6.3 Dirac Propagator

For the computation of S matrix elements and correlation functions, we need the Feynman propagator for the Dirac field, corresponding to a Green's function for the Dirac equation

$$(i\not{\partial} - m)\, S(x-y) = i\,\delta^{(4)}(x-y)\cdot \mathbb{1}_{\text{Dirac}} \qquad (6.28a)$$

A Fourier transformation gives

$$(\not{p} - m)\, S(p) = i \qquad (6.28b)$$

$$\Rightarrow S(p) = \frac{i}{\not{p}-m} = i\,\frac{(\not{p}+m)}{p^2-m^2} \qquad (6.28c)$$

Analogously to the scalar case, there are different versions of the Green's function associated to different boundary conditions. For the description of intermediate (virtual) states, we again need the causal propagator corresponding to the $+i\varepsilon$ prescription for the mass pole. This gives

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}\, e^{-i(x-y)p} \qquad (6.28d)$$

Along the same lines as for the scalar field, we have

$$S_F(x-y) = \langle 0 | T[\Psi_H(x)\,\bar{\Psi}_H(y)] | 0\rangle\,, \qquad (6.28e)$$

Because of the anti-commuting character, we have

$$S_F(x-y) = \begin{cases} \langle 0 | \psi_H(x)\, \bar\psi_H(y) | 0 \rangle & , \ x^0 > y^0 \\ -\langle 0 | \bar\psi_H(y)\, \psi_H(x) | 0 \rangle & , \ x^0 < y^0 \end{cases} \qquad (6.28f)$$

The time-ordering takes care of the ordering of time arguments, the anti-commuting nature of the ladder operators has to be taken care of explicitly.

$S_F(x-y)$ and its corresponding representation in momentum space is an important building block for perturbative computations of amplitudes and diagrams.

6.4 Feynman rules for fermions

As for scalars, the Feynman rules can be derived from the computation of S-matrix elements

$$S = T\, e^{-i\int d^4x\, \mathcal{H}_I} , \qquad (6.29)$$

where $\mathcal{H}_I$ may contain fermionic field operators in the interaction picture. We have already noted above that the minus signs from spinor field reordering has to be taken care of explicitly. Eq. (6.28f) generalizes to higher order operators, e.g.

$$T[\psi_1 \bar\psi_2 \psi_3 \psi_4] = (-1)^3\, \psi_3 \psi_1 \psi_4 \bar\psi_2 \qquad (6.30a)$$

for $x_3^0 > x_1^0 > x_4^0 > x_2^0$

We use the same sign rule for normal-ordered products. With this, we get the relation between contractions, T- and N-ordering and the Dirac propagator,

$$\overbracket{\psi(x)\,\bar\psi(y)} := T[\psi(x)\bar\psi(y)] - N[\psi(x)\bar\psi(y)] \qquad (6.30b)$$

Analogously to the scalar case, it can straightforwardly by verified that

$$\overbracket{\psi(x)\,\bar\psi(y)} = \begin{cases} \{\psi^{+}(x), \bar\psi^{-}(y)\} & , x^0 > y^0 \\ -\{\bar\psi^{+}(y), \psi^{-}(x)\} & , x^0 < y^0 \end{cases} \equiv S_F(x-y) \qquad (6.30c)$$

Similarly, $\overbracket{\psi(x)\,\psi(y)} = 0 = \overbracket{\bar\psi(x)\,\bar\psi(y)}$.

The minus signs also have to be accounted for in contractions, e.g.

$$\contraction{N[}{\psi}{_1\psi_2}{\bar\psi} N[\psi_1\psi_2\bar\psi_3\bar\psi_4] = -\overbracket{\psi_1\bar\psi_3}\, N[\psi_2\bar\psi_4] \qquad (6.30d)$$
$$= -S_F(x_1-x_3)\, N[\psi_2\bar\psi_4].$$

Then, Wick's theorem again reads

$$T[\psi_1\bar\psi_2\psi_3 \ldots] = N[\psi_1\bar\psi_2\psi_3 \ldots] + \text{sum of all possibly contractions.} \qquad (6.31)$$

This is formally identical to the bosonic case, with minus signs taken care of by the conventions specified above.

