3 Gauge Fields on loops & Lattices

3.1 Wegner - Wilson loop

In view of the confinement problem, we would like to study a quark anti-quark pair at large separations. Here, we encounter a problem: e.g., the correlations between a quark field $\psi(x)$ at x and a conjugate quark field $\bar{\psi}(y)$ at y are gauge dependent, since the fields at x and y in general transform differently under gauge transformations $\Omega(x)$.

> CAVE: in order to conform with the literature, we use a different notation in this chapter. Gauge transformations are denoted by $\Omega(x)$, e.g.
>
> $$A_\mu \to \Omega A_\mu \Omega^{-1} - \frac{i}{g} (\partial_\mu \Omega) \Omega^{-1}, \quad \Omega^{-1} = \Omega^\dagger$$

$$\begin{aligned} \psi(x) &\to \Omega(x)\, \psi(x) \\ \bar{\psi}(y) &\to \bar{\psi}(y)\, \Omega^{-1}(y) \end{aligned} \tag{3.1}$$

In order to compare quark fields at different points in a meaningful gauge-invariant way with each other, we need to transport the color information of, say, $\bar{\Psi}(y)$ to point x in a gauge-covariant manner.

Technically speaking, we are looking for a bi-local object $U(y,x)$ which transforms as

$$U(y,x) \to \Omega(y)\, U(y,x)\, \Omega^{-1}(x) \tag{3.2}$$

and which can be used to form gauge-invariant operators, e.g. $\bar{\Psi}(y)\, U(y,x)\, \Psi(x)$. As a normalization, we choose

$$U(x,x) = \mathbb{1}\,. \tag{3.3}$$

For an infinitesimal distance $y = x + dx$, we find

$$\Omega(x+dx)\, U(x+dx, x)\, \Omega^{-1}(x)$$

$$= \Omega(x)\, U(x,x)\, \Omega^{-1}(x) + dx_\mu (\partial_\mu \Omega)(x)\, U(x,x)\, \Omega^{-1}(x)$$

$$+ \Omega(x)\, dx_\mu\, \partial^y_\mu U(y=x, x)\, \Omega^{-1}(x) + \mathcal{O}(dx^2)$$

$$= \mathbb{1} + dx_\mu \left[(\partial_\mu \Omega)\Omega^{-1} + \Omega\, \partial^y_\mu U(y=x,x)\, \Omega^{-1} \right] + \mathcal{O}(dx^2) \tag{3.4}$$

The term in square brackets looks like a gauge-transformed gauge potential, if $\partial^y_\mu U(y=x,x) = ig A_\mu(x)$.

This observation suggests that $\mathcal{U}(x+dx, x)$ can be represented by

$$\mathcal{U}(x+dx, x) = \mathbb{1} + ig\, dx_\mu A_\mu(x) \,, \tag{3.5}$$

and Eq. (3.4) read from right to left shows that this choice has the desired transformation properties.

For finite separations $y \leftrightarrow x$, $\mathcal{U}(y,x)$ can be constructed from a product of $\mathcal{U}(x+dx, x)$'s,

$$y_n = x + \frac{n}{N}(y-x) \,, \quad n = 0, \dots, N$$

$$\mathcal{U}(y,x) = \lim_{N\to\infty} \prod_{n=1}^{N} \mathcal{U}(y_n, y_{n-1}) \,. \tag{3.6}$$

If A_μ was a constant number A and $dx_\mu = \frac{(y-x)}{N}$, we would conclude $\mathcal{U}(y,x) = \lim_{N\to\infty} \left(1 + ig\frac{(y-x)A}{N}\right)^N = e^{ig(y-x)A}$.

For a c-number function $A_\mu(x)$ as in $U(1)$ gauge theories, we find $\mathcal{U}(y,x) = e^{ig\int_x^y dx_\mu A_\mu}$.

