5. Simple Spinor Field theories

Having identified the spinor fields $\xi_\alpha(x)$, $\eta^{\dot\alpha}(x)$ as the simplest non-trivial representation of the Lorentz group, let us try to construct field theories for these spinors by means of Lorentz-invariant Lagrangians

5.1 Kinetic part

Using (4.49), we can immediately write a derivative in spinor space:

$$\partial_\mu \Rightarrow \partial_{\alpha\dot\beta} = (\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu \tag{5.1}$$

Whereas scalar fields involved always two derivatives to form a Lorentz scalar $(\partial_\mu\varphi)(\partial^\mu\varphi)$, it is already possible to write down a single derivative term in the spinor case which is nevertheless bilinear in the fields and thus no total derivative:

$$\eta^{*\alpha}\,(\sigma^\mu)_{\alpha\dot\beta}\,\partial_\mu\eta^{\dot\beta} = \eta^\dagger\,\sigma^\mu\partial_\mu\,\eta \tag{5.2}$$

where $\left((\eta^{\dot\alpha})^* = (\eta^*)^\alpha\right)$

Since the spinor fields are complex, Eq. (5.2) is not guaranteed to be real. Hence, we may try

$$\mathcal{L} \stackrel{?}{=} \eta^\dagger \sigma^\mu \partial_\mu \eta + \text{h.c.} = \eta^\dagger \sigma^\mu \partial_\mu \eta + (\partial_\mu \eta^\dagger) \sigma^\mu \eta$$

$$= \partial_\mu \left(\eta^\dagger \sigma^\mu \eta \right) \tag{5.3}$$

However, this combination projecting on the real part is a total derivative and hence does not give rise to nontrivial equations of motion.

Therefore, the only combination left is the imaginary part

$$\underline{\underline{\mathcal{L}_L^{\text{kin}}}} = \frac{i}{2} \left(\eta^\dagger \sigma^\mu \partial_\mu \eta - (\partial_\mu \eta^\dagger) \sigma^\mu \eta \right)$$

$$=: \frac{i}{2} \underline{\underline{\eta^\dagger \sigma^\mu \overleftrightarrow{\partial}_\mu \eta}} \,. \tag{5.4}$$

This is the simplest possible kinetic term.

Similarly, we obtain for ξ:

$$\underline{\underline{\mathcal{L}_R^{\text{kin}}}} = \frac{i}{2} \xi^*_{\dot\beta} (\bar\sigma^\mu)^{\dot\beta\alpha} \overleftrightarrow{\partial}_\mu \xi_\alpha = \frac{i}{2} \underline{\underline{\xi^\dagger \bar\sigma^\mu \overleftrightarrow{\partial}_\mu \xi}} \tag{5.5}$$

Both Lagrangians exhibit a continuous symmetry of phase transformations,

$$\xi'(x) = e^{i\theta}\,\xi(x) \quad , \quad \xi^{*\prime}(x) = e^{-i\theta}\,\xi^{*}(x)$$

$$\eta'(x) = e^{i\theta'}\,\eta(x) \quad , \quad \eta^{*\prime}(x) = e^{-i\theta'}\,\eta^{*}(x) \quad , \tag{5.6}$$

that leave the action invariant. These symmetries are also called chiral symmetries, each one forming a $U(1)$ group: $U(1)_R$, $U(1)_L$.

5.2 Mass term

Analogously to bosonic field theories, we expect that a mass term in the Lagrangian has to be quadratic in the fields such that excitations propagate according to the relativistic dispersion relation of a point particle. As the kinetic term is linear in derivatives ($\sim$ 4-momenta), we expect the quadratic term in the Lagrangian to be linear in the mass.