Let us discuss further details by way of example, using a Yukawa model as a concrete example of an interacting theory of fermions and scalars. On the fundamental level, Yukawa interactions occur between all standard model fermions and the Higgs boson. On an effective level pion-nucleon interactions are of this type (also meson-quark interactions). We use here the simplest possible version

$$H = H_{\text{Dirac}} + H_{\text{Klein-Gordon}} + \int d^3x \; g \, \bar{\psi} \psi \phi \,. \qquad (6.32a)$$

We study 2-to-2 scattering of fermions as an example:

$$f(p) + f(k) \Rightarrow f(p') + f(k') \,. \qquad (6.32b)$$

To leading ordering, the S matrix reads

$$\langle \vec{p}\,', \vec{k}' | T\left[\tfrac{1}{2!} (-ig)^2 \int d^4x \, \bar{\psi}(x) \psi(x) \phi(x) \int d^4y \, \bar{\psi}(y) \psi(y) \phi(y) \right] | \vec{p}, \vec{k} \rangle \,. \qquad (6.33)$$

We can use Wick's theorem to rewrite the T-product into N-products and contractions. As for the scalar field, we can interpret field operators in N-products acting on the states as contractions:

$$\Psi(x)\,|\vec{p},\underset{\substack{\uparrow\\ \text{Spin}}}{s}\rangle = \int \frac{d^3p'}{(2\pi)^3}\,\frac{1}{\sqrt{2E_{\vec{p}'}}}\sum_{s'} a_{p}^{s'}\,u^{s'}(p')\,e^{-ip'x}\sqrt{2E_{\vec{p}}}\;a_{\vec{p}}^{s\dagger}\,|0\rangle$$

$$= e^{-ipx}\,u^{s}(p)\,|0\rangle\,. \qquad (6.34)$$

Finite contributions are obtained if Ψ acts to the right on fermion states and $\overline{\Psi}$ to the right on antifermion states (and vice versa for actions to the left). For the present case involving only fermions, a typical contraction is

$$\langle \vec{p}\,',\vec{k}'|\,\frac{1}{2!}(-ig)^2\int d^4x\int d^4y\;\overline{\Psi}(x)\Psi(x)\,\Phi(x)\,\overline{\Psi}(y)\Psi(y)\,\Phi(y)\,|\vec{p},\vec{k}\rangle \qquad (6.35)$$

In total, we obtain from the contractions of this type

$$(-ig)^2\int\frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m_\Phi^2}\,(2\pi)^4\,\delta^{(4)}(p'-p-q)\,(2\pi)^4\,\delta^{(4)}(k'-k+q)$$

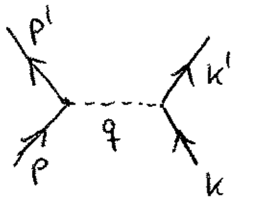
$$\cdot\;\bar{u}(p')\,u(p)\,\bar{u}(k')\,u(k)\,. \qquad (6.36)$$

The factor $\frac{1}{2!}$ cancels against that of vertex permutations. The resulting scattering amplitude is

$$i\mathcal{M} = \frac{-ig^2}{q^2-m_\Phi^2}\,\bar{u}(p')\,u(p)\,\bar{u}(k')\,u(k)\,, \qquad (6.37)$$

with the momentum conservation $p-p' = q = k'-k$.