However, for a matrix-valued $A_\mu = A_\mu^a \tau^a$, we have to take care of the noncommuting property of two A_μ's at neighboring points, the result of which we denote by

$$\underline{\mathcal{U}(y,x) = \mathcal{P}\, e^{ig\int_x^y dx_\mu A_\mu(x)}} \tag{3.7}$$

The $\mathcal{P}$ symbol means "path-ordering". For instance, in a Taylor expansion of (3.7), matrices $A_\mu(x)$ which are attached to a certain point x are ordered from later (left) to earlier (right) points on a path, e.g.

$$\left(\begin{aligned} \text{NB:} \quad \mathcal{P}\left(\int_x^y dx_\mu A_\mu\right)^2 &= \mathcal{P} \int_x^y dx_{1\mu} A_\mu(x_1) \int_x^y dx_{2\mu} A_\mu(x_2) \\ &= \int_x^y dx_{1\mu} \int_{x_1}^y dx_{2\mu} \, A_\mu(x_2) A_\mu(x_1) \\ &\quad + \int_x^y dx_{1\mu} \int_x^{x_1} dx_{2\mu} \, A_\mu(x_1) A_\mu(x_2) \, . \end{aligned} \right) \tag{3.8}$$

Eq. (3.6) is, of course, Path ordered by construction. Also by construction, $\mathcal{U}$ transforms as

$$\mathcal{U}(y,x)[A^\Omega] = \Omega(y)\, \mathcal{U}(y,x)[A]\, \Omega^{-1}(x) \, . \tag{3.9}$$

$\mathcal{U}(y,x)[A]$ constitutes a mapping of paths in coordinate space into the gauge group. In Eq. (3.6) we used specific straight line paths. Whereas $\mathcal{U}(y,x)[A]$ is generally path dependent, the gauge-trafo property (3.9) is only sensitive ~~to~~ the endpoints; any other path in (3.6) would also lead to the desired Eq. (3.9).

Consider now the important case where the path is a closed contour C:

$$C: \left\{ x_\mu(s) \;\middle|\; x_\mu(0) = x_\mu(1) \,,\; s \in [0,1] \right\} . \tag{3.10}$$

A fully gauge-invariant object is then given by

$$\underline{\underline{W(C)}} = \underline{\underline{\operatorname{tr} U(C)}} = \operatorname{tr} \underline{\underline{\mathcal{P}\, e^{ig \oint_C dx_\mu A_\mu(x)}}} \,, \tag{3.11}$$

since $W^\Omega(C) = \operatorname{tr} \Omega(x(1))\, e^{ig \int_0^1 ds \frac{dx_\mu}{ds} A_\mu(x(s))}\, \Omega^{-1}(x(0))$

$$\overset{(3.10)}{=} W(C)$$

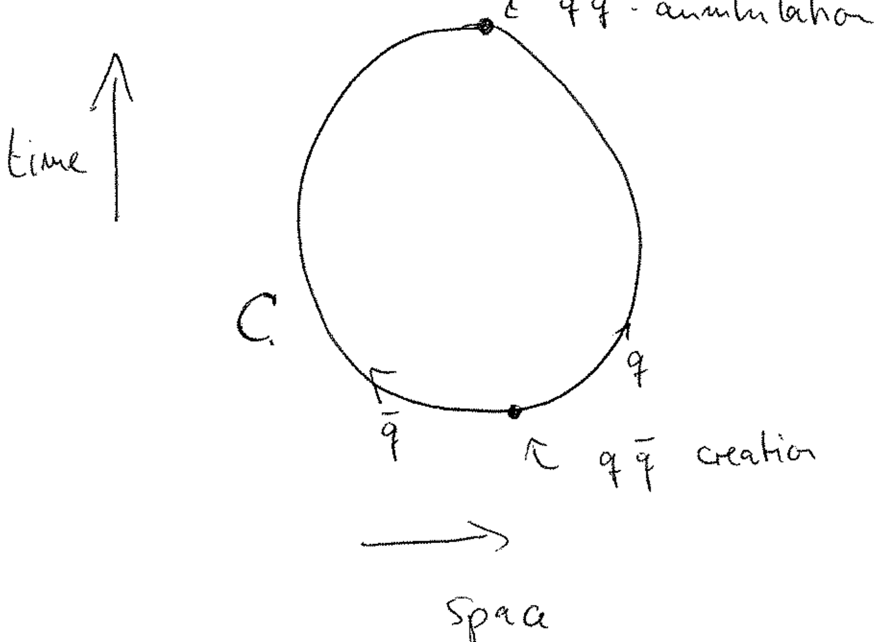
This is the Wegner-Wilson loop which plays a key role in confining gauge theories.
Note that the exponent can be written as

$$ig \oint_C dx_\mu A_\mu(x) \equiv \int d^d x\, j_\mu^a(x) A_\mu^a(x) \,, \tag{3.12}$$

where $j_\mu^a(x) = ig \oint_C dx_\mu^c\, \tau^a\, \delta^{(d)}(x - x^c)$ (3.13)

can be viewed as a source term of a charged particle in fundamental representation (the "i" is due to our Euclidean conventions), propagating

along a closed contour in spacetime.
Alternatively, J_r^a can be viewed as a source term for a quark anti-quark pair, being created at some early time, then propagating a distance and finally annihilating each other again.



