4. Particles and Fields as Representations of the Lorentz group.

Even in absence of any internal symmetries, the symmetries of spacetime are an essential property. In relativistic field theories, these are given by the Poincaré group consisting of spacetime translations and Lorentz transformations, consequences of both of which have already been discussed above. In the following, we detail how Lorentz invariance is connected to a classification of fields. Analogous considerations can also be performed for non-relativistic field theories on the basis of Galilei invariance

4.1 Lorentz transformations

Let us take a closer look at Lorentz transformations, recalling first some essential properties already listed in chapter 1: a Lorentz transformation is a linear operation on spacetime vectors v^μ,

$$v^\mu \to v^{\mu'} = \Lambda^\mu{}_\nu v^\nu , \qquad (4.1)$$

that preserves the scalar product in Minkowski space

$$v^2 = g_{\mu\nu} v^\mu v^\nu \equiv v^\mu v_\mu \quad , \; g = (+,-,-,-) \qquad (4.2)$$

The transformation matrix hence satisfies

$$g_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = g_{\rho\sigma} \, . \qquad (4.3)$$

This generalizes to transformations of arbitrary contravariant tensors

$$T'^{\mu_1 \dots \mu_n} = \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} T^{\nu_1 \dots \nu_n} \qquad (4.4)$$

of rank n.

There are only two constant invariant tensors. One is given by the metric by virtue of (4.3). The other one is the totally anti-symmetric tensor

$$\varepsilon^{\mu\nu\rho\sigma} \quad , \; \varepsilon^{0123} := 1 \qquad (4.5)$$

with the usual rules for the Levi-Civita symbol. According to (4.4), it transforms as

$$\begin{aligned} \varepsilon'^{\mu\nu\rho\sigma} &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \Lambda^\rho{}_\gamma \Lambda^\sigma{}_\delta \, \varepsilon^{\alpha\beta\gamma\delta} \\ &= \varepsilon^{\mu\nu\rho\sigma} \det \Lambda \end{aligned} \qquad (4.6)$$

where the second step makes use of the construction of the determinant using the ε-symbol.

From (4.3), we read off

$$(\det \Lambda)^2 = 1 \quad \Rightarrow \quad \det \Lambda = \pm 1 \tag{4.7}$$

So strictly speaking, ε is only invariant for those Lorentz transformations that have $\det \Lambda = +1$, but changes sign under those with $\det \Lambda = -1$.

From 3d Euclidean space, we are already familiar with transformations that change the sign of ε: these are given by those orthogonal transformations that convert a right-handed basis into a left-handed one. Analogously, this applies to Minkowski space.

From (4.3), we can derive another fact:

$\rho = 0 = \sigma$ $$1 = (\Lambda^0{}_0)^2 - (\Lambda^i{}_0)^2$$

$$\Rightarrow (\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2$$

$$\Rightarrow \underline{\underline{\Lambda^0{}_0 \geq 1}} \quad \text{or} \quad \underline{\underline{\Lambda^0{}_0 \leq -1}} \tag{4.8}$$

The set of all Lorentz-transformations forms the group $O(3,1)$ (more precisely: the Λ's discussed here form a matrix representation of this group) (analogously to orthogonal transformations $O(4)$ in 4-dim Euclidean space, additionally accounting for the metric signatures).

Eqs. (4.7) & (4.8) proof that this set can be decomposed into disconnected components, as there is neither a path (1-parameter family of Λ's) that could possibly continuously interpolate between the $\det\Lambda = +1$ and $\det\Lambda = -1$ transformations nor a path interpolating between the Λ's with $\Lambda^0{}_0 \geq 1$ and $\Lambda^0{}_0 \leq -1$. This makes four disconnected components, out of which those with

$$\det\Lambda = +1 \quad , \Lambda^0{}_0 \geq 1 \qquad (4.9)$$

are called <u>orthochronous</u> <u>proper</u> Lorentz transformations. This is the component that contains the unit element of the group $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu$.

