4.5 Monopoles in Yang-Mills Theories

Comparing Yang-Mills theories with the Georgi-Glashow model, the obvious difference is that there is no Higgs scalar field. However, consider $D=4$ YM theories for the special case of static fields $A_\mu^a(\vec{x},t) \equiv A_\mu^a(\vec{x})$.

Then, the YM action reduces to

$$S_{YM} = T \int d^3\vec{x} \left[\frac{1}{4} F_{ij}^a F_{ij}^a + \frac{1}{2} (D_i A_0^a)(D_i A_0^a) \right]$$

$$\equiv T \int d^3x \, \mathcal{L}_{GG} \Big|_{\lambda=0} \qquad (4.53)$$

with A_0^a playing the role of the Higgs field.

$\Rightarrow$ monopoles are solutions of YM theories with finite energy, but with infinite action ($T \to \infty$); but these solutions are unstable ($\lambda \to 0$).

Let us take a more general viewpoint. In compact $U(1)_3$ as well as in the Georgi-Glashow model, monopoles arise inevitably owing to the structure of the compact / residual gauge group $H = U(1)$ / coset G/H. This gives rise to the idea to classify YM configurations by using an <u>abelian</u> gauge fixing

$$G \; (= SU(N_c)) \xrightarrow{g.f.} H = U(1)^{N_c - 1} \quad (\text{Cartan subgroup})$$

and counting their monopole content in the gauge-fixed theory.

The basis of abelian gauge fixing is the Cartan <u>decomposition</u>:

any Lie algebra has a maximal abelian subalgebra spanned by the maximal set of commuting generators

$$T^{a_0} \in \{T^a\}, \quad (SU(N_c): a = 1, .., N_c^2 - 1, \; a_0 = 1, .., N_c - 1)$$

The T^{a_0}'s generate the maximal abelian subgroup $H \subset G$, inducing the decomposition

$$G = H \otimes G/H. \tag{4.54}$$

The gauge fields can be decomposed as

$$A_\mu = A_\mu^{a_0} T^{a_0} + A_\mu^{\bar{a}} T^{\bar{a}} \equiv \underset{\text{"neutral"}}{A_\mu^{(n)}} + \underset{\text{"charged"}}{A_\mu^{(ch)}}, \tag{4.55}$$

where "neutral" and "charged" refer to their transformation behavior with respect to

$$h = e^{i\,\omega^{a_0} T^{a_0}} \in H \tag{4.56}$$

$$A_\mu^{(ch)} \to h\, A_\mu^{(ch)}\, h^\dagger \qquad \text{(like "bifundamental" matter)}$$

$$A_\mu^{(n)} \to A_\mu^{(n)} - i\,\partial_\mu \omega^{a_0} T^{a_0} \qquad \text{(like a photon)}$$

An abelian gauge fixing now proceeds by fixing the coset with a gauge-fixing condition

$$\mathcal{F}^{(ch)}_{[A]} \equiv \mathcal{F}^{\bar{a}}_{[A]}\, T^{\bar{a}} = 0 \tag{4.57}$$

where $\mathcal{F}^{(ch)}[A^h] = h\,\mathcal{F}^{(ch)}\,h^\dagger$

maintains the abelian symmetry.

Examples:

- diagonalization gauge $\mathcal{F}^{(ch)}(x) = 0$
  where $\mathcal{F}(x)$ is a local function of A_μ,
  e.g. $F_{\mu\nu}F_{\mu\nu}$, F_{12}, ...
  $\leadsto$ $\mathcal{F}$ plays the role of the Higgs field in the GG model

- Polyakov gauge

$$\mathcal{F} = A_0 \,,\quad \partial_0 A_0 = 0 \tag{4.58}$$

(which is as close as possible to Eq. (4.53))

- maximal abelian gauge

$$\mathcal{F}^{(ch)} = D_\mu^{(n)} A_\mu^{(ch)} \qquad (4.59)$$

which fixes $A_\mu^{(ch)}$- fluctuations relative to an $A_\mu^{(n)}$ background (cf. background gauge)

Example (more specific):

diagonalization gauge: choose g such that

$$\mathcal{F}[A^g] = g\,\mathcal{F}\,g^{-1} \overset{!}{=} \mathrm{diag}(\lambda_1, \dots, \lambda_N), \qquad (4.60)$$

which is always possible, since $\mathcal{F} = \mathcal{F}^a \tau^a$ is hermitean.

