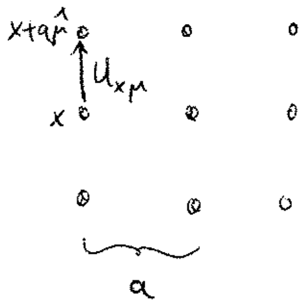
3.3 Gauge fields on the lattice

In Subsect. 3.1, we constructed $U(y,x)$, which acts as a "parallel-transporter" of color information, from a sequence of infinitesimal steps, see Eq. (3.6).
Note, however, that the desired gauge-transformation property $U(y,x) \to \Omega(y)\, U(y,x)\, \Omega^{-1}(x)$ was already present for the infinitesimal step. Since there is an infinitesimal one-to-one correspondence between $U(x+dx,x)$ (for arbitr. dx) and A_μ, c.f. (3.5),

$$U(x+dx,x) = \mathbb{1} + ig\, dx_\mu A_\mu(x) , \qquad (3.29)$$

this suggests that a gauge theory can fully be formulated in a discrete fashion on a spacetime lattice in terms of the variables $U(x+dx,x)$ with full gauge symmetry.

Consider a hypercubic lattice with lattice spacing a. Let us denote the sites by x, and a unit vector pointing into



the μ direction by $\hat{\mu}$. A neighboring site to x thus is denoted by $x + a\hat{\mu}$. To every link between neighboring sites, we associate a parallel transporter

$$U_{x\mu} \equiv U_{x,x+a\hat{\mu}} \quad \in SU(N_c) \,. \qquad (3.30)$$

The inverse is given by

$$U_{x\mu}^{-1} \equiv U_{x\mu}^{+} = U_{x+a\hat{\mu},\,x} \,. \qquad (3.31)$$

Gauge transformations are defined on the sites, $\Omega(x)$, and the links transform as

$$U_{x\mu} \to \Omega(x)\, U_{x,x+a\hat{\mu}}\, \Omega^{-1}(x+a\hat{\mu}) \qquad (3.32)$$

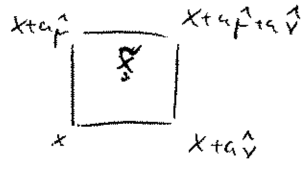
The links encode all gauge-field information and can thus be viewed as the true gauge-field degrees of freedom. The relation between the links and continuum gauge fields is not unique; e.g. equally valid definitions to order a are (c.f. (3.29))

$$U_{x\mu} = U_{x,x+a\hat{\mu}} = \mathbb{1} - i g a A_\mu(x + \tfrac{1}{2}a\hat{\mu})$$

$$\text{or} \quad U_{x\mu} = e^{-i g a A_\mu(x+\frac{1}{2}a\hat{\mu})} . \qquad (3.33)$$

Here we have associated the gauge-field coordinate with the link "center of mass", i.e., lying in between two neighboring sites.

Let us now introduce the "plaquette":



$$U_{\mu\nu} = U_{x\mu}\, U_{x+a\hat{\mu},\nu}\, \underbrace{U_{x+a\hat{\mu}+a\hat{\nu},\, x+a\hat{\nu}}}_{U^+_{x+a\hat{\nu}\,\mu}}\, \underbrace{U_{x+a\hat{\nu},\, x}}_{U^+_{x\nu}} \qquad (3.34)$$

To order a^2, $U_{\mu\nu}$ is given by

$$U_{\mu\nu} = \Big[\big(\mathbb{1} - i g a A_\mu(x+\tfrac{1}{2}a\hat{\mu})\big)\big(\mathbb{1} - i g a A_\nu(x+a\hat{\mu}+\tfrac{1}{2}a\hat{\nu})\big)$$

$$\cdot \big(\mathbb{1} + i g a A_\mu(x+a\hat{\nu}+\tfrac{1}{2}a\hat{\mu})\big)\big(\mathbb{1} + i g a A_\nu(x+\tfrac{1}{2}a\hat{\nu})\big)\Big]$$