The simplest quadratic Lorentz scalars are

$$\eta^{\dot\alpha}\,\varepsilon_{\dot\alpha\dot\beta}\,\eta^{\dot\beta} = \eta^T \varepsilon\, \eta \quad , \quad \xi_\alpha\, \varepsilon^{\alpha\beta}\,\xi_\beta = \xi^T \varepsilon\, \xi \; . \tag{5.7}$$

Explicitly, this yields, e.g.

$$\eta^{\dot\alpha}\,\varepsilon_{\dot\alpha\dot\beta}\,\eta^{\dot\beta} = \eta^1\eta^2 - \eta^2\eta^1 \tag{5.8}$$

If the components η^1 and η^2 are ordinary commuting numbers, this expression is identically zero.
However, with a glimpse into quantum theory, we expect that the connection between spin and statistics eventually implies that the excitations of the spinor fields obey Fermi-Dirac statistics (spin-statistics theorem): in a quantum setting, we will associate η^1 and η^2 with operators that create a spinor excitation above the vacuum. Since these excitations have to obey Fermi-Dirac statistics, these creation operators have to anti-commute, i.e. $\eta^1\eta^2 = -\eta^2\eta^1$.
For operators, this property seems straightforwardly implementable. Nevertheless, here we do not plan to quantize, but stay within classical field theory. Still, we wish to realize the correct statistical properties of the excitations.
Evidently, both requirements cannot be satisfied by pure numbers $\eta^1, \eta^2 \in \mathbb{C}$. Still, there exists a consistent set of numbers, defined in terms of conventional algebraic axioms, that even facilitates the definition of derivatives and integrals, with the special property that these numbers anti-commute. These are the Grassmann numbers.

If we interpret $\eta^1, \eta^2, \xi_1, \xi_2$ to be Grassmann-valued, we have $\eta^1\eta^2 = -\eta^2\eta^1$, and thus (5.8) is nonzero

(Grassmann numbers can be treated abstractly; if we still wish to represent them in terms the body of real numbers, we are lead to matrix representations, see exercises.)

Hence, a real mass term is given by

$$\mathcal{L}_L^m = -\frac{1}{2}\left(m_L \eta^T \varepsilon \eta - m_L^* \eta^\dagger \varepsilon \eta^*\right)$$
$$\mathcal{L}_R^m = -\frac{1}{2}\left(m_R \xi^\dagger \varepsilon \xi^* - m_R^* \xi^T \varepsilon \xi\right), \qquad (5.9)$$

where the mass parameters m_L and m_R may be complex. Here we have used $(\theta\chi)^* = \chi^* \theta^*$ for Grassmann numbers (as is familiar from matrices). Also, $\varepsilon^\dagger = \varepsilon^T = -\varepsilon$ has been used.

These mass terms are called Majorana masses. The Majorana mass breaks the chiral symmetry $U(1)_L$ or $U(1)_R$ completely. If a Majorana mass exists, the corresponding Noether charges are not conserved. In particle physics, no Majorana mass term has been verified (yet). Still, the mass of the neutrinos may be associated with a Majorana mass term; if so, the non-conservation of the Noether charge would translate into violations of lepton number conservation. A possible signature in terms of a neutrinoless double β decay is actively searched for.

In condensed-matter systems, Majorana fermions can arise as an effective degree of freedom. This is currently a very active field of research.

Whereas the above kinetic and mass terms can exist for each representation $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ separately, there is another possible mass term, which exists in the simultaneous presence of the two spinors:

$$\mathcal{L}_D^m = -\left(m\,\xi^\dagger\eta + m^*\eta^\dagger\xi\right) \qquad (5.10)$$

This is the <u>Dirac</u> mass term. It does not break the chiral symmetries completely: choosing $\theta = \theta'$ in (5.6), the spinors transform as

$$\begin{pmatrix}\eta'\\ \xi'\end{pmatrix} = e^{i\theta}\begin{pmatrix}\eta\\ \xi\end{pmatrix}, \quad \begin{pmatrix}\eta^{*\prime}\\ \xi^{*\prime}\end{pmatrix} = e^{-i\theta}\begin{pmatrix}\eta^*\\ \xi^*\end{pmatrix} \qquad (5.11)$$

These simultaneous $U(1)_L$ and $U(1)_R$ transformations form also a $U(1)$ group which is called a "vector" $U(1)_V$. The corresponding Noether charge is positive for η, ξ and negative for η^*, ξ^*. Hence, this symmetry is similar to the $U(1)$ symmetry for a complex scalar. The Noether charge can be associated with "particle number" or electric charge upon coupling to a Maxwell field.