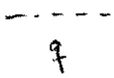
This amplitude can be represented by the following diagram



(6.38)

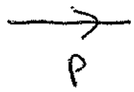
The corresponding Feynman rules are

Propagators

$$\phi(x)\,\phi(y) = $$ 

$$ = \frac{i}{q^2 - m_\phi^2 + i\varepsilon}$$

(no arrow is needed as the propagator is a function of q^2)

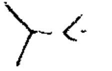
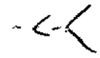
$$\psi(x)\,\bar\psi(y) = $$ 

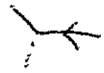
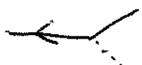
$$ = \frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}$$

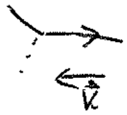
(6.39a)

vertex  $= -ig$

external legs:

$\phi|\vec q\rangle =$  $= 1$ $\qquad$ $\langle \vec q|\phi =$  $= 1$

$\psi|\vec p, s\rangle =$  $= u^s(p)$ $\qquad$ $\langle \vec p, s|\bar\psi =$  $= \bar u^s(p)$

$\bar\psi|\vec k, s\rangle =$ 

$= \bar v^s(k)$ $\qquad$ $\langle \vec k, s|\psi =$ 

$= v^s(k)$

(6.39b)

As before, we have

- momentum conservation at each vertex
- integration over all loop momenta
- combinatorial factors (& minus signs)

(6.40)

There are a few peculiarities in the case of fermions:

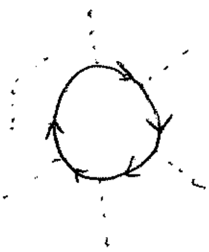
(1) In fact, it turns out that the combinatorial factors in Yukawa theory cancel: the $\frac{1}{m!}$ factor from the Taylor-expansion cancel against the factor $m!$ from the vertex permutations. There are no nontrivial further factors, since a vertex $\phi\bar{\Psi}\psi$ does not contain two identical fields in the sense of Wick contractions.

(2) The arrows on the fermion lines indicate the flux of particle number. In addition, the direction of the momentum flux is relevant, since $S_F(-p) \neq S_F(p)$.

(6.41)

Since the particle number and the momentum flux can be different for external legs in the case of antifermions, an additional arrow for the momentum flux can be useful.

(3) Minus signs occur, if fermionic field operators have to be commuted for contractions. A general rule exists for closed fermion loops:



e.g. for 4 external legs

$$\sim \bar{\psi}\psi\bar{\psi}\psi\bar{\psi}\psi\bar{\psi}\psi$$

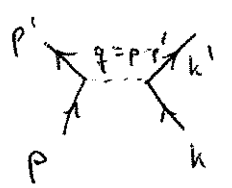
$$= (-1)\ \psi\bar{\psi}\ \psi\bar{\psi}\ \psi\bar{\psi}\ \psi\bar{\psi} = (-1)\,\mathrm{tr}(S_F^4)$$

(6.42)

Closed fermion loops always lead to a minus sign.

6.5 Example: Yukawa potential

As a concrete example, we consider 2-by-2 scattering of distinguishable particles. In (6.37), we determined the matrix element



$$i\mathcal{M} = \frac{-ig^2}{q^2 - m_\varphi^2}\, \bar{u}(p')\, u(p)\, \bar{u}(k')\, u(k) \,. \qquad (6.43)$$

Further diagrams such as  do not exist

due to distinguishability. In the limit of small momenta, we can compare the amplitude with the non-relativistic quantum mechanical scattering amplitude in Born approximation. For this, we expand in small momenta

$$\begin{aligned} p^\mu &\simeq (m, \vec{p}) + \mathcal{O}(\vec{p}^2) \,, & p'^\mu &\simeq (m, \vec{p}') + \mathcal{O}(p'^2) \\ k^\mu &\simeq (m, \vec{k}) + \dots \,, & k'^\mu &\simeq (m, \vec{k}') + \dots \end{aligned} \qquad (6.44a)$$

$$\Rightarrow \quad q^2\ (p-p')^2 = -|\vec{p} - \vec{p}'|^2 + \mathcal{O}(p^4) \qquad (6.44b)$$

The spinors reduce to

$$u^s(p) \simeq \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \qquad (6.45)$$

such that the normalization property (6.7c) remains to hold also for spinors with different (small) momenta

$$\bar{u}^{s'}(p')\, u^s(p) \simeq 2m\, \delta^{s's} . \qquad (6.46)$$

This implies that the spin of the scattering fermions is separately conserved:

$$i\mathcal{M} = \frac{i g^2}{|\vec{p}\,' - \vec{p}|^2 + m_\varphi^2} (4m)\, \delta^{ss'} \delta^{rr'} \qquad (6.47)$$

