Quantum Field Theory

Lecture Notes

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These notes represent a revised version of my notes from the courses (2008 ... 12) originally written in German. It is next to impossible to squeeze a decent QFT course into a summer term.

With the new course on particles & fields (a.k.a. QFT 0) now held in the winter term, a substantial part of the more elementary topics such as aspects of classical field theory, representation theory of the Lorentz group, etc. are now covered in that introductory course. The present lecture notes therefore will go beyond the old one (but might be less detailed in the elementary parts).

Comments are welcome!

1 Introduction

1.1 What is QFT?

Let me try to give partial answers by listing 3 different statements that refer to different equally important aspects of QFT

1. QFT is the generalization of quantum mechanics to an infinite number of degrees of freedom, i.e. an infinite number of particles.

2. QFT is the quantized generalization of classical field theory.

3. (Relativistic) QFT is the consistent unification of quantum mechanics and special relativity.

QFT can be applied to (seemingly) very different fields:

- Statistical (or Euclidean) field theory
  describes statistical systems with many degrees of freedom (often but not necessarily near equilibrium); applications to critical phenomena, phase transitions, universality, etc. (e.g. Ising-type models)

- non-relativistic QFT
  forms the basis for condensed-matter or nuclear many-body physics; partially also for atomic or molecular many-body physics and is closely related if not identical to many-body theory. Typical applications are BCS theory for superconductivity, Bogoliubov-Hartree-Fock theory for quantum gases or Fermi gas- or liquid-models for nuclear matter

- relativistic QFT
  is the basis for the description of all known elementary particles and their interactions (maybe also including

gravity) in particular the

- electromagnetic interaction of charged matter (quarks and leptons such as electrons)

  $\Longrightarrow$ Quantum electrodynamics (QED)

- weak nuclear interaction, manifesting itself, e.g., in the β decay of neutrons and fusion processes in the stellar core. The unification of electromagnetic and weak interaction yields a richly structured QFT: the Glashow - Salam - Weinberg model also entailing the Higgs boson

- strong interaction among the quarks as the building blocks of nucleons (proton, neutron) and mesons.

  $\Longrightarrow$ Quantum chromodynamics (QCD)

These interactions together with the corresponding matter degrees of freedom constitute the Standard Model of particle physics with the underlying conceptual structure being QFT. In this lecture course, we concentrate on relativistic QFTs.

The relation between relativistic QFT and statistical field theory is similar to that between

$$\text{Schrödinger's equation} \overset{t \to i\tau}{\longleftrightarrow} \text{diffusion equation}$$

(quantum mechanics) (thermodynamic systems)

1.2 Units

We concentrate on

relativistic Quantum Field Theory

Special relativistic effects become important for

$$v \lesssim c \simeq 3 \cdot 10^{8} \, \frac{m}{s}$$

quantum phenomena occur for processes with an action

$$S \gtrsim \hbar \simeq 10^{-34} \, Js$$
$$\simeq 6.6 \; 10^{-25} \, GeV s$$

Measuring velocities in units of c and actions in units of $\hbar$ is equivalent to choose

$$\underline{\text{natural}} \text{ units} \qquad c = 1 = \hbar \, . \tag{1.1}$$

Using Einstein's mass energy relation

$$E = m c^2 , \qquad (1.2)$$

we can specify particle masses in units of energy such as GeV. E.g. the proton mass then reads

$$m_p \overset{(1.1)}{\equiv} m_p c^2 \simeq 1 \text{ GeV} . \qquad (1.3)$$

From quantum mechanics, we are familiar with the characteristic wavelength of matter waves (exhibiting interference phenomena): the Compton wave length

$$\lambdabar = \frac{\hbar}{mc} \equiv \frac{1}{m} , \qquad (1.4)$$

which in natural units is thus measured, e.g., in inverse GeV, $(\text{GeV})^{-1}$.

