6. Interacting field theories with spinors

6.1 Yukawa theories

For the construction of scalar theories, we have used a criterion of simplicity. For the interactions this has been partly related to the dimensionality of the interaction terms; e.g. the $\lambda\phi^4$-term in $d=4$ dimensions has a dimensionless coupling constant $[\lambda]=0$.

For a similar argument for spinor theories, we first need the dimensionality of the spinor field. With regard to the kinetic term

$$S_D^{kin} = \int \underbrace{d^4x}_{-4} \; i\bar{\psi}\gamma^\mu \underbrace{\partial_\mu}_{1} \psi \,, \qquad (6.1)$$

we read off that $[\bar{\psi}\psi]=3$ and hence

$$[\psi] = \frac{3}{2} \,. \qquad (6.2)$$

The same result follows from the mass term

$$-\int \underbrace{d^4x}_{-4} \; \underbrace{m}_{1} \; \underbrace{\bar{\psi}\psi}_{\to 3} \,.$$

Recalling that scalar fields have mass dimension $[\phi] = 1$, the simplest interaction term which yields a Lorentz scalar is

$$S_{Yuk} = - \int \underbrace{d^4x}_{\to -4} \; h \; \underbrace{\phi}_{\to 1} \; \underbrace{\bar{\Psi}\Psi}_{\to 3} \, . \tag{6.3}$$

This is the so-called Yukawa interaction describing the interaction of two Dirac spinors with a scalar field. Historically, this has been first use for the description of the pion (scalars) - nucleon (spinors) interaction. The full action of a typical (simple) Yukawa theory is

$$S = \int d^4x \Big[\bar{\Psi} i \not{\partial} \Psi + \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - h \, \phi \bar{\Psi} \Psi - V(\phi) \Big] \, . \tag{6.4}$$

Here we have ignored a possible Dirac mass term which would break the axial symmetry. Actually, also the Yukawa interaction (6.3) breaks the chiral symmetry as

$$\bar{\Psi} \Psi \xrightarrow{\text{axial } U(1)} \bar{\Psi} e^{i\gamma_5 \theta} e^{i\gamma_5 \theta} \Psi = \bar{\Psi} e^{2i\gamma_5 \theta} \Psi \tag{6.5}$$

For generic $\theta \in [0, 2\pi]$, $e^{2i\gamma_5\theta}$ is a non-trivial 4×4 matrix which cannot be compensated by a transformation of a real scalar field $\phi \in \mathbb{R}$.

However, if we choose $\theta = \frac{\pi}{2}$, we have

$$e^{2i\gamma_5\theta} = \cos 2\theta + i\gamma_5 \sin 2\theta \qquad \text{(in general)}$$

$$\theta = \frac{\pi}{2}: \qquad e^{i\pi\gamma_5} = \cos\pi = -1 \tag{6.6}$$

and hence: $\bar{\Psi}\Psi \rightarrow -\bar{\Psi}\Psi$.

If we now combine this specific axial transformation with the $\mathbb{Z}_2$-symmetry of the scalar field $\phi \rightarrow -\phi$ (provided that $V(\phi)$ is $\mathbb{Z}_2$ symmetric), the Yukawa theory of (6.4) is invariant under the discrete symmetry:

$$\begin{aligned} \phi &\rightarrow -\phi \\ \Psi &\rightarrow e^{i\frac{\pi}{2}\gamma_5}\Psi \\ \bar{\Psi} &\rightarrow \bar{\Psi}\, e^{i\frac{\pi}{2}\gamma_5} \end{aligned} \tag{6.7}$$

Note that the Dirac mass term would not be compatible with (6.7).

In turn, if we impose the symmetry (6.7), the spinor field is massless. The mass of the scalar field depends on the parameters in the potential i.e. if we have

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \tag{6.8}$$

The scalar field is massive.