Let us compare this with the scattering amplitude of a non-relativistic QM particle scattering off a potential

$$\langle \vec{p}\,' | iT | \vec{p} \rangle_{QM} \underset{\substack{\uparrow \\ \text{Born approximation}}}{=} -i \tilde{V}(q)\, (2\pi)\, \delta(E_{\vec{p}} - E_{\vec{p}\,'}), \qquad (6.48)$$

$$\vec{q} = \vec{p} - \vec{p}\,'$$

where $\tilde{V}(q)$ is the Fourier transform of the potential. Comparing (6.48) and (6.47), we can read off the potential:

$$\tilde{V}(q) = \frac{-g^2}{|\vec{q}|^2 + m_\varphi^2}, \qquad (6.49)$$

The factor of (2π) makes up for the different normalizations of the $|\vec{p}\rangle$-states.

$$\underline{\underline{V(\vec{x})}} = \int \frac{d^3q}{(2\pi)^3} \frac{-g^2}{|\vec{q}|^2 + m_\varphi^2} e^{i\vec{q}\cdot\vec{x}}$$

$x = |\vec{x}|$

$$= -\frac{g^2}{8\pi^3} \, 2\pi \int_{-1}^{1} du_{\,u=\cos\theta} \int_0^\infty \frac{dq\, q^2}{q^2 + m_\varphi^2} e^{iqxu}$$

$$= -\frac{g^2}{4\pi^2} \int_0^\infty \frac{dq\, q^2}{q^2 + m_\varphi^2} \frac{e^{iqx} - e^{-iqx}}{iqx}$$

$$= -\frac{g^2}{4\pi^2 ix} \int_{-\infty}^{\infty} dq \frac{q\, e^{iqx}}{q^2 + m_\varphi^2}$$

$$= -\frac{g^2}{4\pi^2 ix} \int dq \frac{q\, e^{iqx}}{q^2 + m_\varphi^2}$$

$$= -\frac{g^2}{4\pi^2 ix} \, 2\pi i \, \mathrm{Res}\left(\frac{q\, e^{iqx}}{q^2 + m_\varphi^2}, \, q = +im_\varphi \right)$$

$$= \underline{\underline{-\frac{g^2}{4\pi} \frac{1}{x} e^{-m_\varphi x}}} \qquad \underline{\underline{\text{Yukawa Potential}}} \qquad (6.50)$$

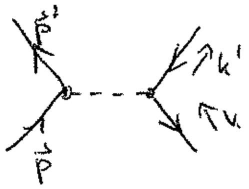
This is an attractive potential with a range of

$$\sim \frac{1}{m_\varphi} \left(= \frac{\hbar}{m_\varphi c} \right)$$

corresponding to the Compton wavelength of the exchanged boson. This is a first example of the fact that classical conservative forces can be fundamentally understood as an exchange of an intermediate particle excitation with a causal propagation behavior. The same picture applies to all known forces.

(NB: Yukawa considered this potential as the basis for a model of the strong interactions among the nucleons. He used the range of this interaction $\sim 1\,\mathrm{fm}$ to predict the mass of the exchange boson: the pion in this case.)

It is instructive (and straightforward) to generalize this computation to the case of one fermion scattering off an antifermion



Then, there is a change in the spinor contraction

$$\bar{v}^{s}(k)\, v^{s'}(k') \simeq -2m\, \delta^{ss'} \qquad (6.51)$$

However there is one further minus sign since the contractions have to be ordered differently as compared to (6.35), in order to account for the antifermion

$$\langle \vec{p}\,', \vec{k}\,' | \int d^4x \int d^4y \, \bar{\psi}(x) \psi(x) \phi(x) \bar{\psi}(y) \psi(y) \phi(y) | \vec{p}, \vec{k} \rangle . \quad (6.52)$$

Compared to (6.35) this is one fermion commutation more giving another factor of minus one. Together with the sign of (6.51), we get the same result for the potential describing fermion anti-fermion scattering in the non-relativistic limit. Hence the Yukawa interaction is also attractive in this case.