For various physical quantities, we hence obtain the following (non-exhaustive) list of natural units

$$\begin{aligned} [\text{mass}] = [\text{energy}] = [\text{momentum}] &\sim \text{GeV} \\ [\text{length}] = [\text{time}] = [(\text{mass})^{-1}] &\sim (\text{GeV})^{-1} \end{aligned} \qquad (1.5)$$

For the conversion of units, the following relation is useful:

$$1 = \hbar c \simeq 197 \text{ MeV fm} \simeq 0.2 \text{ GeV fm}, \quad (1.6)$$

where $\quad 1 \text{ fm} \equiv 1 \text{ Fermi} = 10^{-15} \text{ m}$.

Example :

The lightest bound state of the Strong Interaction is the pion, being a meson (and the lightest hadron),

$$m_\pi \simeq \frac{m_p}{7} \simeq 140 \text{ MeV}, \quad (1.7)$$

with a corresponding Compton wave length of

$$\lambda\!\!\!^{-}_\pi = \frac{1}{m_\pi} \simeq 1.4 \text{ fm}. \quad (1.8)$$

A typical hadronic length scale hence is $R_H \simeq 1$ fm, being thus five orders of magnitude smaller than the atomic length scale $1\,\text{Å} = 10^{-10}$ m. Indeed, we find this hadronic length scale in many quantities; e.g., the charge radius of the proton is $r_p \simeq 0.81$ fm. A typical hadronic cross section in a scattering experiment is of the order

$$\sigma_H \simeq \pi R_H^2 \simeq 3 \text{ fm}^2 = 30 \text{ mb}. \quad (1.9)$$

Similarly, a typical scale for the life time of particles that decay under the influence of the strong interaction can be estimated as

$$\tau = R_H = 1\,\mathrm{fm}/c \simeq \frac{1}{3}\cdot 10^{-23}\,\mathrm{s}. \quad (1.10)$$

1.3 The break-down of relativistic quantum mechanics

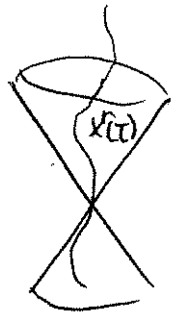
Naively, we may expect that quantum mechanics and special relativity could be consistently unified within a theory of relativistic quantum mechanics, being some relativistic generalization of Schrödinger's equation for an N-particle wave function, e.g. for $N=1$:

$$i\,\partial_t \Psi(\vec{x},t) = H\,\Psi(\vec{x},t) \quad (1.11)$$

with $H = \frac{p^2}{2m} = -\frac{1}{2m}\vec{\nabla}^2$ for a free particle.

Such an attempt at unifying the two concepts, however, typically leads to inconsistencies discussed in the following. Let us start with a free classical particle that propagates <u>causally</u> within its forward light cone along a trajectory

$$x^\mu = x^\mu(\tau) = \left(x^0(\tau), \vec{x}(\tau) \right), \tag{1.12}$$



where τ parametrizes its wordline $x^\mu(\tau)$.

According to Hamilton's principle, the particle moves along a trajectory of stationary action. The action of a particle in special relativity is given by the arc length of its trajectory,

$$S = \int_{\tau_1}^{\tau_2} d\tau\, L = -m \int_{\tau_1}^{\tau_2} d\tau \sqrt{\dot{x}^\mu \dot{x}_\mu} = -m \int_1^2 ds, \tag{1.13}$$

where we have used the Lagrange function $L = -m\sqrt{\dot{x}^\mu \dot{x}_\mu}$, and the invariant length element

$$ds^2 = dx_\mu dx^\mu = dt^2 - d\vec{x}^2 \tag{1.14}$$

and the metric convention $g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$, $a_\mu b^\mu \equiv g_{\mu\nu} a^\mu b^\nu$, and $\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}$. Implicitly, we have assumed that the parametrization $x(\tau)$ has been chosen such that $\dot{x}^2 \geq 0$ is always in the forward lightcone, i.e. "time-like".

The prefactor m is useful for a proper non-relativistic limit.