Now, we know that the $\mathbb{Z}_2$ symmetry in the scalar sector can be broken spontaneously if $V(\phi)$ has minima different from $\phi = 0$, i.e. for

$$V(\phi) = -\frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4!} \phi^4 \tag{6.9}$$

$$\Rightarrow \qquad \phi_{min} = \pm v = \pm \sqrt{\frac{6\mu^2}{\lambda}} \tag{6.10}$$

Let us assume that ϕ picks the value $\phi_{min} = v$ as its ground state. Expanding ϕ about this ground state $\phi(x) = v + \sigma(x)$, we find the action (c.f. Eq. B.8):

$$S = \int d^4x \Big[\bar{\psi}\, i\partial\!\!\!/ \psi + \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) - h v \bar{\psi}\psi - h\sigma \bar{\psi}\psi - \Big[\frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{3!}\lambda v \sigma^3 + \frac{1}{4!}\lambda\sigma^4\Big]\Big] \tag{6.11}$$

Here, we can read off the mass $m_\sigma^2 = 2\mu^2$ of the scalar excitation σ. In addition, we observe the occurrence of a Dirac mass term $-(hv)\bar{\psi}\psi$, such that the Dirac spinors have also acquired a mass

$$m_\psi = hv = h\sqrt{\frac{6\mu^2}{\lambda}}. \tag{6.12}$$

The remaining terms are interactions of Yukawa type $\sim \sigma\bar{\psi}\psi$ or scalar self-interactions.
We conclude that the breaking of the $\mathbb{Z}_2$ symmetry in the scalar sector also extends to the Yukawa sector, spontaneously generating a mass for the Dirac spinor which is otherwise kept zero if the symmetry is preserved. This is a first simple but non-trivial example for the fact that Dirac spinor

masses can be zero on the level of the action but then be generated by spontaneous symmetry breaking in a scalar sector.

The present model is often used as a toy-model for the sector of the Standard model of particle physics involving only the Higgs boson and the top quark (as the heaviest quark). As the model only features a discrete symmetry, no Goldstone bosons occur in the broken phase (as is also true for the standard model, however, by virtue of the Higgs mechanism involving a gauge symmetry).

It is instructive to also study this (toy-) standard model application of the present model on the level of parameters and numbers. On the level of the Lagrangian, we have 3 parameters: h, μ^2, λ. This corresponds to the number of measurable quantities in the top-Higgs sector of the standard model:

Higgs boson mass: $m_H \simeq 125\ \text{GeV}$ (date: 2015)

top quark mass: $m_t \simeq 172\ \text{GeV}$ (date: 2015)

Fermi-constant ~ Higgs vacuum expectation value $v = (\sqrt{2}\, G_F)^{-\frac{1}{2}} \simeq 246\ \text{GeV}$ (date: 2005)

Using the identification with our model parameters:

$$m_H \Leftrightarrow m_\sigma = \sqrt{2\mu^2} = \sqrt{2}\,\mu$$

$$m_t \Leftrightarrow m_\psi = h\,v = h\sqrt{\frac{6\mu^2}{\lambda}} \qquad (6.13)$$

$$v \Leftrightarrow v = \sqrt{\frac{6\mu^2}{\lambda}} \quad \Leftrightarrow \quad \lambda = \frac{6\mu^2}{v^2}$$

We find

$$\mu \simeq 88 \text{ GeV}$$
$$h \simeq 0.70 \qquad (6.14)$$
$$\lambda \simeq 0.77$$

We observe that both coupling constants are of order $\mathcal{O}(1)$. However, λ comes with a factor of $(4!)^{-1}$ in the action. This is not the case for the top-Yukawa coupling h. Even though the top-quark ist very short-lived with a lifetime of $\sim 5 \cdot 10^{-25}$ s and was difficult to discover due to its high mass (discovery 1995 by CDF and DØ at Tevatron, Fermilab), it plays the most important role for the dynamics of the theory at high energies among all the other quarks and leptons.
Of course, for a proper discussion in the context of particle physics, a full quantization of the theory is necessary.

6.2 Yukawa vs. fermionic theories

In the purely scalar case, we have been able to construct a whole class of models by promoting the real scalar $\varphi \in \mathbb{R}$ to a vector φ^a in an internal symmetry space $O(N)$. Naively, one may try to do the same for Yukawa systems by promoting $\varphi \to \varphi^a$ and similarly promoting the Dirac spinor to multiple copies $\psi \to \psi^a$, which are often called "flavors" in the fermionic context, $a = 1 \dots N_f$.

