4.3 Magnetic monopoles & confinement in compact $U(1)$ in $D=3$

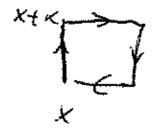
In the continuum, abelian gauge theory is defined by the action

$$S = \frac{1}{4e^2} \int d^D x \, (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \qquad (4.17)$$

One possible lattice formulation is given by (let $a = 1$ lattice spacing)

$$S = \frac{1}{4e^2} \sum_{x,\alpha\beta} F_{x,\alpha\beta}^2 \, ,$$

$$F_{x,\alpha\beta} = A_{x,\alpha} + A_{x+\alpha,\beta} - A_{x+\beta,\alpha} - A_{x,\beta}$$



$$-\infty \leq A_{x,\alpha} \leq \infty \qquad \text{(non-compact)}. \qquad (4.18)$$

An alternative formulation in the spirit of non-abelian theories is given by the definition in terms of link variables:

$$U_{x,\alpha} = e^{i A_{x,\alpha}} \in U(1)$$

$\Rightarrow$ gauge field

$$-\pi \leq A_{x,\alpha} \leq \pi \quad \text{(with identified endpoints)}$$

One may try to link these two formulations by a pseudo-compact notation

$$-\infty \leq A_{x,\alpha} \leq \infty \quad \rightarrow \quad -\infty \leq A_{x,\alpha} + 2\pi\, m_{x,\alpha} \leq \infty$$

with $m_{x,\alpha} \in \mathbb{Z}$ and $-\pi \leq A_{x,\alpha} \leq \pi$.

The partition function of the non-compact theory can then be written as

$$Z_{\text{non-comp.}} = \int_{-\infty}^{\infty} \prod_{x,\alpha} dA_{x,\alpha} \, \exp\left\{ -\frac{1}{4e^2} \sum_{x,\alpha\beta} F_{x,\alpha\beta}^2 \right\}$$

$$= \sum_{\{m_{x,\alpha}\}} \int_{-\pi}^{\pi} \prod dA_{x,\alpha} \, \exp\left\{ -\frac{1}{4e^2} \sum_{x,\alpha\beta} \left(F_{x,\alpha\beta} - 2\pi \bar{n}_{x,\alpha\beta} \right)^2 \right\}$$

(4.19)

where $\bar{n}_{x,\alpha\beta} = m_{x,\alpha} + m_{x+\alpha,\beta} - m_{x+\beta,\alpha} - m_{x,\beta} \in \mathbb{Z}$.

This looks compact but obviously isn't.
(4.19) doesn't have the full periodicity of the compact theory, since the sum includes only those $\bar{n}_{x,\alpha\beta}$ which can be represented by $m_{x,\alpha}$'s. In particular, $\bar{n}$ satisfies the Bianchi identity (and also $F - 2\pi\bar{n}$)

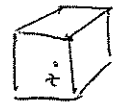
In fact, one obtains the compact theory by summing over all $n_{x,\alpha\beta} \in \mathbb{Z}$:

$$Z_{\text{comp}} = \sum_{\{n_{x,\alpha\beta}\}} \int_{-\pi}^{\pi} \prod_{x,\alpha} dA_{x,\alpha} \, e^{-\frac{1}{4e^2} \sum_{x,\alpha\beta} \left(F_{x,\alpha\beta} - 2\pi n_{x,\alpha\beta} \right)^2}$$

(4.20)

Let us specialize to $D=3$.

Consider the sum over a cube σ_z,



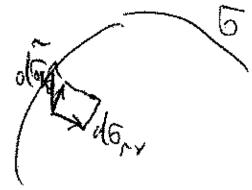
$$\oint_{\partial\sigma_z} n_{x,\alpha\beta} = q_z \in \mathbb{Z} \qquad \left(\begin{array}{l} z: \text{ site on the} \\ \text{dual lattice} \end{array}\right) \tag{4.21}$$

which generally does not vanish.

The meaning of q_z becomes clear from its continuum analogue:

$$\int_{\partial\sigma} d\sigma_{\mu\nu}\, n_{\mu\nu} = q_z ,$$

which should be compared to



$$\int_{\partial\sigma} d\sigma_{\mu\nu} F_{\mu\nu} = \int_{\partial\sigma} d\tilde{\sigma}_\kappa \underbrace{\tfrac{1}{2}\,\varepsilon_{\kappa\mu\nu} F_{\mu\nu}}_{\tilde{F}_\kappa}$$

$$= \int_\sigma d^3x\, \underbrace{\partial_\kappa F_\kappa}_{\text{Bianchi } = 0} = 0 .$$

$\Rightarrow$ q_z measures monopole charges of the compact theory; they are not present in the non-compact theory.

