3. Nonlinear scalar field theories

In the preceding sections, we have already considered scalar field theories with a general potential $V(\phi)$ as an example for a nonlinear generalization of Klein-Gordon theory, cf. Eq. (1.57)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi). \tag{3.1}$$

This class of models has a wide range of applications (in particle physics, many-body physics, statistical physics, etc.) and features a number of physical mechanisms. In the following, we concentrate on their properties related to symmetry and (spontaneous) symmetry breaking.

3.1 $\mathbb{Z}_2$ model

We have already discussed that (3.1) for a real scalar field entails a $\mathbb{Z}_2$-symmetry under

$$\phi \to -\phi \tag{3.2}$$

if $V(\phi) = V(-\phi)$. E.g. for

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{3.3}$$

the equation of motion is

$$\left(\Box + m^2 + \frac{\lambda}{3!} \phi^2 \right) \phi = 0 \tag{3.4}$$

from which it is obvious that for a given solution $\phi_0(x)$ also $-\phi_0(x)$ is a solution of (3.4) (of course, it may not satisfy the same boundary conditions that have been imposed on $\phi_0(x)$. In general, boundary conditions may break (violate) the $\mathbb{Z}_2$ symmetry explicitly.)

In any case, (3.4) has a trivial solution: $\phi = 0$ which is sometimes called the "vacuum solution". Small excitations $\phi \lll 1$ propagate to leading order in a λ-expansion according to the "free" (linear) Klein-Gordon equation $(\Box + m^2)\phi \approx 0 + \mathcal{O}(\lambda)$, justifying to say that excitations on top of the vacuum have a mass m.

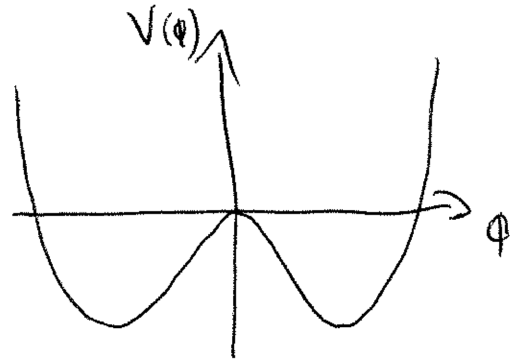
Let us now deform (3.3) a little, and consider the potential

$$V(\phi) = -\frac{1}{2}\mu^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{3.5}$$

At first sight, this looks odd as one may be tempted to say that the theory has a negative mass squared $m^2 \overset{?}{=} -\mu^2$.

This is, however, not true, as we should study the dispersion relation of excitations on top of the vacuum in order to define a propagating mass.

The form of the potential reveals, that $\phi = 0$ is not a stable solution. Any excitation will drive the system towards one of the minima



$$\phi_0 = \pm \sqrt{\frac{6\mu^2}{\lambda}} =: \pm v \,. \qquad (3.6)$$

Hence, the role of the stable vacuum solution is now played by one of the two cases $\phi_0 = \pm v$.

Let us study the excitations on top of the "right" vacuum:

$$\phi(x) = v + \sigma(x) \qquad (3.7)$$

The Lagrangian then reads $\left(v = \sqrt{\frac{6\mu^2}{\lambda}} \right)$

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) - \left[\frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{3!}\lambda v \sigma^3 + \frac{1}{4!}\lambda \sigma^4 \right] \tag{3.8}$$

For small excitations $\sigma <\!\!<\!\!< 1$, the equations of motion then read

$$\left(\Box + (2\mu^2) \right) \sigma = 0 + \mathcal{O}(\lambda) . \tag{3.9}$$

We conclude that these excitations behave like relativistic point particles with a mass $= \sqrt{2}\mu$.
In addition to the quartic $\sim \phi^4$ interaction, σ in (3.8) also exhibits a cubic interaction $\sim \sigma^3$,

$$V_\sigma(\sigma) = \frac{1}{2}(2\mu^2)\sigma^2 + \frac{1}{3!}\lambda v \sigma^3 + \frac{1}{4}\lambda \sigma^4 . \tag{3.10}$$