The other components are related to the orthochronous proper component by a parity transformation (right $\Leftrightarrow$ left-handed basis) and/or a time inversion ($t \to -t$)

Obviously, the infinitesimal Lorentz transformations belong to the orthochronous proper component

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon^\mu{}_\nu \, , \; \varepsilon^\mu{}_\nu \ll 1 \, . \quad (4.10)$$

Expanding (4.3) to first order yields

$$g_{\varrho\sigma} + g_{\mu\sigma}\,\varepsilon^\mu{}_\varrho + g_{\varrho\nu}\,\varepsilon^\nu{}_\sigma + \mathcal{O}(\varepsilon^2) = g_{\varrho\sigma} \qquad (4.11)$$

$$\Longrightarrow \quad \varepsilon_{\nu\mu} + \varepsilon_{\mu\nu} = 0 \qquad (4.12)$$

Thus, $\varepsilon_{\mu\nu}$ is an antisymmetric matrix with 6 independent parameters, 3 of which correspond to Lorentz-boosts (being parametrized by a spatial velocity vector $\vec{v}$) and further 3 describe spatial rotations (e.g. Euler angles).

It is useful to write an infinitesimal Lorentz transformation as

$$v'^\mu = v^\mu + \varepsilon^\mu{}_\nu v^\nu =: \left(\mathbb{1} - \frac{i}{2}\,\varepsilon^{\varrho\sigma} M_{\varrho\sigma}\right)^\mu{}_\nu \, v^\nu \qquad (4.13)$$

where

$$(M_{\varrho\sigma})^\mu{}_\nu = i\left(\delta^\mu_\varrho\, g_{\sigma\nu} - \delta^\mu_\sigma\, g_{\varrho\nu}\right) \qquad (4.14)$$

This way of writing the transformation separates the parameters $\varepsilon^{\rho\sigma}$ from the "generators" $M_{\rho\sigma}$ of the Lorentz symmetry that encode the algebraic structure. For any given set of fixed indices $\rho\sigma$, $M_{\rho\sigma}$ is a 4×4 matrix (with indices $\mu\nu$ in (4.14)). These matrices satisfy

$$\boxed{[M_{\mu\nu}, M_{\rho\sigma}] = -i\left(g_{\mu\rho} M_{\nu\sigma} - g_{\nu\rho} M_{\mu\sigma} - g_{\mu\sigma} M_{\nu\rho} + g_{\nu\sigma} M_{\mu\rho}\right)} \tag{4.15}$$

Eq. (4.15) defines the Lie-Algebra of the Lorentz group $SO(3,1)$ (the "S" means $\det\Lambda = 1$). From an abstract perspective, Eq. (4.14) defines a particular representation of this algebra in terms of 4×4 matrices. Since $M_{\sigma\rho} = -M_{\rho\sigma}$, there are in total 6 generators of this algebra.

Independently of the representation, we obtain finite Lorentz transformations (within the orthochronous proper component) by the exponential map

$$\Lambda = e^{-\frac{i}{2}\varepsilon^{\rho\sigma}M_{\rho\sigma}} \simeq 1 - \frac{i}{2}\varepsilon^{\rho\sigma}M_{\rho\sigma} + \mathcal{O}(\varepsilon^2) \qquad (4.16)$$

4.2 Fields as representations of the Lorentz group

Fields being the fundamental degrees of freedom of a field theory can be classified according to their behavior under Lorentz transformations. So far, we have mainly considered scalar fields which transform trivially,

$$\phi'(x') = \phi(x) \quad , \quad x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad (4.17)$$

We have also already encountered the gauge field $A_\mu(x)$ which transforms as a vector,

$$A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x) \, . \qquad (4.18)$$

For a general N-tupel φ_i, $i = 1 \dots N$, the transformation rule reads

$$\varphi_i'(x') = D(\Lambda)_i{}^j \varphi_j(x) \qquad (4.19)$$

where $D(\Lambda)$ should be an $N \times N$ matrix representation of the Lorentz group. Which representations do exist?