However, it is not fully trivial how to construct a Yukawa interaction from such rather arbitrary building blocks (e.g. try to contract the indices to get a scalar). Moreover, since $\varphi^a \in \mathbb{R}^N$ for $a = 1 \dots N$, φ^a transforms under $O(N)$ whereas ψ^a, $\bar{\psi}^a$ are complex fields and hence

$\bar{\psi}^a \psi^a$ is invariant under the unitary group $U(N_f)$.

So, the symmetries would not fit.

In the above example, we had the action

$$S = \int d^4x \, \Big[\bar{\psi} i \partial\!\!\!/ \psi + \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - h \varphi \bar{\psi} \psi - V(\varphi) \Big], \qquad (6.15)$$

being invariant under $\mathbb{Z}_2$ symmetry. However, the symmetry acted rather differently on ϕ and ψ, c.f. Eq.(6.7).

On the other hand, the symmetry transformation looks equivalent on the level of ϕ and the fermion bilinear:

$$\begin{aligned} \phi &\to -\phi \\ \bar{\psi}\psi &\to -\bar{\psi}\psi \end{aligned} \tag{6.16}$$

In fact, this can become a general construction principle for theories with spinors and further fields to feature invariance under bigger continuous symmetries.

This construction principle becomes even more visible in a certain limit of the above theory. Let us take a look at the equations of motion:

$$\begin{aligned} (i\partial\!\!\!/ - h\phi)\psi &= 0 \\ \partial^2\phi + V'(\phi) + h\bar{\psi}\psi &= 0 \end{aligned} \tag{6.17}$$

Obviously, we have two coupled partial differential equations featuring a high degree of nonlinearity.

Let us study a particularly simple limit: let us assume that ϕ is slowly varying or almost constant in spacetime $\phi \simeq$ const., then, with $\partial^2\phi \simeq 0$, we get

$$h\bar{\psi}\psi + V'(\phi) = 0 \qquad (6.18)$$

For the simple case $V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!}\phi^4$, we have

$$h\bar{\psi}\psi + m^2\phi + \frac{\lambda}{3!}\phi^3 = 0 \qquad (6.19)$$

Let us further assume that $\frac{\lambda}{3!} \ll 1$, then:

$$\phi = -\frac{h}{m^2}\bar{\psi}\psi, \qquad (6.20)$$

which is naturally compatible with the symmetry. Even if we included a full potential $V(\phi)$, (6.18) can in principle be expressed as $\phi = f(\bar{\psi}\psi)$ at least locally connecting the scalar to a fermion bilinear. It is instructive to study the action (6.4) in this limit $\lambda \ll 1$, $\partial\phi \simeq 0$:

$$S = \int d^4x \left[\bar{\psi}\, i\partial\!\!\!/\, \psi - h\phi\bar{\psi}\psi - \frac{1}{2}m^2\phi^2\right] \qquad (6.21)$$

Using the equation of motion (6.20) for ϕ, we get an action depending solely on the spinor field:

$$S = \int d^4x \left[\bar{\Psi} i\partial\!\!\!/ \Psi - h\left(-\frac{h}{m^2}\bar{\Psi}\Psi\right)\bar{\Psi}\Psi - \frac{1}{2}m^2\left(-\frac{h}{m^2}\bar{\Psi}\Psi\right)^2\right]$$

$$= \int d^4x \left[\bar{\Psi} i\partial\!\!\!/ \Psi + \frac{h^2}{2m^2}(\bar{\Psi}\Psi)^2 \right]$$

$$= \int d^4x \left[\bar{\Psi} i\partial\!\!\!/ \Psi + \frac{g}{2}(\bar{\Psi}\Psi)^2 \right] , \; g = \frac{h^2}{m^2} \qquad (6.22)$$

This is the famous Gross-Neveu model, introduced by Gross and Neveu in 1974 in two dimensions as a model with analogies to the strong interactions. The precise statement is that the ~~theories~~ theory defined by (6.22) purely in terms of spinors and that of (6.21) defined in terms of spinors and scalars are completely identical by virtue of the equations of motion (6.20) of the scalar field.