We observe that — while $V(\phi)$ is symmetric under $\phi \Rightarrow -\phi$ — the potential for σ is not, $V_\sigma(\sigma) \neq V_\sigma(-\sigma)$. This is, of course, not too surprising, because we have made a choice in (3.7) and picked the "right" vacuum solution $\phi_0 = +v$. If we had picked the "left" solution, the conclusions about the massive excitation in (3.9) would have been the same, as well as the result that the new potential for σ as the excitation on top of the vacuum $\phi_0 = -v$ would not exhibit a $\mathbb{Z}_2$ symmetry.

The mere fact that the vacuum solution has the property $\phi_0 = \pm v \neq 0$ is already in conflict with the symmetry. In order to be "in the vacuum" the field has to give preference to either a positive amplitude $\phi_0 = +v$ or a negative amplitude $\phi_0 = -v$.

Once, the vacuum solution has made this choice (we say "has broken the symmetry") the symmetry is no longer manifest for excitations on top of the vacuum.

It is useful to introduce some more nomenclature: if the vacuum configuration of a field corresponds to a nonzero amplitude, we say that the field <u>condenses</u>. The value v of the amplitude in the vacuum is called a <u>condensate</u>. As the vacuum configuration no longer respects the symmetry of the Lagrangian, we talk about

spontaneous symmetry breaking.

The attribute "spontaneous" characterizes the situation that the field, in principal, has two (or, in general, several) options to relax towards a vacuum. This should be contrasted with symmetry breaking induced by boundary conditions or non-symmetric terms in the action, which are imposed explicitly in the form of additional conditions or parameters.

3.2 O(N) model

Let us next promote the field φ to an N-component vector field $\varphi^a \in \mathbb{R}$, $a = 1, \ldots, N$ with a Lagrangian

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi^a)(\partial^\mu \varphi^a) - V(\varphi) \qquad (3.11)$$

where

$$V(\varphi) = -\frac{1}{2}\mu^2 \varphi^a \varphi^a + \frac{\lambda}{4!} (\varphi^a \varphi^a)^2 \qquad (3.12)$$

Equivalently, we could use a vector notation

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\varphi})\cdot(\partial^\mu \vec{\varphi}) - \left[-\frac{1}{2}\mu^2 \vec{\varphi}\cdot\vec{\varphi} + \frac{\lambda}{4!}(\vec{\varphi}\cdot\vec{\varphi})^2\right]. \qquad (3.13)$$

It is important to note that these vectors $\vec{\varphi}(x)$ do not "point" along certain directions in space or spacetime, but denote directions in an <u>internal</u> space $\vec{\varphi} \in \mathbb{R}^N$.

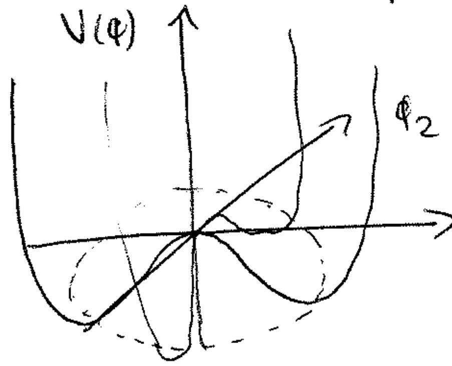
In the form of (3.13), it is easy to see that the model is invariant under transformations that leave the Euclidean scalar product in $\mathbb{R}^N$ invariant. These transformations form the group of orthogonal transformations $O(N)$; i.e. the field vector components φ^a are transformed by $N\times N$ matrices U^{ab}

$$\varphi^a \to U^{ab}\varphi^b, \qquad (3.14)$$

which constitute a matrix representation of $O(N)$. The scalar product is invariant, if th U's satisfy

$$U^{ab}U^{ac} = (U^T)^{ba}U^{ac} = (U^T U)^{bc}$$
$$= \mathbb{1}^{bc} = \delta^{bc}. \qquad (3.15)$$

As the field components ϕ^a are real, the U's correspond to orthogonal $N \times N$ matrices with real components. For the above case with a "negative mass-like parameter" $-\mu^2$, the potential has the form as sketched here → for $N=2$, where the dashed line marks a circle in field space, where the potential is minimal.