Infinitesimally, we have

$$D(\Lambda)_i{}^j = \delta_i{}^j - \frac{i}{2}\varepsilon^{\mu\nu}(S_{\mu\nu})_i{}^j \qquad (4.20)$$

where $S_{\mu\nu}$ is an $N \times N$ matrix for each fixed set of μ, ν. In order to correspond to a Lorentz transformation, $S_{\mu\nu}$ has to satisfy the Lorentz algebra (4.15), $S_{\mu\nu} \equiv D(M_{\mu\nu})$. Our goal is to classify all possible choices of $S_{\mu\nu}$. For this, we first go back to the representation $M_{\mu\nu}$ and introduce

$$\begin{aligned} J_i &:= \frac{1}{2}\varepsilon_{ijk} M^{jk} \\ K_i &:= M_{i0} = -M_{0i} \quad , i,j,k = 1,2,3 \end{aligned} \qquad (4.21)$$

Using (4.15), it is straightforward to verify

$$\begin{aligned} [J_i, J_k] &= i\varepsilon_{ijk} J_k \\ [J_i, K_j] &= i\varepsilon_{ijk} K_k \\ [K_i, K_j] &= -i\varepsilon_{ijk} J_k \end{aligned} \qquad (4.22)$$

$\vec{J}$ satisfies the angular momentum algebra and hence is evidently related to the generator of spatial rotations. $\vec{K}$ in turn corresponds to the generator of Lorentz boosts.

It is instructive change the "basis" of generators once more and introduce

$$\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K}) \quad , \quad \vec{B} = \frac{1}{2}(\vec{J} - i\vec{K}) . \qquad (4.23)$$

These satisfy

$$[A_i, A_j] = i\varepsilon_{ijk} A_k \quad , \quad [B_i, B_j] = i\varepsilon_{ijk} B_k ,$$
$$[A_i, B_j] = 0 . \qquad (4.24)$$

Therefore, the Lorentz algebra is equivalent to two sets of angular momentum algebras "$\vec{A}$ and $\vec{B}$ spins". These spin algebras obviously commute. We conclude that we can classify all possible representations of the Lorentz algebra simply in terms of all possible representations of these angular momentum algebras.

The latter are enumeratable in terms of the eigenvalue of the squared spins $\vec{A}^2$, $\vec{B}^2$. For a given total spin, the eigenvectors can further be labeled by the eigenvalues of on spin component, say A_3 and B_3

$$\vec{A}^2 |A\, a\rangle = A(A+1)\, |A\, a\rangle \quad , \quad A_3 |A\, a\rangle = a\, |A\, a\rangle$$

$$\vec{B}^2 |B\, b\rangle = B(B+1)\, |B\, b\rangle \quad , \quad B_3 |B\, b\rangle = b\, |B\, b\rangle$$

$$a = -A, \dots, A \qquad b = -B, \dots, B \tag{4.25}$$

For a given set of total spin quantum numbers A&B, the representation space is spanned by $|Aa, Bb\rangle = |Aa\rangle \otimes |Bb\rangle$ and is

$$N = (2A+1)(2B+1) \tag{4.26}$$

dimensional. Hence, the index i of the N-tuple field φ_i simply labels all possible values of a and b

$$i = (a, b)\,. \tag{4.27}$$

In this fashion, we have found all possible irreducible representations of the Lorentz algebra. Of course, by means of tensor products, we can combine different representations to form further reducible representations.

4.3 Spinors

Apart from the trivial scalar representation, the simplest representation is a spin $\frac{1}{2}$ representation,

$$\text{e.g.} \quad (A, B) = (0, \tfrac{1}{2}) \tag{4.28}$$

$$\Rightarrow \quad D(\vec{A}) = 0 \quad , \; D(\vec{B}) = \frac{\vec{\sigma}}{2} \quad (\sigma_i : \text{Pauli matrices}) .$$

The corresponding fields have two components,

$$\varphi_i \rightarrow \xi_\alpha \quad , \; \alpha = 1, 2 \tag{4.29}$$

The representations of $\vec{J}$ and $\vec{K}$ are

$$D(\vec{J}) = \frac{\vec{\sigma}}{2} \quad , \; D(\vec{K}) = i \frac{\vec{\sigma}}{2} . \tag{4.30}$$