Denoting the center of a plaquette by $\tilde{x}$, the last line reads $\left(\tilde{x} = x + \frac{1}{2}a(\hat{\mu}+\hat{\nu})\right)$:

$$U_{\mu\nu} = \left[e^{-iga A_\mu(\tilde{x}-\frac{1}{2}a\hat{\nu})} e^{-iga A_\nu(\tilde{x}+\frac{1}{2}a\hat{\mu})} e^{+iga A_\mu(\tilde{x}+\frac{1}{2}a\hat{\nu})} e^{iga A_\nu(\tilde{x}-\frac{1}{2}a\hat{\mu})} \right]$$

$$= \Big[1 - \cancel{g^2 a^2 A_\mu A_\nu} + \cancel{g^2 a^2 A_\mu A_\nu} + g^2 a^2 A_\nu A_\mu - g^2 a^2 A_\mu A_\nu$$

$$+ \frac{i}{2} g a^2 \partial_\nu A_\mu - \frac{i}{2} g a^2 \partial_\mu A_\nu + \frac{i}{2} g a^2 \partial_\nu A_\mu - \frac{i}{2} g a^2 \partial_\mu A_\nu \Big]$$

$$+ \mathcal{O}(a^3)$$

all $A_\mu A_\mu$ $A_\nu A_\nu$ terms cancel

$$= \left[1 - i g a^2 \left(\partial_\mu A_\nu - \partial_\nu A_\mu - ig [A_\mu, A_\nu] \right) \right] + \mathcal{O}(a^3)$$

$$= \left[1 - i g a^2 F_{\mu\nu} + \mathcal{O}(a^3) \right]$$

$$= e^{-iga^2 F_{\mu\nu}} + \mathcal{O}(a^3) \qquad (3.35)$$

Now consider (β: normalization)

$$S_p = \beta \left(1 - \frac{1}{N_c} \mathrm{Re}\, \mathrm{tr}\, e^{-iga^2 F_{\mu\nu}} + \dots \right)$$

$$= \beta \Big(1 - \frac{1}{N_c} \underbrace{\mathrm{tr}\, \mathbb{1}}_{= N_c} + \frac{1}{N_c} g a^2 \mathrm{Re}\, \underbrace{\mathrm{tr}\, i F_{\mu\nu}}_{= 0}$$

$$+ \frac{1}{N_c} g^2 a^4 \mathrm{Re}\, \mathrm{tr}\, F_{\mu\nu} F_{\mu\nu} + \mathcal{O}(a^6) \Big)$$

no summation

$$\Rightarrow \quad S_p = \frac{\beta}{N_c} g^2 a^4 \operatorname{tr} F_{\mu\nu} F_{\mu\nu} + \mathcal{O}(a^6) \tag{3.36}$$

Summing over all possible plaquettes, we define the Wilson action on a lattice,

$$S[U] = \sum_p S_p = \beta \sum_{\mu\nu} \left(1 - \frac{1}{N_c} \operatorname{Re} \operatorname{tr} U_{\mu\nu} \right) \tag{3.37}$$

$$\xrightarrow{a \to 0} \quad \beta \frac{2g^2}{N_c} \int d^4x \, \frac{1}{2} \operatorname{tr} F_{\mu\nu} F_{\mu\nu} \,,$$

with summation

which corresponds to the Yang-Mills action if we choose

$$\beta = \frac{N_c}{2g^2} \,. \tag{3.38}$$

(NB: The real part "Re" is introduced to keep the action real also to higher orders in a).

The quantum gauge theory is finally defined by integrating over all possible values for the gauge variables U:

$$Z = \int \mathcal{D}U \, e^{-S[U]} , \qquad (3.39)$$

where $\mathcal{D}U$ denotes the invariant group measure, $\mathcal{D}U = \prod_{\{\text{all links}\}} dU$, and dU denotes the Haar measure.