The conjugate momentum is

$$p_\mu \equiv - \frac{\partial L}{\partial \dot{x}^\mu} = m \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} \tag{1.15}$$

sign convention due to Minkowski metric with "-1" in spatial components.

Not all components of p_μ are independent: energy and spatial 3-momentum are related, because

$$p_\mu p^\mu \equiv p^2 = E^2 - \vec{p}^{\,2} \overset{(1.15)}{=} m^2 \frac{\dot{x}_\mu \dot{x}^\mu}{\dot{x}_\mu \dot{x}^\mu} = m^2, \tag{1.16}$$

corresponding to the relativistic one-particle dispersion relation $m^2 = E^2 - \vec{p}^{\,2}$, or $E^2 = \vec{p}^{\,2} + m^2$.

Let us try to define a relativistic vesion of the Hamilton function,

$$H = - p_\mu \dot{x}^\mu - L = - m \frac{\dot{x}^2}{\sqrt{\dot{x}^2}} + m \sqrt{\dot{x}^2} = 0. \tag{1.17}$$

This Hamilton function obviously vanishes! (This is related to the fact that the action S is invariant under reparametrizations of $x^\mu(\tau)$.)

Eq. (1.17) implies that there is no Lorentz-invariant generator of a relativistic time evolution.

This is not too surprising, as time is a component of a 4-vector and not a scalar quantity under Lorentz transformations.

As an alternative, we can give up exlicit covariance. For this, we choose a reference frame and use the corresponding Galilei-Newton time in this frame as the parameter τ:

$$\tau := t \equiv x^0 . \tag{1.18}$$

Then, the Hamilton function simply corresponds to the energy as usual,

$$H_t = p^0 = \sqrt{\vec{p}^2 + m^2} =: E_{\vec{p}} , \tag{1.19}$$

obtained, e.g., from the constraint (1.16). (Here, we have chosen the positive root for the energy to be bounded from below.)

We obtain a relativistic version of Schrödinger's equation

$$i \partial_t \Psi(\vec{x},t) = H_t \Psi(\vec{x},t) = \sqrt{-\vec{\nabla}^2 + m^2} \; \Psi(\vec{x},t), \tag{1.20}$$

where the square root of the Laplacian is understood in the sense of Hilbert's spectral theory.

This equation admits plane-wave solutions,

$$\Psi_{\vec{p}}(\vec{x},t) = e^{-iE_{\vec{p}}t + i\vec{p}\cdot\vec{x}} , \qquad (1.21)$$

which should correspond to solutions for a free relativistic particle. In fact, this solution can also be written in fully covariant form, $\Psi_{\vec{p}}(\vec{x},t) = e^{-ip^\mu x_\mu}$, illustrating that the solution is form-invariant in any reference frame. Though we have violated manifest Lorentz invariance by picking a specific reference frame to construct a Hamiltonian, Lorentz invariance is automatically restored again on the level of the solutions.

As an important test, a relativistic quantum theory has to maintain causality. Classically, causality is guaranteed, if a particle moves in the forward light cone,

$$x^2 = t^2 - \vec{x}^2 > 0 \qquad \text{for } m > 0, \qquad (1.22)$$

i.e. $\left(\frac{dx}{dt}\right)^2 = 1 - v^2 > 0 \Rightarrow v^2 < 1$

(for a particle trajectory passing through the origin).

Let us check this for the relativistic equation (1.20), assuming that we start with a pointlike wave function localized at the origin,

$$\Psi(\vec{x}, t=0) = \delta^{(3)}(\vec{x}) . \qquad (1.23)$$

Causality then implies that the probability for measuring the particle beyond the light cone has to vanish,

$$\Psi(\vec{x}, t) = 0 \qquad \text{for} \quad \vec{x}^2 > t^2 . \qquad (1.24)$$

To check (1.24), we study the Green's function, also called "the propagator", in analogy to the non-relativistic case:

$$\begin{aligned} U(\vec{x}, t) &:= \langle \vec{x} | \, e^{-iH_t t} \, | \vec{0} \rangle \\ &= \int \frac{d^3 p}{(2\pi)^3} \, e^{-i\sqrt{\vec{p}^2 + m^2}\, t + i\vec{p}\cdot\vec{x}} , \end{aligned} \qquad (1.25)$$

where $U(\vec{x}, 0) = \delta^{(3)}(\vec{x})$. Using polar momentum coordinates

$$\begin{aligned} U(\vec{x}, t) &= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^{\infty} dp \, p^2 \, e^{-i\sqrt{\vec{p}^2 + m^2}\, t} \int_{-1}^{1} d(\cos\theta) \, e^{ipx\cos\theta} \\ &= \frac{1}{4\pi^2 i x} \int_{-\infty}^{\infty} dp \, p \, e^{ipx - i\sqrt{p^2 + m^2}\, t} , \quad x = |\vec{x}| \end{aligned} \qquad (1.26)$$

The integral can be computed exactly with the aid of

Bessel functions. Here, it suffices to use a stationary-phase approximation:

$$\int dp\ e^{if(p)} = \int dp\ e^{i\left[f(p_0) + \frac{1}{2} f''(p_0)(p-p_0)^2 + O(|p-p_0|^3)\right]}$$

$$\simeq \frac{\sqrt{2\pi}}{\sqrt{if''(p_0)}}\ e^{if(p_0)}, \quad \text{using } f'(p_0) = 0, \tag{1.27}$$

i.e., p_0 is a stationary point of the phase $f(p)$.

In the present case, we have

$$f(p) = px - E_{\vec{p}}\, t \quad \Rightarrow \quad p_0 = \pm \frac{imx}{\sqrt{x^2-t^2}} \tag{1.28}$$

Normalizability of the propagator requires to choose the + sign,

$$\Rightarrow \quad f(p_0) = im\sqrt{x^2-t^2} \tag{1.29}$$

Hence, we obtain for the propagator

$$U(\vec{x},t) \sim e^{-m\sqrt{x^2-t^2}} \neq 0 \quad \text{for } x^2 > t^2 \tag{1.30}$$

We have to conclude that the probability to find the particle beyond the light cone is nonzero in this theory. Hence, causality is violated!

This probability is, however, exponentially small with a width corresponding to the Compton wave length $\lambda = \frac{1}{m}$.

We conclude that relativistic quantum mechanics becomes inconsistent if we localize down to or even below its Compton wave length. This can also be illustrated with the following heuristic argument: complete localization in spacetime, e.g. at $t=0$ with uncertainty $\Delta x \to 0$ as in (1.23), goes along with a large momentum uncertainty $\Delta p \gtrsim \frac{1}{\Delta x}$. Assuming that typical momenta are of the same size as the uncertainty, $p \simeq \Delta p$, and using that $p \equiv |\vec{p}| = (E_{\vec{p}}^2 - m^2)^{1/2} < E_{\vec{p}}$, we have

$$\Delta x \gtrsim \frac{1}{\Delta p} \simeq \frac{1}{p} > \frac{1}{E_{\vec{p}}} > 0 \quad . \qquad (1.31)$$

Therefore, localization appears to require negative energies as well in order to achieve $\Delta x \to 0$. These will in fact be associated later with anti-particles.

Let us conclude this section with a few comments: though we have shown the causality violations for our toy theory of relativistic quantum mechanics defined in (1.20), analogous conclusions also hold for the Klein-Gordon as well as Dirac's theory.

Second, we have implicitly assumed in the above argument that the particle is subject to some interactions. Otherwise, we would, of course, not be able to perform any measurement. With hindsight, it is actually possible to define a relativistic quantum theory of particles as long as they are completely non-interacting. Though this turns out to be mathematically consistent and in agreement with all required principles, it is physically rather irrelevant due to the absence of interactions.
Already for a very weakly interacting system, the above argument exemplifies the problems of a relativistic quantum theory and calls for a more powerful concept.