Of course, beyond the limit $\lambda \to 0$ and for non vanishing scalar kinetic terms, the equivalence is only approximate.

Incidentally in the quantized version, the exact equivalence between (6.22) and (6.21) persists to hold. Moreover the equivalence can even hold upon inclusion of interactions and derivative terms for properties of the long-range physics. This is an example of <u>universality</u>.

In turn, if we had started with the Gross-Neveu model (6.22), we could have used the inverse construction, defining a scalar field

$$\varphi = -g\,\bar{\psi}\psi \tag{6.23}$$

to write the action as

$$S = \int d^4x \left[\bar{\psi}\, i\partial\!\!\!/\psi - \varphi\bar{\psi}\psi - \frac{1}{2}\frac{1}{g}\varphi^2 \right] \tag{6.24}$$

Writing $g = \frac{h^2}{m^2}$ and rescaling $\varphi \to h\varphi$ would have lead to (6.21) again. This construction (which also exists for the quantum theory) that converts a non-linear fermionic theory into a bilinear (Gaußian) action is known as Hubbard-Stratonovich transformation.

Let us use this construction to introduce Yukawa models with higher symmetries. E.g. it is straightforward to upgrade the spinor content to N_f flavors ψ^a, $a = 1, \dots N_f$:

$$S = \int d^dx \left[\bar{\psi}^a i\partial\!\!\!/ \psi + \frac{g}{2} (\bar{\psi}^a \psi^a)^2 \right] \qquad (6.25)$$

This theory is invariant under flavor rotations,

$$\begin{aligned} \psi^a &\to U^{ab} \psi^b \\ \bar{\psi}^a &\to \bar{\psi}^b U^{\dagger ba} \end{aligned} \qquad (6.26)$$

such that $U^\dagger U = \mathbb{1}$, i.e. $U \in U(N_f)$

In absence of a mass term $\sim \bar{\psi}^a \psi^a$ (which would also be $U(N_f)$ invariant), the model also has the discrete $\mathbb{Z}_2$ axial symmetry (6.7), transforming $\bar{\psi}^a\psi^a \to -\bar{\psi}^a\psi^a$.

The structure of the interaction suggests to introduce a scalar field

$$\phi = -g\, \bar{\psi}^a \psi^a, \qquad (6.27)$$

Leading, as before, to the equivalent action

$$S = \int d^4x \left[\bar{\psi}^a i\partial\!\!\!/ \psi^a - \phi \bar{\psi}^a \psi^a - \frac{1}{2}\frac{1}{g}\phi^2 \right] \quad (6.28)$$

Now, we can add kinetic terms and interaction terms for the scalar field to arrive at a new Yukawa theory for N_f spinor flavors:

$$S_{Yuk} = \int d^4x \Big[\bar{\psi}^a i\partial\!\!\!/ \psi^a - h\phi \bar{\psi}^a \psi^a + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) \Big] \quad (6.29)$$

The model still has the full $U(N_f)$ flavor symmetry. However, the scalar sector is the same as before. In order to preserve the $\mathbb{Z}_2$ symmetry of the fermionic system, we only need a real scalar $\phi \in \mathbb{R}$ and a $\mathbb{Z}_2$ symmetric potential $V(-\phi) = V(\phi)$.

Upon spontaneous symmetry breaking by a suitable potential with a minimum at $\phi_{min} = v \neq 0$, all flavors of fermions acquire the same mass term $\quad -m_\psi \bar{\psi}^a \psi^a \quad , \; m_\psi = h v \; . \qquad (6.30)$

Most importantly, the breakdown of the $\mathbb{Z}_2$ symmetry does not imply the breakdown of flavor symmetry. The mass term preserves the $U(N_f)$ symmetry.

In order to arrive at a more complex scalar sector, the axial / chiral symmetry on the fermionic side has to be more complex as well.

