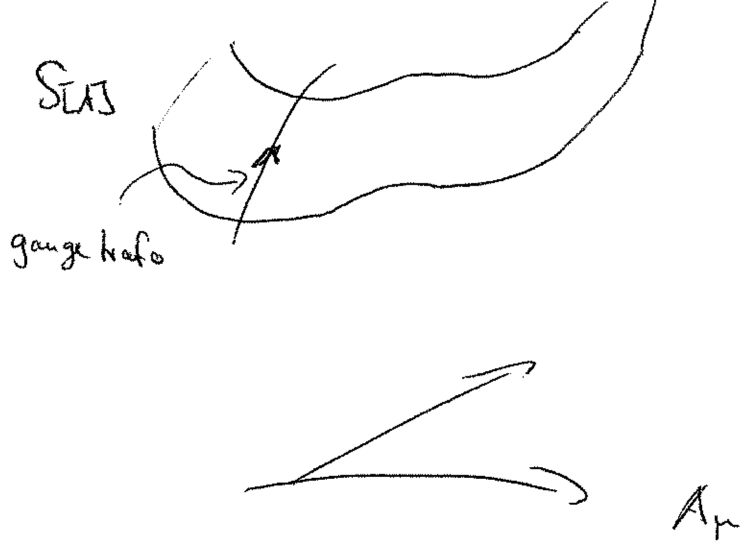
$$A_\mu(x) \to A_\mu(x) + \partial_\mu \lambda(x) \, , \tag{7.86}$$

where $\lambda(x)$ is a local gauge function which – apart from differentiability – is arbitrary. (7.86) leaves the physical observables invariant:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to \partial_\mu A_\nu - \partial_\nu A_\mu + \underbrace{\partial_\mu \partial_\nu \lambda - \partial_\nu \partial_\mu \lambda}_{=0} = F_{\mu\nu} \, . \tag{7.87}$$

Hence, a naive functional integral $\int \mathcal{D}A$ would contain "too many" configurations, since an integral over A_μ would also include an integration over all redundant gauge copies of the same physical field content.

Since the action only depends on $F_{\mu\nu}$, a deformation of the field $A_\mu \rightarrow A_\mu + \delta A_\mu$, that corresponds to a gauge transformation $\delta A_\mu = \partial_\mu \lambda$, does not change the action. Hence, the Maxwell action must feature "flat directions" as a functional on configuration space.



These flat directions correspond precisely to the zero eigenvalues of the Maxwell operator. In order to quantize the photon field, we have to get rid of this redundancy in configuration space. This is possible by requiring a gauge constraint which fixes a part of A_μ without constraining the physical degrees of freedom. One example is a gauge condition known already from the classical theory, e.g.

$$\partial_\mu A^\mu = 0, \qquad \text{(Lorenz gauge)} \qquad (7.88)$$

which is a covariant gauge condition.

We can add this constraint on the classical level by means of the Lagrange multiplier technique

$$\mathcal{L} \to \mathcal{L}_{Maxwell} + \mathcal{L}_{gf}$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 \qquad (7.89)$$

(we choose the square of $\partial_\mu A^\mu$ such that the action becomes extremal on the space of configurations satisfying $\partial_\mu A^\mu = 0$.)

Here, α should be viewed as a Lagrange multiplier.

The functional integral hence reads

$$Z[J] = \int \mathcal{D}A \; e^{iS_{Maxwell} + i S_{gf} - i \int J_\mu A^\mu}$$

$$= \int \mathcal{D}A \; e^{i\int (-\frac{1}{4}) F_{\mu\nu}F^{\mu\nu} - i\int \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 - i \int J_\mu A^\mu} \; . \qquad (7.90)$$

In the limit $\alpha \to 0$ (together with the $i\varepsilon$ prescription) the gauge-fixing term can also be read as a representation of a δ functional:

$$\lim_{\alpha \to 0} e^{-i\frac{1}{2\alpha}\int (\partial_\mu A^\mu)^2} \sim \delta[\partial_\mu A^\mu] \qquad (7.91)$$

In other words, gauge fixing can be viewed as a way to cut out those configurations in field space that satisfy the gauge condition $\partial_\mu A^\mu = 0$. [NB: we encounter a subtlety here: a true cut of this type would involve a δ functional of the form $\delta\left[A\big|_{\partial_\mu A^\mu=0}\right]$. The transition $\delta[A|] \to \delta[\partial_\mu A^\mu]$ should be viewed as a variable change which involves a Jacobian. In Maxwell's theory, this Jacobian is independent of the field and thus a constant factor which is irrelevant in most (but not all) cases. However, in non-abelian gauge theories, the Jacobian becomes field dependent and thus is relevant in most cases $\Rightarrow$ Faddeev-Popov quantization.)

The limit $\underline{\underline{\alpha \to 0}}$ is known as the Landau gauge. In fact, for the existence of the functional integral, this limit is not important. The Maxwell operator can be inverted for all $|\alpha| < \infty$. Within perturbation theory, it is possible to prove (using Ward-Takahashi identities) that physical amplitudes are independent of α. Hence, α can also be chosen in a convenient fashion, e.g., $\underline{\underline{\alpha = 1}}$ (known as Feynman gauge).