5.3 The Dirac spinor

Whereas the kinetic terms as well as the Majorana mass term can be formulated for each $SL(2,\mathbb{C})$ spinor ξ or η (Weyl spinors) separately, the Dirac mass term requires the simultaneous presence of both Weyl spinors and provides for a bilinear coupling. Hence, it is useful to introduce the combined 4-spinor

$$\Psi(x) = \begin{pmatrix} \eta^{\dot\alpha}(x) \\ \xi_\alpha(x) \end{pmatrix}, \tag{5.12}$$

which is a <u>Dirac spinor</u>, obviously belonging to the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation of the Lorentz group. We obtain a compact notation for the Lagrangians by also summarizing the (generalized) Pauli matrices as

$$\gamma^\mu := \begin{pmatrix} 0 & (\bar\sigma^\mu)^{\dot\alpha\beta} \\ (\sigma^\mu)_{\alpha\dot\beta} & 0 \end{pmatrix} \tag{5.13}$$

Or, more explicitly

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \quad , \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \qquad (5.14)$$

These are the Dirac matrices. They occur here in the so-called chiral representation (several other representations are also used in the literature). Independently of the representation, the γ matrices satisfy (c.f. (4.44))

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = \{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \cdot \mathbb{1} \qquad (5.15)$$

Eq. (5.15) can be viewed as the defining property of the Dirac matrices. Mathematically, the Dirac matrices form the base elements of a Clifford algebra, i.e. an algebra of elements that close under the anti-commutator.

The generator of Lorentz transformations in the Dirac representation can also be constructed from those of the Weyl spinors, c.f. (4.50):

$$\underbrace{\sigma^{\mu\nu}}_{\text{now } 4\times4} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} (\sigma^{\mu\nu})_\alpha{}^\beta & 0 \\ 0 & \underbrace{(\bar{\sigma}^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}}_{2\times2} \end{pmatrix}. \qquad (5.16)$$

Hence, the Lorentz transformed spinor reads

$$\Psi'(x') = D_{(\frac{1}{2},0)\oplus(0,\frac{1}{2})}(\Lambda)\ \Psi(x) = \left(e^{-\frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}}\right)\Psi(x) =: A\ \Psi(x). \tag{5.17}$$

The 4x4 matrix A is the direct analogue of the transformation matrix a. From (4.51) and (4.52), we can read off

$$\left(A\,\Psi(x)\right)^{\dot\alpha}_{\ \alpha} = \begin{pmatrix} (\varepsilon a^* \varepsilon^T)^{\dot\alpha}{}_{\dot\beta} & 0 \\ 0 & a_\alpha{}^\beta \end{pmatrix}\begin{pmatrix} \eta^{\dot\beta}(x) \\ \xi_\beta(x) \end{pmatrix}. \tag{5.18}$$

Here, we have used the SL(2,C) spinor indices $\dot\alpha, \alpha$ in order to make the SL(2,C) content explicit. Of course, working directly with the Dirac spinor, it is more natural to summarize the components for $\dot\alpha = 1,2$, $\alpha = 1,2$ into one index $\Psi^\gamma(x)$, $\gamma = 1,2,3,4$, of the 4-component spinor $\Psi(x)$.

