2 Aspects of classical field theory

In the introductory section, we have essentially derived (or motivated) the Lagrangean formulation of classical field theory in almost complete analogy to classical mechanics. Let us continue to use this analogy to apply further concepts of classical mechanics to field theory, starting with the

2.1 Hamiltonian formulation

Let us use the Klein-Gordon field as a simple example for the following section. As in (1.57), we generalize the mass term to a full potential:

$$S[\phi] = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) ,$$
$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - V(\phi) . \tag{2.1}$$

Let us first try to find a relativistic (covariant) Hamiltonian, naively generalizing the rules of classical mechanics to field theory. For this, we first define a field momentum conjugate to the field amplitude:

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi} \overset{(2.1)}{=} \partial_\mu \phi \tag{2.2}$$

The corresponding Hamiltonian is then obtained by a Legendre transform:

$$\mathcal{H}_{cov} = \Pi_\mu \underbrace{\partial^\mu \phi}_{=\Pi^\mu} - \underbrace{\mathcal{L}}_{=\frac{1}{2}\Pi_\mu \Pi^\mu - V(\phi)} = \frac{1}{2}\Pi_\mu \Pi^\mu + V(\phi) \tag{2.3}$$

At first glance, this looks similar to point particle Hamiltonians a la $H = \frac{p^2}{2m} + V(x)$. However, there is a problem: with (2.2), the "kinetic" term corresponds to $\frac{1}{2}\Pi_\mu \Pi^\mu = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\vec{\nabla}\phi)^2$. Because of the minus sign, $\mathcal{H}_{cov}$ is not bounded from below even for bounded potentials $V(\phi)$. Hence, $\mathcal{H}_{cov}$ cannot be interpreted as an energy quantity related to a given field configuration.

This is not too surprising, since $\mathcal{H}_{cov}$ by construction is invariant under Lorentz transformations, whereas the field energy is expected to transform as a 0-component of a 4-vector (as for a point particle).

In order to preserve the energy interpretation for the Hamiltonian, we give up manifest covariance for a moment and choose a fixed reference frame with a time t, $x^\mu = (t, \vec{x})$, such that the Lagrangian reads

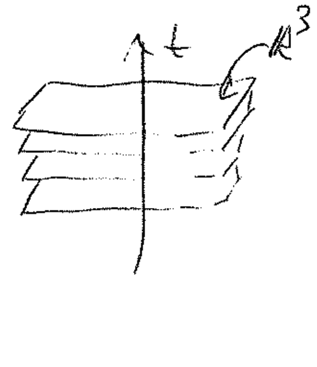
$$\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)(\partial^\mu \varphi) - V(\varphi) = \frac{1}{2} \dot{\varphi}^2 - (\vec{\nabla}\varphi)^2 - V(\varphi) \tag{2.4}$$

Now, we define the canonical momentum as in classical mechanics:

$$\Pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}(\vec{x})} = \dot{\varphi}(\vec{x}), \tag{2.5}$$

where the notation should indicate that this definition holds at every space point $\vec{x}$, while the time t is considered as an evolution parameter as in classical mechanics. The Hamiltonian formulation thus induces a foliation of spacetime

$$\mathbb{M} \to \mathbb{R}^3 \otimes \mathbb{R}$$



Again, we obtain the Hamiltonian by a Legendre transformation

$$\mathcal{H} = \pi \underbrace{\dot{\phi}}_{=\pi} - \underbrace{\mathcal{L}}_{\frac{1}{2}\pi^2 - (\vec{\nabla}\phi)^2 - V(\phi)} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + V(\phi) \tag{2.6}$$

For potentials bounded from below, this is a manifestly bounded function of the field and the momentum. Its units correspond to those of an energy density. Hence, the three terms can be interpreted as the energy densities stored in or required for the time evolution $\sim \pi^2$, spatial field variations $\sim (\vec{\nabla}\phi)^2$, or in the excitation of field amplitudes $\sim V(\phi)$.

As will be detailed in the exercises, the equation of motion follow now directly from the corresponding Hamilton equations in complete analogy to classical mechanics.

The construction can be briefly summarized as follows:

$\Phi(\vec{x})$ and $\Pi(\vec{x})$ span the phase space.