$\Rightarrow$ The Yukawa interaction is always attractive independently of the charge of the particles.

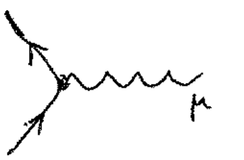
6.6 Feynman rules for Quantum Electrodynamics (QED)

On the level of Feynman diagrams, the step from Yukawa theory to QED seems rather small. (This simplicity is indeed somewhat misleading, since QED features a local gauge symmetry which has to be taken into account more carefully; see following sections). In QED, the exchange boson is not a scalar but a vector boson A^μ, the 4-potential of electrodynamics, $A^\mu = (\varphi, \vec{A})$. The interaction Hamiltonian reads

$$H_I = \int d^3x \; e \, \bar{\Psi} \gamma_\mu \Psi A^\mu , \qquad (6.53)$$

representing a coupling of A^μ to the Noether current of phase rotations $j_\mu \sim \bar{\Psi}\gamma_\mu \Psi$ with coupling constant e.

Though the details involve a variety of subtleties, the Feynman rules can almost be guessed. In addition to the fermionic conventions, we have

vertex:  $\hat{=} \; -ie\gamma^\mu$

photon propagator:  $= -\dfrac{i g_{\mu\nu}}{q^2 + i\varepsilon}$ (6.54)

external photon legs: $\overbrace{A_\mu |\vec{p}\rangle}$ $=$  $= \varepsilon_\mu(p)$

$\overbrace{\langle \vec{p} | A_\mu}$ $=$  $= \varepsilon^*_\mu(p)$

The symbol $\varepsilon_\mu(p)$ denotes the photon polarization vector. From classical electrodynamics, we take over the result that photons (classical waves) are transversely polarized such that $p_\mu \varepsilon^\mu(p) = 0$. Writing $\varepsilon^\mu = (0, \vec{\varepsilon})$, we have $\vec{p}\cdot\vec{\varepsilon}(p) = 0$, hence $\vec{\varepsilon}$ parametrizes two independent degrees of freedom. For $\vec{p} \parallel \hat{e}_z$, linear polarizations are, e.g., given by $\vec{\varepsilon} \sim \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, whereas circular polarizations (right-handed / left-handed photons) are given by $\vec{\varepsilon} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\\pm i\\0\end{pmatrix}$.

Concerning the photon propagator, it is clear that $\langle 0|T[A_\mu A_\nu]|0\rangle$ carries two 4-vector indices. We will see below that the tensor structure carrying these indices is more involved, but gauge dependent. For a certain (legitimate) choice, this gauge dependence boils down to just $-g_{\mu\nu}$ (so-called Lorenz-Feynman gauge). Naively, it seems that the photon propagator carries an opposite sign compared to the scalar propagator $\sim \frac{i}{p^2-m^2+i\varepsilon}$. However, the physical propagating modes are transverse modes that can be parametrized by the spatial components of $A^\mu \to \vec{A}$. For these spatial components, we have $g_{\mu\nu}\big|_{\mu,\nu=1,2,3=i,j} = -\delta_{ij}$. Hence, the sign of the propagator for these modes is the same as for the scalar (however, it is and remains different for the time-like components A^0, implying that A^0 states have negative norm. Such negative-norm states can be a serious problem for any QFT, since they violate basic defining principles of Quantum mechanics. Nevertheless, in QED these problems are tamed by gauge invariance.

The absence of a photon mass in the propagator is clear from the fact that the field equations for the gauge potential can be brought into the form

$$\partial^2 A_\mu = 0 , \tag{6.55}$$

which holds in Lorenz gauge. This is the Klein-Gordon equation for the massless case for each component of the gauge field.

Some elementary QED examples are discussed in the exercises.