With the aid of another definition,

$$\bar A := \gamma_0 A^\dagger \gamma_0 , \tag{5.19}$$

together with (4.45), it is straightforward to verify that

$$\bar{A}\, \gamma^\mu A = \Lambda^\mu{}_\nu \gamma^\nu . \qquad (5.20)$$

This equation emphasises the relation between the Lorentz transformations of the Dirac spinor and that of the "4-vector" of Dirac matrices γ^μ, as the transformations of the spinor indices of the γ's (LHS) can be written as a Lorentz transformation of the vector index (RHS).

The "bar" symbol in (5.19) is used to denote <u>Dirac conjugation</u>. In addition to complex conjugation, it involves a multiplication with γ_0 for each index. It is useful to think of γ_0 as a "spin metric", i.e., it relates spinor space to its corresponding dual vector space. The elements of this dual space are Dirac conjugated spinors:

$$\bar{\psi} := \psi^\dagger \gamma_0 . \qquad (5.21)$$

In fact, this spinor occurs naturally, if we

Consider the kinetic Lagrangian for the Dirac spinor

$$\mathcal{L}_D^{kin} := \mathcal{L}_L^{kin} + \mathcal{L}_R^{kin} = \frac{i}{2}\,\bar{\Psi}\,\gamma^\mu \overset{\leftrightarrow}{\partial}_\mu \Psi$$

$$= \frac{i}{2}\left(\bar{\Psi}\gamma^\mu \partial_\mu \Psi\right) - \frac{i}{2}\left((\partial_\mu \bar{\Psi})\gamma^\mu \Psi\right)$$

$$= i\,\bar{\Psi}\gamma^\mu \partial_\mu \Psi - \frac{i}{2}\,\partial_\mu\left(\bar{\Psi}\gamma^\mu\Psi\right) . \qquad (5.22)$$

Hence, the action can be written as

$$\underline{\underline{S_D^{kin}}} = \underline{\underline{\int d^4x \;\; i\,\bar{\Psi}\,\gamma^\mu \partial_\mu \Psi}} . \qquad (5.23)$$

Similarly, the Dirac mass term (for a real mass $m = m^*$) can compactly be written as

$$\mathcal{L}_D^{m} = -m\,\bar{\Psi}\Psi . \qquad (5.24)$$

For an analysis of chiral symmetries in the Dirac notation, it is useful to introduce

$$\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \qquad (5.25)$$

where the first equality holds in general, and the second is particular for the chiral representation.

In the chiral representation, it is obvious that γ_5 can be used to define chiral projectors

$$P_R = \frac{1+\gamma_5}{2} \quad , \quad P_L = \frac{1-\gamma_5}{2} \, , \qquad (5.26)$$

satisfying

$$P_{R,L}^2 = P_{R,L} \quad , \quad P_R P_L = 0 = P_L P_R$$
$$\text{and} \quad P_R + P_L = \mathbb{1} \, , \qquad (5.27)$$

such that

$$\psi_R := P_R \psi = \begin{pmatrix} 0 \\ \xi_\alpha \end{pmatrix} \quad , \quad \psi_L := P_L \psi = \begin{pmatrix} \eta^{\dot\alpha} \\ 0 \end{pmatrix} . \qquad (5.28)$$

Before we analyse the chiral symmetries, let us briefly verify the manifest Lorentz invariance of the Dirac theory:

$$S_D = \int d^4x \left(i \bar\psi \gamma^\mu \partial_\mu \psi - m \bar\psi \psi \right) \qquad (5.29)$$

Since $\psi' = A\psi$, it follows $\bar\psi' = \bar\psi \bar A$ (using $\gamma_0^2 = \mathbb{1}$).