The gauge-fixed Maxwell operator finally reads

$$K^{\alpha}_{M\,\mu\nu} = \left(-\partial^2 g_{\mu\nu} + \partial_\mu\partial_\nu - \frac{1}{\alpha}\partial_\mu\partial_\nu\right) \qquad (7.92a)$$

The corresponding Green's function equation for the photon propagator reads

$$K^{\alpha}_{M\,\mu\nu}\, D^{\nu\kappa} = -\left(\mathbb{1}\right)_{\mu}{}^{\kappa}$$

$$\Rightarrow \left(-\partial^2 g_{\mu\nu} + \partial_\mu \partial_\nu - \frac{1}{\alpha}\partial_\mu\partial_\nu\right) D^{\nu\kappa} = -\delta^{(4)}(x-y)\,\delta_\mu{}^{\kappa} \qquad (7.92b)$$

Fourier transformation yields:

$$\left(p^2 g_{\mu\nu} - \left(1-\frac{1}{\alpha}\right)p_\mu p_\nu\right) D^{\nu\kappa} = -\delta_\mu{}^{\kappa} \qquad (7.93)$$

$$\Rightarrow D^{\nu\kappa}(p) = -\frac{1}{p^2}\left(g^{\nu\kappa} - (1-\alpha)\frac{p^\nu p^\kappa}{p^2}\right) \qquad (7.94)$$

Check:

$$\left(p^2 g_{\mu\nu} - \left(1-\frac{1}{\alpha}\right)p_\mu p_\nu\right)\left(-\frac{1}{p^2}\right)\left(g^{\nu\kappa} - (1-\alpha)\frac{p^\nu p^\kappa}{p^2}\right)$$

$$= -\delta_\mu{}^{\kappa} + \left(1-\frac{1}{\alpha}\right)\frac{p_\mu p^\kappa}{p^2} + (1-\alpha)\frac{p_\mu p^\kappa}{p^2} - \left(1-\frac{1}{\alpha}\right)(1-\alpha)\frac{p_\mu p^\kappa}{p^2}$$

$$= -\delta_\mu{}^{\kappa} + \left(\cancel{1} - \cancel{\frac{1}{\alpha}} + \cancel{1} - \cancel{\alpha} - \cancel{1} + \cancel{\alpha} + \cancel{\frac{1}{\alpha}} - \cancel{1}\right)\frac{p_\mu p^\kappa}{p^2}$$

$$= -\delta_\mu{}^{\kappa} \qquad \checkmark$$

(In fact, for Feynman gauge $\alpha = 1$, we encounter the propagator which we guessed in (6.54) for QED: $\sim\!\sim\!\sim = i\, D^{\nu\kappa}_{(p)}\big|_{\alpha=1} = -\frac{i g^{\nu\kappa}}{p^2}$)

Now, we can compute the generating functional for the free quantized photon field:

$$Z[J] = \int \mathcal{D}A \; e^{iS_{\text{Maxwell}} + iS_{gf} - i\int J_\mu A^\mu}$$

$$= \int \mathcal{D}A \; e^{-\frac{i}{2}(A, K_M^\alpha A) - i\int JA}$$

$$= Z[0] \; e^{-\frac{i}{2}(J, DJ)} \qquad (7.95)$$

where we have used a formal notation also including the Lorentz indices and followed the same steps as, e.g. in Eqs. (7.66) and (7.67). The explicit form of the exponent reads

$$-\frac{i}{2} \int d^4x \, d^4y \; J_\mu(x) \, D^{\mu\nu}(x-y) \, J_\nu(y)$$

$$\overset{FT}{=} -\frac{i}{2} \int d^4p \; J_\mu(-p) \, D^{\mu\nu}(p) \, J_\nu(p) \qquad (7.96)$$

For perturbative computations, the photon propagator $i D^{\mu\nu}(p)$ plays the role of the Feynman propagator $i\Delta_F(p)$ for scalar theories. As already mentioned, the gauge parameter can (in perturbation theory) be chosen for reasons of convenience.

Popular choices are

Feynman gauge $\alpha=1$: $iD^{\alpha=1}_{\mu\nu} = -\frac{ig_{\mu\nu}}{p^2}$ (7.97)

Landau gauge $\alpha=0$: $iD^{\alpha=0}_{\mu\nu} = -\frac{i}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)$ (7.98)

Yennie gauge $\alpha=-3$: $iD^{\alpha=-3}_{\mu\nu} = -\frac{i}{p^2}\left(g_{\mu\nu} - 4\frac{p_\mu p_\nu}{p^2}\right)$ (7.99)

In Feynman gauge, already introduced in (6.54), the propagator has the simplest Lorentz structure.

In Landau gauge, the propagator is transversal,

$$D^{\alpha=0}_{\mu\nu}(p)\,p^\nu = -\frac{1}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)p^\nu = -\frac{1}{p^2}\left(p_\mu - p_\mu\right) = 0$$

which is a physical property of plane wave propagation in electrodynamics.

In Yennie gauge, the propagator is traceless,

$$D^\mu{}_\mu\big|_{\alpha=-3} = -\frac{1}{p^2}\left(\delta^\mu_\mu - 4\right) = 0$$

which is convenient in IR studies of QED.