6.3 Models with continuous chiral symmetry

In the exercises, we had already studied a fermionic model with continuous chiral symmetry:

$$S_{NJL} = \int d^4x \left(\bar{\psi} i \partial\!\!\!/ \psi - \frac{g}{2} \left((\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right) \right) \qquad (6.31)$$

This is the famous Nambu–Jona-Lasinio model for the case of one fermion flavor $N_f = 1$.

The model has been invented by Nambu and Jona-Lasinio (and independently by Vaks and Larkin) in 1961 by transferring ideas from the

BCS theory of superconductivity to the description of nucleons and mesons in elementary particle physics. Up to the present day it is frequently used as an effective low-energy model of the strong interactions (low-energy QCD)

The model is invariant under

$$U_V(1): \quad \psi \to e^{i\alpha}\psi \,, \quad \bar\psi \to \bar\psi e^{-i\alpha} \tag{6.32}$$
$$U_A(1): \quad \psi \to e^{i\alpha\gamma_5}\psi \,, \quad \bar\psi \to \bar\psi\, e^{+i\alpha\gamma_5}$$

as discussed in detail in the exercises. Hence it is also invariant under both chiral symmetries $U_L(1)$, $U_R(1)$, which are a linear combination of (6.32).

In the spirit of the Hubbard-Stratonovich transformation, it is natural to introduce two scalar fields,

$$\varphi_1 = -g\,(\bar\psi\psi) \quad , \quad \varphi_2 = -ig\,(\bar\psi\gamma_5\psi) \tag{6.33}$$

to rewrite (6.31) as

$$S_{NJL} = \int d^4x \left[\bar\psi\, i\partial\!\!\!/\,\psi - \varphi_1 \bar\psi\psi - i\varphi_2 \bar\psi\gamma_5\psi - \frac{1}{2}\frac{1}{g}\left(\varphi_1^2 + \varphi_2^2\right)\right] \tag{6.34}$$

Since $\bar\psi\psi$ as well as $\bar\psi\gamma_5\psi$ are separately invariant under $U_V(1)$, the fields φ_1 and φ_2 transform trivially under this symmetry: $\varphi_{1/2} \to \varphi_{1/2}$. Since the Noether charge of this $U_V(1)$ corresponds to particle number, this implies that φ_1 and φ_2 do not carry particle number ($\hat{=}$ electric charge, hence are neutral).

In order to identify their transformation under $U_A(1)$, we note that

$$e^{i\alpha\gamma_5} = \mathbb{1} \cos\alpha + i\gamma_5 \sin\alpha \tag{6.35}$$

This implies that

$$\begin{aligned}
\bar\Psi\Psi &\to \bar\Psi e^{i\alpha\gamma_5} e^{i\alpha\gamma_5}\Psi = \bar\Psi e^{2i\alpha\gamma_5}\Psi \\
&= \bar\Psi\Psi \cos 2\alpha + i \sin 2\alpha\, \bar\Psi\gamma_5\Psi \\
\bar\Psi\gamma_5\Psi &\to \bar\Psi e^{i\alpha\gamma_5}\gamma_5 e^{i\alpha\gamma_5}\Psi \\
&= \bar\Psi\gamma_5\Psi \cos 2\alpha + i \sin 2\alpha\, \bar\Psi\Psi
\end{aligned} \tag{6.36}$$

We observe that the combination

$$\varphi_1 \bar\Psi\Psi + i\varphi_2 \bar\Psi\gamma_5\Psi$$

is invariant under $U_A(1)$, iff φ_1 and φ_2

transform as

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \qquad (6.37)$$

Interpreting φ_a, $a = 1,2$ as an element of $\mathbb{R}^2$, Eq. (6.37) corresponds to an $SO(2)$ rotation in the φ_1, φ_2 plane. This rotation also leaves the scalar mass term $\sim (\varphi_1^2 + \varphi_2^2)$ invariant as it corresponds to the scalar product in $\mathbb{R}^2$. Since the symmetry groups $SO(2) \simeq U(1)$ are isomorphic to one another the complex transformations of ψ and the real transformations of φ_1, φ_2 fit perfectly. Note, however that a full axial rotation in $U_A(1)$ from $\alpha = 0$ to $\alpha = 2\pi$ covers the $SO(2)$ rotations twice: $2\alpha = 0$ to $2\alpha = 4\pi$.
In the language of Noether charges this implies that the scalar carries twice the axial charge of the spinor.