Let us explicitly study the kinetic part:

$$\overline{\Psi}'\gamma^\mu \underline{\partial'_\mu}\Psi' = \overline{\Psi}\bar{A}\,\gamma^\mu\,\Lambda_\mu{}^\nu\partial_\nu A\Psi$$

$$= \overline{\Psi}\,\underbrace{\bar{A}\gamma^\mu A}\,\Lambda_\mu{}^\nu\partial_\nu\Psi$$

$$\overset{(5.20)}{=} \overline{\Psi}\,\Lambda^\mu{}_\rho\,\gamma^\rho\,\underbrace{\Lambda_\mu{}^\nu\partial_\nu}_{g_{\mu\sigma}\Lambda^\sigma{}_\nu\partial^\nu}\Psi$$

$$= \overline{\Psi}\,\underbrace{g_{\mu\sigma}\Lambda^\mu{}_\rho\Lambda^\sigma{}_\nu}_{=g_{\rho\nu}}\gamma^\rho\partial^\nu\Psi = \underline{\overline{\Psi}\gamma^\nu\partial_\nu\Psi} \qquad (5.30)$$

Of course, the invariance was already clear from the $SL(2,\mathbb{C})$ construction. But this example shows manifestly that invariant scalars arise if both vector as well as Dirac spinor indices are all fully contracted. From the invariance of the Dirac mass term in the $SL(2,\mathbb{C})$ construction, it follows that $\bar{A}A = \mathbb{1}$ (which can be verified straightforwardly) such that

$$\overline{\Psi}'\Psi' = \overline{\Psi}\Psi \qquad (5.31)$$

transforms as a <u>scalar</u>.

Similarly, we can justify that $\bar{\psi}\gamma_\mu\psi$ transforms as a vector and $\bar{\psi}\sigma_{\mu\nu}\psi$ as a tensor under Lorentz transformations. Since we have

$$\gamma_5 \to -\gamma_5 \quad \text{under parity} \quad x^i \to -x^i,$$

the combination $\bar{\psi}\gamma_5\psi$ is a pseudoscalar and $\bar{\psi}\gamma_5\gamma_\mu\psi$ a pseudovector under Lorentz transformations. Note that only the "open" Lorentz indices are relevant for this classification. With respect to spinor space, all these expressions are "scalars" anyway.

Concerning the chiral transformations $U(1)_L \times U(1)_R$ of ξ and η, these can equivalently be represented by their linear combinations:

$$\begin{aligned} \theta = \theta' : \; U(1)_V \; &: \; \psi' = e^{i\theta}\psi \;, \; \bar{\psi}' = \bar{\psi}\, e^{-i\theta} \\ \theta = -\theta' : \; U(1)_A \; &: \; \psi' = e^{i\gamma_5\theta}\psi \;, \; \bar{\psi}' = \bar{\psi}\, e^{i\gamma_5\theta} \end{aligned} \tag{5.32}$$

As discussed in the exercises, γ_5 anticommutes with all γ_μ's:

$$\{\gamma^\mu, \gamma_5\} = 0 \tag{5.33}$$

With this property, we can verify the invariance of the kinetic term under the $U(1)_A$, the so-called axial transformations:

$$
\begin{aligned}
U(1)_A: \quad \bar{\psi}' \gamma^\mu \partial_\mu \psi' &= \bar{\psi}\, e^{i\gamma_5\theta} \gamma^\mu \partial_\mu\, e^{i\gamma_5\theta} \psi \\
&= \bar{\psi} \gamma^\mu e^{-i\gamma_5\theta} \partial_\mu e^{i\gamma_5\theta} \psi = \bar{\psi} \gamma^\mu \partial_\mu \psi \,.
\end{aligned} \tag{5.34}
$$

The mass term $\sim -m\bar{\psi}\psi$, however, is not invariant under axial transformations.

By contrast, both kinetic and mass term are invariant under the vector transformations $U(1)_V$ in agreement with the observations in the $SL(2,\mathbb{C})$ formalism.