For general N, this minimum corresponds to an $(N-1)$-dimensional sphere S^{N-1}, which is defined by

$$\phi_0^a \phi_0^a = v^2 = \frac{6\mu^2}{\lambda} . \qquad (3.16)$$

In contrast to the $\mathbb{Z}_2$ model there are not merely two points, but a continuum of possible vacuum solutions.

Let us choose a specific one

$$\vec{\phi}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ v \end{pmatrix} , \quad v = \sqrt{\frac{6\mu^2}{\lambda}} . \qquad (3.17)$$

Then, the $O(N)$ symmetry is spontaneously broken, Since a generic $O(N)$ transformation would rotate $\vec{\Phi}_0$ to a different point on S^{N-1}.

Still, there is a subset of $O(N)$ transformations that leaves $\vec{\Phi}_0$ invariant. This is the set of rotations about the $\vec{\Phi}_0$-axis in field space. It is possible to show that this subset forms again a group, namely $O(N-1)$. We say that the ground-state (3.17) breaks the group $O(N)$ spontaneously to $O(N-1)$.

Now, it is interesting to study the excitations on top of the vacuum, which we parameterize by

$$\vec{\Phi}(x) = \begin{pmatrix} \vec{\pi}(x) \\ v + \sigma(x) \end{pmatrix} , \quad \pi^i , \; i = 1, \dots, N-1 \tag{3.18}$$

In terms of the fields $\pi^i(x)$, $\sigma(x)$, the Lagrangian reads

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \pi^i)(\partial^\mu \pi^i) + \frac{1}{2} (\partial_\mu \sigma)(\partial^\mu \sigma) - V(\sigma, \pi^i) \tag{3.19}$$

where

$$V(\sigma,\pi^i) = \frac{1}{2}(2\mu^2)\sigma^2 + \sqrt{\frac{\lambda}{6}}\,\mu\,\sigma^3 + \sqrt{\frac{\lambda}{6}}\,\mu\,(\pi^i)^2\sigma$$

$$+ \frac{\lambda}{4!}\sigma^4 + \frac{\lambda}{12}(\pi^i)^2\sigma^2 + \frac{\lambda}{4!}\left[(\pi^i)^2\right]^2 \qquad (3.20)$$

Here, we observe:

- a scalar excitation $\sigma(x)$ with mass

$$m_\sigma^2 = 2\mu^2 . \qquad (3.21)$$

- The π^i and σ fields are interacting as well as self interacting. This means that the field equations for π^i and σ are mutually coupled and non linear.

- The Lagrangian is invariant under transformations of π^i by orthogonal $(N-1)\times(N-1)$ matrices

$$\pi^i \Rightarrow U^{ij}\pi^j \qquad \text{where } U \in O(N-1). \qquad (3.22)$$

This reflects the residual $O(N-1)$ symmetry.

- The π - field remains massless, as there is no pure quadratic term in π^i.

The last point is particularly important: the spontaneous breaking of a continuous global symmetry $O(N) \to O(N-1)$ yields $N-1$ massless bosons (here: scalars).

The latter are called Nambu-Goldstone bosons (or only "Goldstone bosons"), where the nomenclature comes from a QFT / particle physics context. The phenomenon, however, is equally important in classical field theory, e.g. in applications to statistical models (e.g. spin waves).