We can summarize the parameters $\varepsilon^{\mu\nu}$ of the Lorentz transformation into two 3-vectors:

$$(\varepsilon_{23}, \varepsilon_{31}, \varepsilon_{12}) =: -\vec{\theta} \; , \; (\varepsilon_{10}, \varepsilon_{20}, \varepsilon_{30}) = \vec{\omega} \tag{4.31}$$

such that the representation of the Lorentz transformation is given by

$$D(\Lambda) = e^{i\vec{\theta} \cdot D(\vec{J}) + i\vec{\omega} \cdot D(\vec{K})} , \tag{4.32}$$

or explicitly

$$a_\alpha^{\ \beta} := D(\Lambda)_\alpha^{\ \beta} = \left[e^{\frac{i}{2}\vec{\theta}\cdot\vec{\sigma} - \frac{1}{2}\vec{\omega}\cdot\vec{\sigma}} \right]_\alpha^{\ \beta} \qquad (4.33)$$

$$\Rightarrow \xi'_\alpha(x') = a_\alpha^{\ \beta}\, \xi_\beta(x) . \qquad (4.34)$$

As can be verified explicitly, the matrix a is a 2×2 matrix with complex entries and satisfies

$$\det a = 1 \qquad (4.35)$$

Thus it has 6 real parameters which are exhausted by $\vec{\theta}$ and $\vec{\omega}$. The set of matrices of this type form the matrix group

$$\underset{\substack{\uparrow \\ \det = 1}}{S}\ \underset{\substack{\uparrow \\ \text{linear} \\ \text{transformations}}}{L}\ (\underset{\substack{\uparrow \\ 2\times 2 \text{ matrix}}}{2}\ ,\ \underset{\substack{\uparrow \\ \text{complex components}}}{\mathbb{C}}) \qquad (4.36)$$

We call the field $\xi_\alpha(x)$ also an "$SL(2,\mathbb{C})$ spinor".

The above equations (4.32) & (4.33) describe a homomorphism between the Lorentz group $SO(3,1)$ and $SL(2,\mathbb{C})$, where $SL(2,\mathbb{C})$ covers each element of $SO(3,1)$ twice (as is already familiar from $SU(2) \Leftrightarrow SO(3)$ in quantum mechanics):

let $\vec{w}=0$. If we rotate θ_1 by 2π, we have $\Lambda_\mu{}^\nu = \delta_\mu^\nu$, whereas $a \to -a$ in $SL(2,\mathbb{C})$.

The identity is reached again after a 4π rotation.

To close this section, we can also study the complex conjugate spinor $(\xi_\alpha)^* \equiv \xi_{\dot\alpha}$ ("dotted" spinor), which transforms as

$$\eta'_{\dot\alpha}(x') = a^*{}_{\dot\alpha}{}^{\dot\beta}\, \eta_{\dot\beta}(x) \qquad \left(a^*{}_{\dot\alpha}{}^{\dot\beta} \equiv \left(a_\alpha{}^\beta\right)^* \right) \tag{4.37}$$

From the complex conjugate form of a in (4.33) we can deduce backwards that this corresponds to a representation

$$D(\vec{A}) = -\frac{\vec{\sigma}^*}{2}, \quad D(\vec{B}) = 0 \tag{4.38}$$

which is an $(A,B) = (\frac{1}{2},0)$ representation.

Since the dimension of a representation of the Lorentz group is given by $N = (2A+1)(2B+1)$, 4-vectors (being related to integer spins) have to be related to the mixed representation:

$$2 \times 2^* : \quad (0, \tfrac{1}{2}) \times (\tfrac{1}{2}, 0) = (\tfrac{1}{2}, \tfrac{1}{2}) \tag{4.39}$$

In practise, this implies that there must be a relation between an object with indices $(\alpha, \dot{\beta})$ and one with index μ. For this, we define the auxiliary objects

$$(\sigma_\mu)_{\alpha\dot{\beta}} = (\mathbb{1}, \vec{\sigma}) \quad , \quad (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} = (\mathbb{1}, -\vec{\sigma}) \tag{4.40}$$

It is suggestive to use the 2d ε-tensor as a metric in spinor space, e.g.