5.4 Dirac Equation

Since the Dirac spinor is a complex object (complex-Grassmann-valued), we can use the same trick as for complex scalar fields and treat Ψ and $\bar{\Psi}$ as formally independent for the variational principle. Hence, we obtain the equation of motion by varying the action (5.29) e.g. with respect to $\bar{\Psi}$:

$$0 = \frac{\delta}{\delta \bar{\Psi}} S_D = \left(i\gamma^\mu \partial_\mu - m \right) \Psi(x) = 0 \qquad (5.35)$$

This is the Dirac equation. In the following, let us just recall a few basic properties of this relativistic spinor theory. In order to verify that m indeed has the meaning of mass in the sense of a relativistic point particle, let us multiply (5.35) with $(-i\gamma^\nu \partial_\nu - m)$:

$$0 = (-i\gamma^\nu\partial_\nu - m)(i\gamma^\mu\partial_\mu - m)\Psi = \Big(\underbrace{\gamma^\mu\gamma^\nu}_{= \underbrace{\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}}_{= g^{\mu\nu}} + \frac{1}{2}\underbrace{[\gamma^\mu,\gamma^\nu]}_{\text{antisym}}} \overbrace{\partial_\mu \partial_\nu}^{\text{sym.}} + m^2 \Big) \Psi$$

$$= \left(\partial^2 + m^2 \right) \Psi(x) . \qquad (5.36)$$

Hence, the solutions of the Dirac equation also satisfy the Klein-Gordon equation and thus the solutions obey the relativistic energy-momentum relation with mass m.

This suggests as an ansatz

$$\psi(x) = u(p)\, e^{-ipx} \qquad \text{where } p^2 = m^2 \tag{5.37}$$

In the chiral basis, the spinor $u(p)$ has to satisfy

$$\left[\begin{pmatrix} 0 & \bar{\sigma}\cdot p \\ \sigma\cdot p & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right] u(p) = 0 \tag{5.38}$$

We observe that $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_\mu p_\nu \underbrace{\tfrac{1}{2}\left(\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu\right)}_{= g^{\mu\nu}} = p^2 = m^2$

and hence the Dirac equation is solved by

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\xi \\ \sqrt{p\cdot\sigma}\,\xi \end{pmatrix} \tag{5.39}$$

where ξ is an arbitrary $SL(2,\mathbb{C})$ spinor.

Check:

$$\left[\begin{pmatrix} 0 & \bar{\sigma}\cdot p \\ \sigma p & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right] \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\xi \\ \sqrt{p\cdot\sigma}\,\xi \end{pmatrix} = \begin{pmatrix} \bar{\sigma}\, p \sqrt{p\cdot\sigma}\,\xi - m\sqrt{p\cdot\bar{\sigma}}\,\xi \\ \sigma\cdot p\sqrt{p\bar{\sigma}}\,\xi - m\sqrt{p\cdot\sigma}\,\xi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{p\bar{\sigma}}\left(\sqrt{(\bar{\sigma}\, p)(\sigma\, p)} - m\right)\xi \\ \sqrt{p\sigma}\left(\underbrace{\sqrt{(\sigma\, p)(\bar{\sigma}\, p)} - m}_{=0}\right)\xi \end{pmatrix} = 0$$

Possible base spinors are $\xi^s = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ (times a Grassmann-valued number) such that (5.39) represents two solutions corresponding to spin-up ξ^1 or spin-down ξ^2 along the 3-direction, i.e. eigenvalues to $p_3\sigma_3 = \begin{pmatrix}p_3 & 0\\0 & -p_3\end{pmatrix}$.

The solutions are normalized to

$$\bar{u}^r(p)\, u^s(p) = 2m\,\delta^{rs} \qquad (5.40)$$

or $$u^{\dagger r}(p)\, u^s(p) = 2E_{\vec{p}}\,\delta^{rs}, \quad E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$$

which is straightforwardly verifiable.