The number of Goldstone bosons is related to the symmetry breaking pattern, more specifically to the "number of broken generators". The latter are those generators of $O(N)$ that generate transformations that would not have the chosen vacuum invariant.

This statement is quantifiable:

$$\# \text{ of } O(N) \text{ generators} \qquad n_{O(N)} = \frac{1}{2} N(N-1)$$

$$\# \text{ of } O(N-1) \text{ generators} \qquad n_{O(N-1)} = \frac{1}{2}(N-1)(N-2)$$

$$\Rightarrow n_{O(N)} - n_{O(N-1)} = N-1 \mathrel{\hat{=}} \# \text{ of } \pi^i \text{ fields} . \tag{3.23}$$

The present example is a special case of the more general Goldstone theorem, see below, relating the appearance of Goldstone bosons and their number to the number of spontaneously broken generators (it is not restricted to the present $O(N)$ case).

The notation in terms of σ and π fields is taken over from low-energy models of Quantum Chromodynamics (QCD): QCD has an approximate chiral symmetry (to be discussed later). In the case, where only "up" and "down" quarks are considered, the symmetry corresponds to independent "flavor" rotations, i.e. unitary transformations, of left- and right-handed components of the Dirac spinor fields. The symmetry group is

$$SU(2)_L \times SU(2)_R \underset{\substack{\uparrow \\ \text{isomorphic to}}}{\simeq} O(4) \qquad (3.24)$$

The σ field is also often called a "radial" excitation, as it characterizes field excitations orthogonal to the S^{N-1} sphere (orthogonal to the dashed line in the above figure), while the

π^i fields are excitations within the S^{N-1} sphere. The σ excitation has to go "uphill" in the potential $V(\sigma, \pi^i)$, and thus is massive. In QCD it is supposed to correspond to a heavy scalar mesonic resonance $(\sim \mathcal{O}(1\,\mathrm{GeV}))$. The π^i excitations are excitations within S^{N-1}, i.e. a "flat" direction in the potential landscape. In QCD, π^1, π^2, π^3 correspond to the light pions with a mass $\sim 135\,\mathrm{MeV}$. This small mass arises from the fact that the chiral symmetry is only approximate in QCD. It is also explicitly broken by the quark mass terms.

In the literature, $O(N)$ models in the form discussed here are also called "linear sigma models".

3.3 Goldstone theorem

The connection between the appearance of massless Goldstone bosons and spontaneously broken symmetries is generally formulated within Goldstone's theorem. It holds both in classical field theory as well as in quantum field theory. In both cases, the proof is essentially identical except for the fact that the classical potential has to be replaced by the effective potential in QFT (NB: the effective potential already includes the effects of all quantum fluctuations.)

We start from the action that we write as

$$S[\phi] = \int d^4x \left(-V(\phi) + \text{terms with derivatives} \right). \tag{3.25}$$

We assume that the derivative terms – if nonzero – only result in deviations from the extremum of the action, such that the ground state is homogeneous and thus determined by the minimum of the potential. In other words, we assume that $V(\phi)$ is minimized by ϕ_0^a = const. in space and time. Then

$$\frac{\partial}{\partial \phi^a} V \Big|_{\phi^a(x) = \phi_0^a} = 0 \tag{3.26}$$

Expanding about this minimum, we get

$$V(\varphi) = V(\varphi_0) + \frac{1}{2}(\varphi-\varphi_0)^a(\varphi-\varphi_0)^b \frac{\partial}{\partial\varphi^a\partial\varphi^b}V(\varphi_0) + \dots \tag{3.27}$$

since the linear term vanishes by virtue of (3.26). The coefficient of the quadratic term

$$m^2_{ab} := \frac{\partial^2}{\partial\varphi^a\partial\varphi^b}V(\varphi_0) \tag{3.28}$$

is a symmetric matrix, the eigenvalues of which specify the masses of the fields. Since φ_0 is a minimum, these masses cannot be negative.