$$(\bar{\sigma}_\mu)_{\dot{\alpha}\beta} := \varepsilon_{\dot{\alpha}\dot{\gamma}}\, \varepsilon_{\beta\delta}\, (\bar{\sigma}_\mu)^{\dot{\gamma}\delta} \tag{4.41}$$

Then it can straightforwardly be checked that σ_μ and $\bar{\sigma}_\mu$ are related by

$$(\bar{\sigma}_\mu)_{\dot{\alpha}\beta} = \left[(\sigma_\mu)_{\alpha\dot{\beta}} \right]^* \tag{4.42}$$

With these definitions, it also follows that

$$\frac{1}{2}\operatorname{tr}\left(\bar{\sigma}^\mu \sigma_\nu\right) = \delta^\mu_\nu \;, \quad \left(\sigma^\mu\right)_{\alpha\dot\beta}\left(\bar\sigma_\mu\right)^{\dot\gamma\delta} = 2\,\delta_\alpha^\delta\,\delta_{\dot\beta}^{\dot\gamma}$$

and (4.43)

$$\sigma_\mu \bar\sigma_\nu + \sigma_\nu \bar\sigma_\mu = \bar\sigma_\mu \sigma_\nu + \bar\sigma_\nu \sigma_\mu = 2 g_{\mu\nu} \qquad (4.44)$$

Using the explicit representation (4.33) for a Lorentz transformation $a_\alpha{}^\beta$, we obtain the important formula

$$\sigma_\mu \Lambda^\mu{}_\nu = a \sigma_\nu a^+ . \qquad (4.45)$$

This equation connects the Lorentz transformation of a 4-vector, $\Lambda^\mu{}_\nu$, with the transformation matrices a and a^+ of a spinor and its complex conjugate.

This suggests to define the spinor representation of a 4-vector

$$\underline{x} := x^\mu \sigma_\mu = \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix}, \qquad (4.46)$$

Eq. (4.45) now gives us the transformation properties

$$\underline{X}' = \underbrace{x'^{\mu}}_{\Lambda^{\mu}{}_{\nu} x^{\nu}} \sigma_{\mu} = \sigma_{\mu} \Lambda^{\mu}{}_{\nu} x^{\nu} \overset{(4.45)}{=} a \sigma_{\nu} a^{+} x^{\nu}$$

$$= a \underline{X} a^{+} \qquad (4.47)$$

In turn, we can construct a 4-vector out of two independent spinors ξ_{α}, $\eta_{\dot{\alpha}}$:

$$V_{\mu} : \xi^{\alpha} (\sigma_{\mu})_{\alpha\dot{\beta}} \eta^{\dot{\beta}} . \qquad (4.48)$$

By an argument inverse to (4.47), it is possible to show that V_{μ} transforms as a 4-vector under Lorentz transformations if ξ_{α} and $\eta_{\dot{\alpha}}$ transform as spinors.

The general relation between a vector and a mixed spinor object is hence given by

$$V_{\alpha\dot{\beta}} = V^{\mu} (\sigma_{\mu})_{\alpha\dot{\beta}} \quad , \quad V^{\mu} = \frac{1}{2} (\bar{\sigma}^{\mu})^{\dot{\beta}\alpha} V_{\alpha\dot{\beta}} \qquad (4.49)$$

So far, we have written the Lorentz transformations a and a^* of the $SL(2,\mathbb{C})$ spinors explicitly in terms of Pauli matrices. However, there is also a representation of the generators in terms of objects that satisfy the Lorentz algebra directly. These are given by

$$(\sigma^{\mu\nu})_\alpha{}^\beta := \frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu\right)_\alpha{}^\beta$$

$$(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} := \frac{i}{2}\left(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu\right)^{\dot{\alpha}}{}_{\dot{\beta}} \qquad (4.50)$$

Each of these to objects satisfy the Lorentz algebra (4.15) with $M^{\mu\nu} \to \sigma^{\mu\nu}$ or $\bar{\sigma}^{\mu\nu}$. So we have $D_{\frac{1}{2}}(M^{\mu\nu}) \triangleq \sigma^{\mu\nu}$ or $\bar{\sigma}^{\mu\nu}$.