Given a parameterization of U in terms of coordinates ω^a on group space, i.e. $U = e^{-i\omega^a \tau^a}$, the Haar measure corresponds to the reparametrization invariant measure with respect to coordinate transformations,

$$dU = \nu \sqrt{\det g} \prod_a d\omega^a , \qquad (3.40)$$

where

$$g_{ab} = \underbrace{(2)}_{\text{for } SU(N_c>1)\text{ ; for } U(1) : (1)} \operatorname{tr}\left(\frac{\partial U}{\partial \omega^a} \frac{\partial U^+}{\partial \omega^b} \right) \qquad (3.41)$$

denotes the induced metric on group space, and the normalization ν can be choosen such that

$$\int dU = 1 . \tag{3.42}$$

The metric transforms covariantly under coordinate transformations, $\omega'^a = f^a(\omega)$,

$$g_{cd} = g_{ab} \frac{\partial \omega'^a}{\partial \omega^c} \frac{\partial \omega'^b}{\partial \omega^d} , \tag{3.43}$$

implying that Jacobian factors from coordinate transformations cancel out explicitly in (3.40),

$$\Rightarrow dU' = dU . \tag{3.44}$$

A special case of coordinate transformations is given by left and right translations in group space,

$$U' = \Omega U \quad , \quad U' = U \Omega , \tag{3.45}$$

(i.e., $U'(\omega') = \Omega U(\omega)$ can be viewed as a coordinate transformation $\omega \to \omega'(\omega)$.)

Hence the Haar measure is simultaneously invariant under left and/or right transformations,

$$d(\Omega U) = d(U\Omega) = dU. \tag{3.46}$$

A simple illustration of all this is given by the Haar measure for $U(1)$:

$$U = e^{-i\omega}, \quad g_{ab} = 1$$
$$\int dU = \int_{-\pi}^{\pi} \frac{d\omega}{2\pi} = 1 \tag{3.47}$$

Consider $\Omega = e^{i\alpha}$ (e.g. left translation)

$$\Rightarrow \quad U'(\omega') = e^{-i\omega'} = \Omega U(\omega) = e^{-i(\omega-\alpha)}$$

$$\Rightarrow \quad \omega' = \omega - \alpha \tag{3.48}$$

Of course, explicit representations can be worked out for the Haar measure of $SU(N_c)$, but this will not be a matter of concern here.

Let us finally remark that the partition function

$$Z = \int \mathcal{D}U \, e^{-S[U]} \quad \text{is finite} \tag{3.49}$$

for finite lattices and compact gauge groups; hence, correlators and observables can immediately be computed,

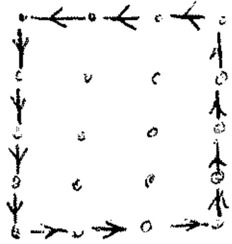
$$\langle \mathcal{O}(U) \rangle = \frac{1}{Z} \int \mathcal{D}U \, \mathcal{O}(U) \, e^{-S[U]}. \tag{3.50}$$

In particular, gauge fixing is not necessary for non-perturbative lattice computations.

3.4 Wegner-Wilson loop in strong-coupling expansion

The Wegner-Wilson loop on the lattice is simply given by the (trace over the) product of link variables along the contour,

$$W(C) = \mathrm{tr} \prod_C U ,$$



C

(3.51)

and its expectation value reads

$$\langle W(C) \rangle = \frac{1}{Z} \int \mathcal{D}U \; \mathrm{tr} \prod_C U \; e^{-S[U]} . \qquad (3.52)$$

With S being $\sim \frac{1}{g^2}$,

$$S[U] \overset{(3.37)}{\underset{(3.38)}{=}} \frac{N_c}{2g^2} \sum_{r\nu} \left(1 - \frac{1}{N_c} \mathrm{Re}\, \mathrm{tr}\, U_{r\nu}\right) , \qquad (3.53)$$

We may try to compute (3.53) in a strong coupling expansion $\frac{1}{g^2} \to 0$ (this is reminiscent to a high-temperature expansion in statistical systems $Z \sim e^{-\beta H}, \beta \to 0$).