(3.14)

The Wegner-Wilson loop expectation value therefore is nothing but the generating functional for a special source ${}^{C}J_r^a$:

$$\langle W(C)\rangle = \frac{1}{Z[0]}\, Z[{}^{C}J] = \frac{1}{Z[0]}\int \mathcal{D}A\, \Delta_{FP}[A]\, e^{-S_{YM}-S_{gf}+\int {}^{C}J_r^a \cdot A_r^a},$$

where the path-ordering prescription is implicitly understood. (3.15)

The meaning of the Wegner-Wilson loop can heuristically be understood by the quantum mechanical analog; here, the connection between the functional integral (path integral) and the Hamiltonian formalism is most immediate; in Euclidean QM, we get for the partition function of a particle, moving in $d=3$ for a given time T in the presence of some interaction with J:

$$\underline{\underline{Z[J]}} = \int d^3\vec{x}_i \, \langle \vec{x}_i | e^{-H(J)T} | \vec{x}_i \rangle$$

$$= \int d^3\vec{x}_i \int_{\vec{x}(0)=\vec{x}_i}^{\vec{x}(T)=\vec{x}_i} \mathcal{D}x \; e^{-\int_0^T d\tau \, \mathcal{L}(\vec{x},\dot{\vec{x}};J)}$$

$$= \mathrm{Tr}\, e^{-H(J)T}$$

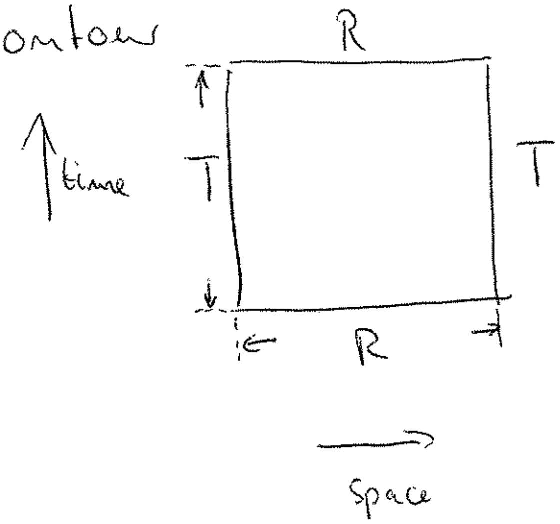
$$\underset{\text{of states}}{\overset{\text{complete set}}{=}} \sum_{n=0}^{\infty} \langle n | e^{-H(J)T} | n \rangle = \sum_{n=0}^{\infty} e^{-E_n(J)T}$$

$$= e^{-E_0(J)T} \left(1 + \sum_{n=1}^{\infty} e^{-(\underbrace{E_n - E_0}_{>0})T} \right) \qquad (3.16)$$

For large times $T \to \infty$, the partition function is dominated by the ground-state energy:

$$E_0(J) = -\lim_{T\to\infty} \frac{1}{T} \ln Z[J] \qquad (3.17)$$

Transfering this reasoning to Quantum Gauge Theory suggests that the Wegner-Wilson loop expectation value (3.15) is related to the energy associated with the creation and annihilation of a $q\bar{q}$ pair. Choosing as a contour



corresponding to a $q\bar{q}$ pair that remains static at a distance R for some time T, we expect the ground-state energy to dominate $\langle W(C)\rangle$ in the limit $T \to \infty$ and to correspond to the static potential $V(R)$ between the $q\bar{q}$ pair:

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \ln \langle W(C)\rangle . \qquad (3.18)$$

(NB: The connection between $V(R)$ and $\langle W(C)\rangle$ can indeed more rigorously be shown in QFT with the aid of the transfer matrix formalism.)