Hence, the Lorentz transformation can be written as

$$\xi'_\alpha(x') = a_\alpha{}^\beta \xi_\beta(x) = \left[e^{-\frac{i}{4}\varepsilon^{\mu\nu}\sigma_{\mu\nu}}\right]_\alpha{}^\beta \xi_\beta(x) \qquad (4.51)$$

or for $\eta^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\eta_{\dot{\beta}}$ as

$$\eta'^{\dot{\alpha}}(x') = \left(\varepsilon a^* \varepsilon^T\right)^{\dot{\alpha}}{}_{\dot{\beta}}\,\eta^{\dot{\beta}}(x) = \left[e^{-\frac{i}{4}\varepsilon^{\mu\nu}\bar{\sigma}_{\mu\nu}}\right]^{\dot{\alpha}}{}_{\dot{\beta}}\,\eta^{\dot{\beta}}(x) \qquad (4.52)$$

4.5 Some aspects of spinor calculus

For a given spinor ξ_α, we wish to identify the dual spinor ξ^α such that the inner product of the two forms a scalar product which is invariant under Lorentz transformations. As already suggested in the preceding section, this metric is given by the anti-symmetric tensor in two dimensions,

$$\varepsilon^{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = \varepsilon_{\alpha\beta} = \varepsilon_{\dot\alpha\dot\beta} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4.53)$$

such that

$$\xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta \quad , \quad \eta^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\eta_{\dot\beta} . \quad (4.54)$$

The resulting Lorentz invariance of the inner product $\xi^\alpha\xi_\alpha = \varepsilon^{\alpha\beta}\xi_\beta\xi_\alpha$ will be discussed in the exercises.

Since ε is anti-symmetric, some <u>care</u> is necessary, as some manipulations seem non-obvious if compared to vector calculus in $\mathbb{R}^3$ or $\mathbb{M}$. For instance,

$$\begin{aligned} \xi_\alpha &= -\varepsilon_{\alpha\beta}\xi^\beta \quad , & \eta_{\dot\alpha} &= -\varepsilon_{\dot\alpha\dot\beta}\eta^{\dot\beta} \\ &= \xi^\beta\varepsilon_{\beta\alpha} & &= \eta^{\dot\beta}\varepsilon_{\dot\beta\dot\alpha} \end{aligned} \quad (4.55)$$

because: $\quad -\varepsilon_{\alpha\beta}\xi^\beta = -\underbrace{\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma}}_{=-\delta_\alpha^\gamma}\xi_\gamma = \xi_\alpha .$

In (4.55), we observe that no explicit sign appears if the indices are arranged such that they are contracted from upper-left to lower-right, or "NW – SO".
north west – south east

i.e. if we wish to drop the indices in our notation, we have to agree on this convention:

$$\xi \zeta := \xi^\alpha \zeta_\alpha = - \xi_\alpha \zeta^\alpha \,. \qquad (4.56)$$

Another useful notation is inspired by matrix multiplication rules (e.g. also the scalar product of two Euclidean vectors $\vec{x}$ and $\vec{y}$, $\vec{x}\cdot\vec{y}$, can be viewed as a matrix multiplication where the left vector is considered as transposed vector $\vec{x}\cdot\vec{y} = \vec{x}^T\vec{y}$), where we consider the left-hand spinor (not the dual spinor!) as a transposed spinor:

$$\xi \zeta = \xi^\alpha \zeta_\alpha = \varepsilon^{\alpha\beta} \xi_\beta \zeta_\alpha = \xi_\beta \varepsilon^{\alpha\beta} \zeta_\alpha$$

$$= - \xi_\beta \varepsilon^{\beta\alpha} \zeta_\alpha = - \xi^T \varepsilon \zeta = \xi^T \varepsilon^T \zeta \qquad (4.57)$$

where

$$\varepsilon^T = - \varepsilon \qquad (4.58)$$

In this latter notation, we can write Lorentz transformations in the following manner:

$$\xi'_\alpha = a_\alpha{}^\beta \xi_\beta \Rightarrow \xi' = a\,\xi \qquad (4.59a)$$

or $$\xi'^T = \xi^T a^T \qquad (4.59b)$$