Using functional differenciation, we can define Poisson brackets for general phase space functionals $A[\Phi,\Pi]$, $B[\Phi,\Pi]$:

$$\{A,B\} = \int d^3z \left(\frac{\delta A}{\delta\Phi(\vec{z})} \frac{\delta B}{\delta\Pi(\vec{z})} - \frac{\delta B}{\delta\Phi(\vec{z})} \frac{\delta A}{\delta\Pi(\vec{z})} \right). \quad (2.7)$$

The fundamental Poisson brackets read

$$\begin{aligned} \{\Phi(\vec{x}), \Pi(\vec{y})\} &= \delta^{(3)}(\vec{x}-\vec{y}) \\ \{\Phi(\vec{x}), \Phi(\vec{y})\} &= 0 = \{\Pi(\vec{x}), \Pi(\vec{y})\} \end{aligned} \quad (2.8)$$

The canonical equations of motion then read as usual

$$\dot{\Phi}(\vec{x}) = \{\Phi(\vec{x}), H\}, \quad \dot{\Pi}(\vec{x}) = \{\Pi(\vec{x}), H\}, \quad (2.9)$$

where we have used the Hamilton functional

$$H = \int d^3\vec{y}\, \mathcal{H} \quad (2.10)$$

($\mathcal{H}$ hence is also called the Hamiltonian density)

Inserting (2.6) into (2.9) leads to the field equation

$$0 = \ddot{\varphi} - \vec{\nabla}^2\varphi + V'(\varphi) \equiv \Box\varphi + V'(\varphi) \quad (2.11)$$

as expected. We emphasize that (2.11) is a covariant field equation, even though the Hamiltonian construction is not manifestly covariant at intermediate stages.

2.2 Symmetries and conservation laws

In classical mechanics, symmetries can be closely related to conserved quantities as is captured by Noether's theorem. In fact, the same relation persists in classical field theory:

Let us consider an infinitesimal deformation of the field

$$\varphi(x) \rightarrow \varphi'(x) = \varphi(x) + \delta\varphi(x), \quad (2.12)$$

where $\delta\varphi(x)$ is considered to be an infinitesimal continuous deformation (finite deformations can be generated from successive infinitesimal deformations).

Eq. (2.12) is considered to be a symmetry transformation if the field equations remain invariant.

On the level of the Lagrangian, this implies that $\mathcal{L}$ is allowed to change only up to a total derivative:

$$\begin{aligned} &\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta\mathcal{L} \,, \\ &\text{where} \quad \delta\mathcal{L} = \partial_\mu K^\mu \,. \end{aligned} \qquad (2.13)$$

Then, the action changes by a surface term

$$\delta S = \int d^4x\, \delta\mathcal{L} = \int_{\underset{\uparrow \text{ spacetime volume}}{\Omega}} d^4x\, \partial_\mu K^\mu = \int_{\partial\Omega} d\sigma_\mu K^\mu \qquad (2.14)$$

If K^μ is sufficiently localized (which we assume in the following), δS vanishes. This implies that the action is invariant under (2.12) & (2.13) and so are the equations of motion.

Noether's theorem now relates this invariance to a conserved quantity.

Let $\phi \to \phi + \delta\phi$ with $\delta\mathcal{L} = \partial_\mu K^\mu$ be a symmetry transformation. Then, there is a 4-current,

Noether current: $$J^\mu = \Pi^\mu \delta\phi - K^\mu \tag{2.15}$$

where $$\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} , \tag{2.16}$$

which is conserved, $$\partial_\mu J^\mu = 0 , \tag{2.17}$$
if ϕ satisfies the equations of motion.

Proof:

Varying the Lagrangian yields

$$\partial_\mu K^\mu = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \underbrace{\delta(\partial_\mu\phi)}_{=\partial_\mu\delta\phi}$$

$$= \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right]\delta\phi + \partial_\mu\Big(\underbrace{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}}_{=\Pi^\mu}\delta\phi\Big) \tag{2.18}$$

Using the equations of motion, the term in [ ]-brackets vanishes, and we find

$$0 = \partial_\mu\left(\Pi^\mu\delta\phi - K^\mu\right) =: \partial_\mu J^\mu \tag{2.19}$$

$\square$

If in addition the Noether current vanishes sufficiently fast towards spatial infinity $|\vec{x}| \to \infty$, we find

$$0 = \int d^3x \; \partial_\mu J^\mu = \partial_t \int d^3x \, J^0 - \int_{\mathbb{R}^3} d^3x \; \vec{\nabla} \cdot \vec{J}$$

$$= \partial_t \int d^3x J^0 - \underbrace{\int_{\partial \mathbb{R}^3} d\vec{\sigma} \cdot \vec{J}}_{\to 0} = \partial_t \int d^3x \, J^0 =: \dot{Q} . \qquad (2.20)$$

The corresponding integral over the zero component of the current is called the Noether charge,

$$Q = \int d^3x \; J^0 , \qquad (2.21)$$

which by virtue of (2.20) is conserved.