Next, we assume that the theory has a continuous symmetry (obeyed by the action as well as the quantization procedure in QFT) with the transformed field of the form

$$\varphi^a \to \varphi^a + \delta\varphi^a, \tag{3.29}$$

where $\delta\varphi^a$ can be some function of all fields $\delta\varphi^a = \delta\varphi^a(\varphi)$. Considering only constant fields, the invariance of the action implies invariance of the potential,

$$V(\varphi) = V(\varphi + \delta\varphi) \tag{3.30}$$

$$\Rightarrow \quad \delta\varphi^a \frac{\partial}{\partial\varphi^a} V(\varphi) = 0 . \tag{3.31}$$

Differentiating with respect to φ^b and setting $\varphi = \varphi_0$, we get

$$0 = \left.\frac{\partial(\delta\varphi^a)}{\partial\varphi^b}\right|_{\varphi_0} \underbrace{\left(\frac{\partial V(\varphi_0)}{\partial\varphi^a}\right)}_{=0} + \delta\varphi^a(\varphi_0)\, m_{ab}$$

$$= \delta\varphi^a(\varphi_0)\, m_{ab} \tag{3.32}$$

If the transformation leaves φ_0 unchanged, then $\delta\varphi^a(\varphi_0) = 0$, and (3.32) is trivially satisfied. A spontaneously broken symmetry is precisely one for which $\delta\varphi^a(\varphi_0) \neq 0$. In this case, $\delta\varphi^a(\varphi_0)$ is an eigenvector of the mass matrix with eigenvalue zero.

This proves Goldstone's theorem: every continuous symmetry of the theory that is not a symmetry of the ground state φ_0 gives rise to a massless excitation corresponding to a Nambu – Goldstone boson.

3.4 Hidden symmetry & the Higgs mechanism

Though the Goldstone theorem has many applications in field theory in condensed-matter as well as particle physics, it hampered progress in particle physics for quite a while around ~1960. While the use of symmetries appeared technically and aesthetically helpful in the construction of models for the weak (and strong) interactions, these symmetries had to be broken in order to match with the data. If the breaking happens spontaneously, Goldstone's theorem seemed to imply the necessary occurence of massless excitations – which, however, were not observed. On the contrary, the potentially existing bosons seemed to be rather heavy.

The essential breakthrough was stimulated by Anderson's description of superconductivity and the Meissner - Ochsenfeld - Effekt in condensed-matter physics and then was transferred to nonabelian models and particle physics by Brout, Englert, Higgs, Hagen, Kibble and Guralnik, leading to what is now known as the electroweak Higgs sector of the standard model of particle physics.

We will study here the essentials with the aid of a simpler model: scalar QED (or abelian Higgs model):

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi - \frac{\lambda}{4!} 4 (\varphi^* \varphi)^2 , \tag{3.33}$$

where $\varphi = \frac{1}{\sqrt{2}} (\varphi_1 + i \varphi_2) \in \mathbb{C}$ is a complex charged scalar field (e.g. the charged pions). The gauge field A_μ occurs in the covariant derivative

$$D_\mu = \partial_\mu + ie A_\mu \tag{3.34}$$

and the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. (3.35)

The theory is symmetric under <u>local</u> $U(1)$ transformations (gauge transformations)

$$\varphi(x) \longrightarrow e^{-ie\Lambda(x)} \varphi(x) , \quad e^{-ie\Lambda(x)} \in U(1)$$

$$A_\mu(x) \longrightarrow A_\mu + \partial_\mu \Lambda(x) , \tag{3.36}$$

where $\Lambda(x)$ is an arbitrary smooth function of spacetime.

With $\mu^2 > 0$, the potential part of (3.33), $V = -\mu^2 \varphi^* \varphi + \frac{\lambda}{4!} 4 (\varphi^* \varphi)^2$ has a "mexican hat" shape such that the minima of V satisfy

$$\varphi_0^* \varphi_0 = \frac{1}{2} v^2 \qquad , v = \sqrt{\frac{6\mu^2}{\lambda}} \qquad (3.37)$$

as before (the factor $\frac{1}{2}$ takes care of the different normalization of the scalar fields $\in \mathbb{C}$).