In general $n_{x,\alpha\beta}$ can be decomposed into

$$n_{\alpha\beta} = \partial_\alpha m_\beta - \partial_\beta m_\alpha + \varepsilon_{\alpha\beta\gamma}\,\partial_\gamma \phi \qquad \left(\begin{array}{c}\text{continuum}\\ \text{notation}\end{array}\right) \tag{4.22}$$

such that

$$q_z = \int_{\partial\sigma_z} d\sigma_{\mu\nu}\, m_{\mu\nu} = \int_{\partial\sigma_z} d\tilde{\sigma}_\kappa\, \underbrace{\tfrac{1}{2}\, \varepsilon_{\kappa\mu\nu}\, \varepsilon_{\mu\nu\lambda}}_{\delta_{\kappa\lambda}}\, \partial_\lambda \varphi$$

$$= \int_{\sigma_z} d^3x\, \underbrace{\partial^2\varphi}_{\rho_M} \qquad (4.23)$$

where $\rho_M := \partial^2\varphi$ can be identified with a monopole density.

The action can thus be written as

$$\int d^3x\, \left(F_{\alpha\beta} - 2\pi\, m_{\alpha\beta}\right)^2$$

$$= \int d^3x\, \Big(\big(\underbrace{F_{\alpha\beta} - 2\pi\,(\partial_\alpha m_\beta - \partial_\beta m_\alpha)}_{F' \text{ satisfies Bianchi}}\big) - 2\pi\, \varepsilon_{\alpha\beta\gamma}\, \partial_\gamma \varphi \Big)^2$$

$$= \int d^3x\, \Big[(F'_{\alpha\beta})^2 - \underbrace{4\pi\, F'_{\alpha\beta}\, \varepsilon_{\alpha\beta\gamma}\, \partial_\gamma \varphi}_{= 8\pi\, \tilde{F}'_\gamma \partial_\gamma \varphi \ \xrightarrow[\text{Bianchi}]{\text{i.b.p.}}\ 0} + 4\pi^2\, \underbrace{\varepsilon_{\alpha\beta\gamma}\, \varepsilon_{\alpha\beta\delta}}_{= 2\delta_{\gamma\delta}}\, \partial_\gamma\varphi\, \partial_\delta\varphi \Big]$$

$$= \int d^3x\, \left[F'^{\,2}_{\alpha\beta} + 8\pi^2\, (\partial\varphi)^2 \right]$$

$$= \int d^3x\, \Big[F'^{\,2}_{\alpha\beta} + 8\pi^2\, \rho_M\, \underbrace{\frac{1}{-\partial^2}}_{=G}\, \rho_M \Big] \qquad (4.24)$$

$=G$: Green's function "propagator"

This implies for the partition function

$$Z_{comp} = \Bigg[\int_{-\infty}^{\infty} \prod_{x,\alpha} dA'_{x,\alpha} \; e^{-\frac{1}{4e^2} \sum_{x,\alpha\beta} F^{12}_{x,\alpha\beta}} \Bigg] \cdot \sum_{\{q_z\}} e^{-\frac{2\pi^2}{e^2} \sum_{z,z'} q_z G_{zz'} q_{z'}}$$

$$A' = A - 2\pi m$$

$$= Z_{non-comp} \cdot Z_{monopoles} \qquad (4.25)$$

The factor Z_{monop} describes the partition function of a gas of monopoles, which are subject to Coulomb interactions

$$\sum_{z,z'} q_z G_{zz'} q_{z'} \xrightarrow{\text{contin}'} \frac{1}{4\pi} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|} + \underbrace{\frac{\text{const}}{4\pi} \sum_a q_a^2}_{\text{self-energy}} \qquad (4.26)$$

In the following, we consider only the monopole partition function (now for general lattice spacing a):

$$Z_{monop.} = \sum_{q_z=-\infty}^{\infty} e^{-\frac{2\pi^2}{a e^2} \sum_{z,z'} q_z G_{zz'} q_{z'}}$$

$$\sum_{z,z'} q_z G_{zz'} q_{z'} = \sum_{z \neq z'} q_z G_{zz'} q_{z'} + \underbrace{G(0)}_{\substack{\text{e.g. } \simeq 0.253 \text{ depends} \\ \text{on lattice regularization}}} \sum_z q_z^2$$

The action can be made local by introducing an auxiliary field χ_z:

$$Z_{monop} = \sum_{q_z=-\infty}^{\infty} \int_{-\infty}^{\infty} \prod_z d\chi_z \; e^{-\frac{e^2 a}{4\pi^2} \sum_z \frac{1}{2} (\partial_\mu \chi_z)^2 + i \sum_z q_z \chi_z} \cdot e^{-\frac{2\pi^2}{e^2 a} G(0) \sum_z q_z^2} \tag{4.27}$$

χ_z has the meaning of a dual photon field.