Let us illustrate the significance of Noether's theorem with the aid of two examples

Example 1: translations

Translations are part of the space-time symmetries which together with the Lorentz transformations form the Poincaré group.

translation invariant systems do not feature a distinguished point in spacetime. A translation

$$x^\mu \to x^{\mu'} = x^\mu - a^\mu \,, \quad a^\mu = \text{const} \tag{2.22}$$

acts on the field as

$$\phi(x) \to \phi'(x) = \phi(x-a). \tag{2.23}$$

For infinitesimal translations, we get

$$\begin{aligned} \phi(x-a) &= \phi(x) - a_\mu \partial^\mu \phi(x) + \mathcal{O}(a^2) \\ \Rightarrow \quad \delta\phi(x) &= - a_\mu \partial^\mu \phi(x) . \end{aligned} \tag{2.24}$$

Similarly, we get for the Lagrangian

$$\begin{aligned} \mathcal{L}(x) \to \mathcal{L}(x-a) &= \mathcal{L}(x) - a_\mu \partial^\mu \mathcal{L}(x) + \mathcal{O}(a^2) \\ \Rightarrow \quad \delta\mathcal{L} &= - a_\mu \partial^\mu \mathcal{L}(x) \equiv \partial_\mu K^\mu \\ \Rightarrow \quad K^\mu &= - a^\mu \mathcal{L} . \end{aligned} \tag{2.25}$$

From this, we get the Noether current

$$\begin{aligned} J^\mu &= \Pi^\mu \delta\phi - K^\mu = \Pi^\mu (-a_\nu \partial^\nu \phi) + a^\mu \mathcal{L} \\ &= - a_\nu \left(\Pi^\mu \partial^\nu \phi - g^{\nu\mu} \mathcal{L} \right) =: - a_\nu T^{\mu\nu} \,, \end{aligned} \tag{2.26}$$

where we have defined the canonical energy-momentum tensor

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu} \mathcal{L} \tag{2.27}$$

which by Noether's theorem satisfies

$$\partial_\mu T^{\mu\nu} = 0 \tag{2.28}$$

The 00-component corresponds to the Hamiltonian density,

$$T^{00} = \pi^0 \partial^0 \phi - \mathcal{L} \equiv \pi \dot{\phi} - \mathcal{L} = \mathcal{H} . \tag{2.29}$$

The associated conserved Noether charge

$$\dot{Q} = \partial_t \int d^3x\, J^0 \Rightarrow \partial_t \int d^3x\, T^{0\nu} =: \frac{d}{dt} P^\nu = 0 \tag{2.30}$$

can be interpreted as the physical 4-momentum of the field (not to be confused with the canonical momentum π^μ),

$$P^\mu := \int d^3x\, T^{0\mu} = \int d^3x \left(\pi\, \partial^\mu \phi - g^{0\mu} \mathcal{L} \right), \tag{2.31}$$

the components of which read

$$P^0 = \int d^3x \, T^{00} = H \qquad \text{(energy)}$$

$$P^i = \int d^3x \, \pi \, \partial^i \varphi \, . \qquad \text{(3-momentum)} \qquad (2.32)$$

(e.g. in Maxwell's theory, P^i is related to the Poynting vector.)

Example 2: complex scalar field

In addition to spacetime symmetries also <u>internal</u> symmetries can induce conservation laws. Let us consider the case of a complex scalar field

$$\mathcal{L} = \partial_\mu \varphi^* \partial^\mu \varphi - m^2 \varphi^* \varphi \, . \qquad (2.33)$$

The Lagrangian is invariant under phase rotations, $\delta\mathcal{L}=0$

$$\varphi \to e^{i\alpha} \varphi \quad , \quad \varphi^* \to e^{-i\alpha} \varphi^* \qquad (2.34)$$

for $\alpha = \text{const} \in \mathbb{R}$. Infinitesimally, we have

$$\varphi \to \varphi - i\alpha\varphi = \varphi + \delta\varphi \, , \quad \varphi^* \to \varphi^* + i\alpha\varphi^* = \varphi^* + \delta\varphi^* \, . \qquad (2.35)$$

Since $\delta\mathcal{L}=0$, we have $K^\mu = 0$ as well.

Correspondingly, the Noether current is

$$J^\mu = \Pi^\mu \delta\varphi + \Pi^{*\mu} \delta\varphi^* = -i\alpha \left(\varphi \partial^\mu \varphi^* - \varphi^* \partial^\mu \varphi \right)$$
$$= -2\alpha \, \mathrm{Im} \left(\varphi^* \partial^\mu \varphi \right) . \tag{2.36}$$

Apart from the (irrelevant) factor α, we obtain the Klein-Gordon current

$$j^\mu = \frac{J^\mu}{\alpha} = -2 \, \mathrm{Im} \left(\varphi^* \partial^\mu \varphi \right) , \tag{2.37}$$

and the corresponding Noether charge

$$Q = \int d^3x \, j^0 = i \int d^3x \left(\varphi^* \partial^0 \varphi - \varphi \partial^0 \varphi^* \right) . \tag{2.38}$$

Both expressions (2.37) & (2.38) are familiar from relativistic quantum mechanics: after reinterpreting the "negative energy states" as antiparticles, j^μ corresponds to the electromagnetic current generated by a Klein-Gordon wave function, and Q to its electric charge, which upon coupling to a Maxwell field generate $\vec{E}$ and $\vec{B}$ fields.