These symmetry considerations allow us to finally construct a Yukawa theory that exhibits the chiral symmetry of the NJL model.

$$S_{Yuk\text{-}NJL} = \int d^4x \left[\bar{\psi}\, i\not{\partial} \psi + \frac{1}{2} (\partial_\mu \varphi_a)(\partial^\mu \varphi_a) - h \left(\varphi_1 \bar{\psi}\psi + i \varphi_2 \bar{\psi} \gamma_5 \psi \right) - V(\varphi) \right] \qquad (6.38)$$

where $V(\varphi)$ depends on φ_a only through the scalar product $\varphi_a \varphi_a$. Note that the symmetry fixes both Yukawa interactions to have the same coupling h.

Let us now study the predictions of this model for the particle / mass spectrum in the phase with spontaneous symmetry breaking if $V(\varphi)$ develops a vacuum expectation value at

$$\varphi_{0a} \varphi_{0a} = v^2 \qquad (6.39)$$

Parametrizing the field as

$$\begin{pmatrix} \varphi_1(x) \\ \varphi_2(x) \end{pmatrix} = \begin{pmatrix} v + \sigma(x) \\ \pi(x) \end{pmatrix}, \qquad (6.40)$$

the action (6.38) becomes (for $V(\varphi) = -\frac{1}{2}\mu^2 \varphi_a \varphi_a + \frac{\lambda}{4!} (\varphi_a \varphi_a)^2$)

$$S_{Yuk-NJL} = \int d^4x \Big[\bar{\Psi}\, i\partial\!\!\!/ \Psi + \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) + \frac{1}{2}(\partial_\mu \pi)(\partial^\mu \pi)$$

$$- h v \bar{\Psi}\Psi \quad - h\left(\sigma \bar{\Psi}\Psi + i\pi \bar{\Psi}\gamma_5 \Psi\right)$$

$$- V(\sigma, \pi) \Big] \,, \qquad (6.41)$$

where $V(\sigma,\pi)$ is the same potential that we have studied in the context of $O(N)$ models in Eq. (3.20) for the case of only one π field. Hence, we obtain the mass spectrum

$$\begin{aligned} m_\Psi &= h v \\ m_\sigma &= \sqrt{2\mu^2} \\ m_\pi &= 0 \,. \end{aligned} \qquad (6.42)$$

The masslessness of the π is in agreement with Goldstones theorem. The fermions become massive. As the π-field couples to $\bar{\Psi}\gamma_5\Psi$ which is a pseudoscalar fermion bilinear, also π must trasform as a pseudoscalar (i.e. with a minus sign under parity)

In their original publications Nambu & Jona-Lasinio associated the ψ's with the nucleon (proton / neutron), the π-field with the/a light pion and thus predicted the sigma meson as a heavy nucleon / anti-nucleon bound-state. Of course, quarks had not yet been invented in 1961.
In the modern use of the NJL model, ψ denotes the quarks and hence m_ψ is interpreted as the constituent quark mass $m_\psi \simeq 300\,\mathrm{MeV}$.
With regard to the Hubbard - Stratonovich transformation

$$\varphi_1 = v + \sigma \sim -g(\bar\psi\psi) ,$$

the non-vanishing expectation value of φ_1 is also interpreted as a nonvanishing chiral condensate $\langle\bar\psi\psi\rangle$ (in quantum notation).
Since the mesons $(\sigma, \pi, \ldots)$ are bound states and not fundamental in contrast to the quarks. The formation of a bilinear condensate is sometimes also called "dynamical symmetry breaking". Quantitatively, the vacuum expectation value is related to the pion decay constant $v \simeq 93\,\mathrm{MeV}$.