By means of Weyl permutations, the eigenvalues can be ordered: $\lambda_1 > .. > \lambda_N$

$\Rightarrow$ g is uniquely determined up to a residual abelian symmetry $g \to hg$, $h \in H$ as desired.

This construction fails if $\lambda_i(x) = \lambda_{i+1}(x)$ at one point (line, surface...) x. This then is a singular point with a higher residual symmetry, or a "defect".

Example (even more specific):

$G = SU(2)$:

$$\mathfrak{F}^a = |\mathfrak{F}| \cdot n^a \underbrace{(\theta,\varphi)}_{\substack{\text{polar angles} \\ \text{in color space}}} \quad , \quad |\mathfrak{F}| = \sqrt{\mathfrak{F}^a \mathfrak{F}^a} \tag{4.61}$$

The diagonalization condition (4.60) requires

$$|\mathfrak{F}|\, n^a \sigma^a = |\mathfrak{F}|\, g\, \sigma^3 g^{-1} \qquad \left(\tau^a \overset{SU(2)}{=} \frac{\sigma^a}{2}\right) \tag{4.62}$$

Solution for g:

$$g = e^{i\theta\, \hat{\varphi} \cdot \frac{\vec{\sigma}}{2}} \quad , \quad \hat{\varphi} = \sin\varphi\, \hat{e}_1 + \cos\varphi\, \hat{e}_2$$

The solution becomes singular, if $\mathfrak{F}$ is chosen such that $|\mathfrak{F}|(x) = 0$ at some point x ($\Rightarrow \lambda_1 = \lambda_2 = 0$, leaving θ and φ undetermined), or if $\theta = \pi$ (φ remains undetermined)

The singularity manifests itself in the gauge potential:

$$A_\mu^g = g A_\mu g^{-1} + g \partial_\mu g^{-1},$$

consider the photonic component

$$A_\mu^{(n)} \rightarrow A_\mu^{(n)} + \left(g \partial_\mu g^{-1}\right)^{(n)} \tag{4.63}$$

$$= (1 - \cos\theta)\, \partial_\mu \varphi$$

The last line looks like an abelian monopole field (cf. (4.14)) but this time in color space!

Now, let $(\mathcal{F}|(x_0)=0$ at x_0 (and $\neq 0$ in a neighborhood of x_0). Enclosing x_0 with a sphere S^2 at a given time, this defines a mapping

$$\underset{\text{in space}}{S^2} \longrightarrow \{(\theta,\varphi)\} = \underset{\substack{\text{in}\\ \text{color space}}}{S^2} = SU(2)/U(1)\otimes Z_2$$

which is classified by $m \in \Pi_2(S^2)$.

For $m \neq 0$, $(g\partial_\mu g^{-1})^{(m)}$ carries a monopol charge $\sim m$, and the Dirac string at $\theta=\pi$ in color space is mapped to $|m|$ Dirac strings in coordinate space.

In summary, abelian gauge fixing is applicable everywhere except for localized defect positions (points, lines, ...).
The abelian ~~defects~~ gauge fields become singular at these defects, exhibiting e.g. magnetic monopoles (vortices, domain walls..).

This brings us to the <u>conjecture</u> for the monopole-based confinement scenario:

The relevant low-energy configurations which induce confinement are those which – if brought into an abelian gauge – carry monopole charges. Confinement then is caused in this gauge along the lines of the monopole-gas or dual-Meissner-effect picture).

Let us conclude this section with an important remark:

The extended residual symmetry at the defect has an important consequence: since $\lambda_i = \lambda_{i+1}$, the gauge transformation that mixes λ_i with λ_{i+1} corresponds to

a variation of $\mathcal{F}^a$ which leaves

$\mathcal{F}^a$ invariant: $\delta \mathcal{F}^a = 0$. I.e. the Jacobian of general variations $\frac{\delta \mathcal{F}^a}{\delta \omega^b}$ has one eigenvalue $= 0$. Since $\frac{\delta \mathcal{F}^a}{\delta \omega^b}$ is the Faddeev-Popov operator, the Faddeev-Popov determinant

$\Delta_{FP} = 0$, demonstrating that the defect lives right at the Gribov horizon.

4.6 Abelian projection & abelian dominance in SU(2)