Now we can study the weak-coupling limit $e^2 a \to 0$, where only monopoles with charges $|q_z| \leq 1$ need be taken into account:

$$\sum_{q_z=-1}^{1} e^{-\frac{2\pi^2}{e^2 a} G(0) \sum_z q_z^2 + i \sum_z q_z \chi_z}$$

$$= \underbrace{1}_{q_z=0 \; \forall z} + \underbrace{\sum_z e^{-\frac{2\pi^2 G(0)}{e^2 a}} \left[e^{i\chi_z} + e^{-i\chi_z} \right]}_{\substack{q_z = 1,-1 \text{ for one } z \\ \text{and } q_{z'}=0 \text{ for } z' \neq z}} + \underbrace{\frac{1}{2}}_{\substack{\text{removes double} \\ \text{counting for } z \leftrightarrow z'}} \sum_{z,z'} \left(e^{-\frac{2\pi^2 G(0)}{e^2 a}} \right)^2 \left[e^{i\chi_z + i\chi_{z'}} + e^{i\chi_z - i\chi_{z'}} + e^{-i\chi_z + i\chi_{z'}} + e^{-i\chi_z - i\chi_{z'}} \right]$$

$$+ \dots$$

$$= 1 + \sum_z \left(2 e^{-\frac{2\pi^2 G(0)}{e^2 a}} \right) \cos\chi_z + \frac{1}{2} \left(\sum_z \left(2 e^{-\frac{2\pi^2 G(0)}{e^2 a}} \right) \cos\chi_z \right)^2 + \dots$$

$$= \exp\left[2 e^{-\frac{2\pi^2 G(0)}{e^2 a}} \sum_z \cos\chi_z \right]. \tag{4.28}$$

$$\Rightarrow Z_{monop} \overset{e^2 a \to 0}{\simeq} \int \prod_z d\chi_z \; e^{-\frac{e^2 a}{4\pi^2} \sum_z \left(\frac{1}{2} (\partial_\mu \chi_z)^2 - M_0^2 \cos\chi_z \right)}$$

$$\text{where} \quad M_0^2 = \frac{8\pi^2}{e^2 a} \, e^{-\frac{2\pi^2 G(0)}{e^2 a}} \,. \tag{4.29}$$

In continuum notation, this goes into

$$\sum_z a^3 \to \int d^3x \quad , \quad \partial_\mu \to a\,\partial_\mu \quad , \quad M = \frac{M_0}{a}$$

$$\Rightarrow Z_{monop} \simeq \int \mathcal{D}\chi \; e^{-\frac{e^2}{4\pi^2} \int d^3x \left(\frac{1}{2} (\partial_\mu \chi)^2 - M^2 \cos\chi \right)} \tag{4.30}$$

which is the partition function of the

Sine - Gordon model

This concludes the weak-coupling mapping from a monopol gas to a "dual-photon" Sine-Gordon system.

In fact, in this system, the WW-loop expectation value can be computed; for this, we have to translate the WW loop into the "language" of the sign - Gordon model:

the monopole distribution generates an EM field

$$\partial_\mu \tilde{F}_\mu(x) = 2\pi \, \rho_M(x) \qquad \left(q_z/a^3 \to \rho_M \right)$$

$$\Rightarrow \quad \tilde{F}_\mu(x) = \frac{1}{2}\int d^3x' \, \frac{(x-x')_\mu}{|x-x'|^3}\, \rho_M(x') \qquad (4.31)$$

Now consider the flux through a surface S with boundary $\partial S = C$:

$$\oint_C dx_\mu A_\mu(x) = \int_S d\tilde{\sigma}_\mu \tilde{F}_\mu(x)$$
$$= \int d^3x' \int_S d\tilde{\sigma}_\mu \frac{1}{2}\frac{(x-x')_\mu}{|x-x'|^3}\rho_M(x')$$
$$=: \int d^3x' \, \eta_S(x')\, \rho_M(x')$$

(4.32)

where $\eta_S(x') = -\frac{1}{2}\frac{\partial}{\partial x'_\mu}\int_S d\tilde{\sigma}_\mu \frac{1}{|x-x'|}$.