In addition, there are also "negative frequency" solutions

$$\psi(x) = v(p)\, e^{+ipx}, \quad p^2 = m^2, \quad p^0 > 0 \qquad (5.41)$$

where $v(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\;\eta^s\\ -\sqrt{p\cdot\bar{\sigma}}\;\eta^s\end{pmatrix}$ with spin base vectors η^s, $s=1,2$

The latter are normalized to

$$\bar{v}^r(p)\, v^s(p) = -2m\,\delta^{rs}, \quad v^{\dagger r}(p)\, v^s(p) \qquad (5.42)$$

The u and v spinors are also mutually orthogonal,

$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0 \qquad (5.43)$$

In particle-physics processes, one is often interested in spin-summed results (e.g. if the spin of a single particle is not measured by the detector). For these, let us finally mention the following spin sums

$$\sum_s u^s(p)\,\bar{u}^s(p) = \gamma\cdot p + m$$
$$\sum_s v^s(p)\,\bar{v}^s(p) = \gamma\cdot p - m \qquad (5.44)$$

The frequently occurring combination $\gamma_\mu p^\mu = \gamma\cdot p$ is often abbreviated by the Feynman slash $\underline{\underline{\gamma_\mu p^\mu = \not{p}}}$

5.5 Rarita - Schwinger spinors

So far, we have encountered the trivial spin-0 (scalar fields), and the nontrivial spin-$\frac{1}{2}$ (Weyl spinors, Dirac spinors), and spin-1 (vector fields, photon) representations of the Lorentz group. In classical field theory, it is straightforward to construct higher-spin representations and their corresponding free theories (interacting theories which satisfy all consistency criteria can be more difficult ...).

As an example, let us study the spin-$\frac{3}{2}$ case. More concretely, we wish to compose a field Ψ_μ such that it unifies properties of a Dirac spinor (with 4 spinor components with suppressed indices) as well as a vector field with index $\mu = 0,1,2,3$. So, in total Ψ_μ has 16 complex components. Since vectors belong to the $(\frac{1}{2},\frac{1}{2})$ representation, and Dirac spinors to the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ representation, the general object Ψ_μ is an element of the tensor product space

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)\right)$$
$$= \left[\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\tfrac{1}{2},0\right)\right] \oplus \left[\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(0,\tfrac{1}{2}\right)\right] \qquad (5.45)$$

Now, recall from the summation of angular momenta that the tensor product of two spin-$\frac{1}{2}$ gives a spin-1 as well as a scalar spin-0 component:

$$\tfrac{1}{2} \otimes \tfrac{1}{2} = 1 \oplus 0$$

Or, using the notation that counts the dimensions of the Hilbert spaces,

$$\underline{2} * \underline{2} = \underline{3} + \underline{1} \, . \qquad (5.46)$$

Hence, Eq. (5.45) yields

$$\left(\tfrac{1}{2},\tfrac{1}{2}\right) \otimes \left(\left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)\right)$$
$$= \left(1,\tfrac{1}{2}\right) \oplus \left(0,\tfrac{1}{2}\right) \oplus \left(\tfrac{1}{2},0\right) \oplus \left(\tfrac{1}{2},1\right)$$
$$= \underbrace{\left[\left(1,\tfrac{1}{2}\right) \oplus \left(\tfrac{1}{2},1\right)\right]}_{\text{Rarita-Schwinger}} \oplus \underbrace{\left[\left(\tfrac{1}{2},0\right) \oplus \left(0,\tfrac{1}{2}\right)\right]}_{\text{Dirac spinor}} . \qquad (5.47)$$

We observe that this tensor product contains Dirac spinors as well as the new terms $\left[\left(1,\tfrac{1}{2}\right) \oplus \left(\tfrac{1}{2},1\right)\right]$, and thus is reducible into a Dirac part that we already know, and a new part which we will call a Rarita-Schwinger spinor (incidentally, Rarita and Schwinger's original $\frac{3}{4}$-page paper deals with the full reducible object).

It is, in fact, easy to get rid off the Dirac part by noting that the object $(\gamma^\mu \psi_\mu)$ is a "scalar" with respect to the Lorentz index structure but still features a Dirac index. Hence for a general ψ_μ, the object $\chi = \gamma^\mu \psi_\mu$ transforms as a Dirac spinor and thus corresponds to the $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$ part of ψ_μ.