For this, we need some elementary integrals over group space; by construction, we have

$$\int dU \overset{(3.42)}{=} 1 \,. \qquad (3.54)$$

Furthermore, it holds that ($N_c = 3$)

link attached to x

$$\int dU \; U_{ij}(l_{x\mu}) = 0 \qquad (3.55)$$

color matrix indices

which can be checked by choosing a specific parametrization.
From these two equations, it follows that

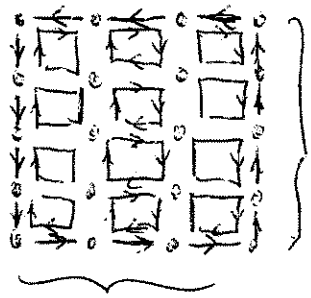
$$\int dU \; U_{ij}(l_{x\mu}) \, U^{\dagger}_{kl}(l_{y\nu}) = \frac{1}{3} \, \delta_{il} \, \delta_{jk} \, \delta_{l_{x\mu} l_{y\nu}} \,. \qquad (3.56)$$

A strong-coupling expansion of (3.52) corresponds to a Taylor expansion of the exponential $e^{-S[U]} \sim e^{\frac{1}{2g^2}\sum_{\mu\nu} \mathrm{Re}\,\mathrm{tr}\, U_{\mu\nu}}$.

To 0th order, we get

$$\langle W(C) \rangle^{0} = \frac{1}{Z} \int \mathcal{D}U \; \mathrm{tr} \prod_{C} U = 0 \qquad (3.57)$$

since at any link the rule (3.55) holds. We observe that a link out of the Wegner-Wilson loop can only contribute if there is another link on top of it, the product of which has a singlet component according to (3.56). This happens to be the case when a link on the contour is multiplied by a conjugate link being part of a plaquette. To lowest non-vanishing order in $\frac{1}{g^2}$, this implies that the Wegner-Wilson loop area has to be tiled completely by plaquettes



$J = T/a$

$I = R/a$

For an area of side length $I \times J$, this makes $2 \cdot I \cdot J + I + J$ pairs of links, contributing a factor $\left(\frac{1}{3}\right)^{2 \cdot I \cdot J + I + J}$ to the WW-loop expectation value (according to Eq. (3.56). The contraction of the Kronecker-δ's from (3.56) as well as from the prefactor of the action (3.53), we obtain further contributions of this form. The final result reads

$$\langle W(I,J)\rangle^{1} = 3\left(\frac{\beta}{18}\right)^{IJ}\left(1+\mathcal{O}(\beta)\right) \qquad , \ \beta = \frac{N_c}{2g^2} \overset{SU(3)}{=} \frac{3}{2g^2}$$

$$= 3\left(\frac{\beta}{18}\right)^{\frac{T}{a}\frac{R}{a}}\left(1+\mathcal{O}(\beta)\right) \tag{3.58}$$

to lowest order in $1/g^2$.

For the static potential, we thus obtain

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln \left(3 \left(\frac{\beta}{18}\right)^{\frac{TR}{a^2}} + \dots \right)$$

$$\simeq \left(-\ln \frac{\beta}{18}\right) \frac{1}{a^2} R \equiv \sigma R, \qquad (3.59)$$

with $\sigma = -a^{-2} \ln \frac{\beta}{18}$.

The strong-coupling expansion therefore produces a linearly confining potential. The strong-coupling expansion therefore gives analytical insight into the structure of the theory at large bare coupling.

However, it turns out to be difficult to relate the strong-coupling expansion to the parameter region where the renormalized coupling takes on physical values.

The result (3.59), though conceptually highly interesting, thus does not serve as a proof of confinement.