In the language of electrodynamics, η_S describes a surface S of dipols.

The Wegner-Wilson loop for a monopol distribution thus reads

$$W(C) = e^{i\oint_C dx_\mu A_\mu} = e^{i\int d^3x\, \eta_S(x)\rho_M(x)} \qquad (4.33)$$

$$\left(\simeq e^{i\sum_z \eta_{S,z} q_z} \quad \text{lattice} \right)$$

Now note that $W(C)$ occurs in the calculation of $\langle W(C)\rangle$ in exactly the same fashion as the factor $e^{i\sum_z q_z \chi_z}$ in the monopole partition function, see Eq. (4.27). Hence, $\langle W(C)\rangle$ can be obtained by a shift of χ in the partition function:

$$\langle W(C)\rangle = \frac{1}{Z_{monop}} \int \mathcal{D}\chi \; e^{-\frac{e^2}{4\pi^2}\int d^3x \left(\frac{1}{2}\left[\partial_\mu(\chi-\eta_S)\right]^2 - M^2\cos\chi\right)} \qquad (4.34)$$

Now, the calculation can approximately be done by a saddle-point approximation. For this, we consider the EoM:

$$\partial^2\chi_{cl} = \underbrace{-\partial^2\eta_S}_{\substack{S \text{ in } x,y \text{ plane} \\ = 2\pi\,\delta'(z)\,\Theta_S(x,y)}} + M^2\sin\chi_{cl} . \qquad (4.35)$$

$$\underbrace{\Theta_S(x,y)}_{\begin{cases} =1 & \text{for } x,y \text{ inside } S \\ =0 & \text{otherwise}\end{cases}}$$

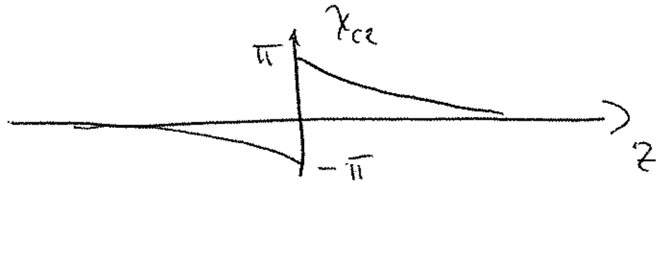
Near S, η_S is almost homogeneous in x,y direction:

$$\partial^2\eta_S \simeq \partial_z^2\eta_S \quad \Rightarrow \quad \eta_S(x) \simeq \pi \,\mathrm{sign}\, z \;\Theta_S(x,y) \qquad (4.36)$$

discontinuity of 2π at $z=0$

To the same approximation, the solution for χ_{cl} reads:

$$\chi_{cl} = \mathrm{sign}\, z \cdot 4\cdot \arctan\left(e^{-M|z|}\right)\Theta_S(x,y) \qquad (4.37)$$



Inserting this into the classical action yields

$$\underbrace{\int d^3x}_{\int_S dx\,dy \int_{-\infty}^{\infty} dz} \left(\frac{1}{2} \left(\partial_\mu (\chi_{cl} - \eta_S) \right)^2 + M^2 (1 - \cos \chi_{cl}) \right) = 8M \cdot \text{area}(S) + \dots$$

$$\Rightarrow \quad \langle W(C) \rangle \underset{\text{approx}}{\overset{\text{saddle point}}{\simeq}} e^{-\frac{e^2}{4\pi^2} 8M \,\text{area}(S)} \qquad (4.39)$$

which proves monopole-induced confinement in compact $U(1)$ in $d=3$.

NB:

- compact $U(1)_3$ is confining already in the weak-coupling limit
- confinement is also observed in the strong-coupling expansion. Since no phase transition occurs in between, both ends describe the same physics.
- the vacuum of compact $U(1)_3$ rather resembles a monopole gas than a dual

superconductor. Even though monopoles induce confinement, it is not really linked with the picture of a dual Meissner effect.

- the photon γ does not remain massless, but acquires a mass $\simeq M$ (being exponentially small).

This highlights the generally expected connection between

$$\text{Confinement} \iff \text{mass gap}$$