In turn, those fields ψ_μ that satisfy the irreducibility condition

$$\gamma^\mu \psi_\mu = 0 \qquad (5.48)$$

do not contain Dirac spinor elements and hence transform as $(1,\frac{1}{2}) \oplus (\frac{1}{2},1)$ representation of the Lorentz group.

The irreducibility condition (5.48) has important consequences for the construction of a Lagrangian. For instance, one might naively try to write down a symmetrically looking mass term:

$$\bar{\psi}^{\mu} \psi_{\mu} = \bar{\psi}_{\mu} g^{\mu\nu} \psi_{\nu} = \frac{1}{2} \bar{\psi}_{\mu} \{\gamma^{\mu}, \gamma^{\nu}\} \psi_{\nu}$$

$$= \frac{1}{2} \bar{\psi}_{\mu} \gamma^{\mu} \underbrace{\gamma^{\nu} \psi_{\nu}}_{=0} + \frac{1}{2} \bar{\psi}_{\mu} \underbrace{\gamma^{\nu} \gamma^{\mu}}_{= \gamma^{\mu}\gamma^{\nu} - [\gamma^{\mu}, \gamma^{\nu}]} \psi_{\nu}$$

$$= \frac{1}{2} \bar{\psi}_{\mu} \gamma^{\mu} \gamma^{\nu} \psi_{\nu} - \frac{1}{2} \bar{\psi}_{\mu} [\gamma^{\mu}, \gamma^{\nu}] \psi_{\nu}$$

$$= i \bar{\psi}_{\mu} \sigma^{\mu\nu} \psi_{\nu} \qquad , \quad \sigma^{\mu\nu} = \frac{i}{2} [\gamma^{\mu}, \gamma^{\nu}] \tag{5.49}$$

We observe the mass term, in fact, has to be antisymmetric in the Lorentz indices of the Rarita-Schwinger field. A similar argument applies to the building block of a kinetic term:

$$\bar{\psi}_{\mu} \gamma_{\nu} \partial_{\kappa} \psi_{\lambda} ,$$

implying that all indices must be contracted in an antisymmetric fashion. This is possible with the aid of the ε tensor. In order to preserve parity invariance, we amend this building block with another γ_5 factor. The resulting Lagrangian for the Rarita-Schwinger field reads

$$\mathcal{L} = -\frac{1}{2}\,\overline{\Psi}_\mu\left(\varepsilon^{\mu\nu\kappa\lambda}\,\gamma_5\gamma_\nu\,\partial_\kappa \;-\; i m\,\sigma^{\mu\lambda}\right)\Psi_\lambda \qquad (5.50)$$

Correspondingly, the field equation yields

$$\left(\varepsilon^{\mu\nu\kappa\lambda}\,\gamma_5\gamma_\nu\,\partial_\kappa - i m\,\sigma^{\mu\lambda}\right)\Psi_\lambda = 0 \qquad (5.51)$$

Spin-$\frac{3}{2}$ fields are indeed known and used in physics for the description of spin-$\frac{3}{2}$ bound states in the theory of the strong interactions. An example is given by the Δ resonances of the nucleon which are bound states of 3 quarks with all spins $\frac{1}{2}$ aligned to yield a spin $\frac{3}{2}$ state (Δ^- : ddd , Δ^0: udd
Δ^+ : uud, Δ^{++}: uuu),

each having a lifetime $\sim 5\cdot 10^{-24}$ s and commonly decaying to (p^+, n^0) and (π^+, π^-, π^0) depending on the charge state.

Elementary particles of spin-$\frac{3}{2}$ which are not bound-states have not been observed so far. In fact, a straightforward perturbative quantization of spin-$\frac{3}{2}$ fields leads to inconsistencies ("perturbatively nonrenormalizable").

These inconsistencies can be (partly) resolved in supersymmetric theories, where the Rarita-Schwinger spinor becomes the superpartner of the graviton and is called gravitino.