The fact that the symmetry is a local symmetry is an essential difference to the purely scalar cases, say the $O(2)$ model, considered before: e.g. choosing φ_0 to point into the φ_2 direction everywhere is not a meaningful statement, since the local transformation (3.36) can change φ_0 independently from one point to another.

The gauge symmetry (3.36) indeed suggests to parametrize $\varphi(x)$ differently then before,

$$\varphi(x) = \frac{1}{\sqrt{2}} e^{i\frac{\pi(x)}{v}} \left(v + \sigma(x)\right)$$

$$= \frac{1}{\sqrt{2}} \left(v + \sigma(x) + i\,\pi(x)\right) + \mathcal{O}(\pi^2) \tag{3.38}$$

The second line is reminiscent to the linear parametrization used before, however, the complete parametrization in the first line is nonlinear.

For a given field configuration $\varphi(x)$, $A_\mu(x)$, we are free to perform a gauge transformation (in the $\mathbb{Z}_2$ model, this corresponded to choose the "right" minimum without loss of generality; or in the $O(N)$ model, we chose $\varphi_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v \end{pmatrix}$).

Here, we choose a special gauge transformation with

$$\Lambda(x) = \frac{\pi(x)}{e v} \tag{3.39}$$

Then:

$$\varphi(x) \to \varphi'(x) = e^{-ie\Lambda(x)} \varphi(x) \underset{(3.39)}{\overset{(3.38)}{=}} \frac{1}{\sqrt{2}} \left(v + \sigma(x)\right)$$

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \frac{1}{ev} \partial_\mu \pi(x) \tag{3.40}$$

In terms of the new fields $\sigma(x), \pi(x), A'_\mu(x)$, the Lagrangian now reads

$$\mathcal{L} = -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} + \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) + \frac{1}{2} e^2 v^2 A'_\mu A^{\mu\prime}$$
$$+ \frac{1}{2} e^2 (A'_\mu)^2 \sigma(2v+\sigma) - \frac{1}{2}(2\mu^2)\sigma^2 \qquad (3.41)$$
$$+ \mathcal{O}(\sigma^3, \sigma^4) \,.$$

We observe:

- σ occurs as a massive scalar as in the purely scalar models
- Additionally, the photon A'_μ has acquired a mass term as in Proca theory
- Most surprisingly, $\pi(x)$ has vanished completely!

This last observation is, in fact, compatible with the counting of propagating degrees of freedom: in the initial formulation, say, with a standard scalar mass parameter $V = +m^2 \varphi^* \varphi \dots$, we had 2 real scalar fields (φ_1, φ_2) and 2 photon polarization modes (2 transverse modes): $2+2=4$

Now, we find 1 real scalar (σ) and 3 polarization modes of a "massive" photon (2 transverse & 1 longitudinal).

The would-be Nambu-Goldstone boson π has been "eaten up" by the photon. This highlights the essentials of the Higgs (Anderson, Brout, Englert, Kibble, Hagen, Guralnik) mechanism.

We finally emphasize that the above analysis involved a special choice of gauge (fixed by hand). The observations made above become particularly transparent in this gauge choice. By choosing a gauge, the gauge symmetry is in some sense explicitly broken by hand. By a somewhat unfortunate nomenclature, the Higgs mechanism is sometimes referred to as the "spontaneous breaking of gauge symmetry". In a strict sense, this is nonsense, as gauge symmetry cannot be broken according to Elitzur's theorem.

The point here is that particular gauges are convenient to identify the excitations. The gauge symmetry is still intact and we could try to look for the same physics in a different gauge. These circumstances are therefore better referred to by the name "hidden symmetry".