Confinement in gauge theories is therefore signaled by

$$V(R) \overset{\text{conf.}}{=} \sigma R \quad \text{for large } R$$

$$\Rightarrow \quad \underline{\langle W(C)\rangle} \sim e^{-\sigma T R} = \underline{e^{-\sigma A}} \qquad (3.19)$$

where A is the area encircled by the contour C : $C = \partial A$.

Eq. (3.19) expresses the famous <u>area law</u> of the Wegner-Wilson loop which serves as an important criterion <u>for confinement</u>.

3.2 Wegner-Wilson loop in QED (i.e. U(1) non-compact gauge theory)

As an illustration, let us compute the Wegner-Wilson loop VEV in U(1) gauge theory in Feynman gauge:

$$\langle W(C)\rangle = \frac{1}{Z}\int \mathcal{D}A \; e^{-\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu} - \frac{1}{2\alpha}\int(\partial_\mu A_\mu)^2 + \int {}^{C}\!J_\mu A_\mu} \tag{3.20}$$

Using $\alpha = 1$ (Feynman gauge) and

$$\frac{1}{4}\int F_{\mu\nu}F_{\mu\nu} = \frac{1}{2}\int A_\mu\left(-\partial^2\delta_{\mu\nu} + \partial_\mu\partial_\nu\right)A_\nu$$

$$\frac{1}{2}\int(\partial_\mu A_\mu)^2 = \frac{1}{2}\int A_\mu\left(-\partial_\mu\partial_\nu\right)A_\nu\,, \tag{3.21}$$

we get

$$\langle W(C)\rangle = \frac{1}{Z}\int \mathcal{D}A \; e^{-\frac{1}{2}\int A_\mu(-\partial^2)A_\mu + \int {}^{C}\!J_\mu A_\mu} \tag{3.22}$$

$$= \frac{1}{Z}\int \mathcal{D}A \; e^{-\frac{1}{2}\int\left(A_\mu - \frac{1}{(-\partial^2)}{}^{C}\!J_\mu\right)(-\partial^2)\left(A_\mu - \frac{1}{(-\partial^2)}{}^{C}\!J_\mu\right)} \cdot e^{+\frac{1}{2}\int {}^{C}\!J_\mu \frac{1}{(-\partial^2)}{}^{C}\!J_\mu}$$

The seeming source dependence in the first line can be shifted away by a substitution $A_\mu \to A_\mu + \frac{1}{(-\partial^2)} J_\mu$.
The first line is thus exactly $= 1$. We obtain

$$\langle W(C) \rangle = e^{\frac{1}{2} \int {}^C J_\mu \frac{1}{(-\partial^2)} {}^C J_\mu} \tag{3.23}$$

The symbol $\frac{1}{(-\partial^2)}$ denotes nothing but the Green's function of the 4-dim. Laplacian

$$(-\partial^2)\, G = \mathbb{1} \;, \text{ i.e., } -\partial_x^2 \, G(x,y) = \delta(x,y) \tag{3.24}$$

which can be determined as

$$G(x,y) = \frac{1}{4\pi^2} \frac{1}{|x-y|^2} \qquad \left(\hat{=} \frac{1}{(-\partial^2)} \right).$$

(Euclidean photon propagator in spacetime.)
In $U(1)$ gauge theory, the source term for a static $e^+ e^-$ pair reads (c.f. (3.13)):

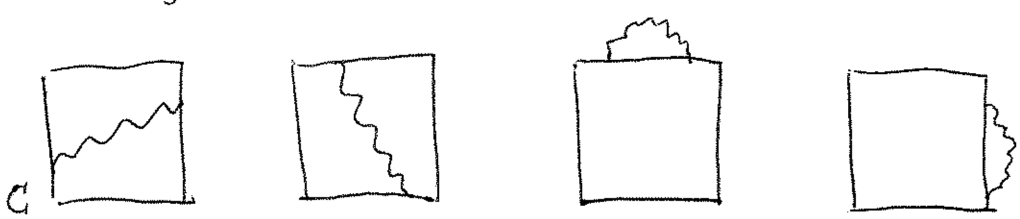
$$ {}^C J_\mu(x) = i e \oint_C dx_\mu^C \, \delta^{(4)}(x - x^C) \tag{3.25}$$

where C denotes the rectangular contour on p. 74.

The exponent in (3.23) thus reads

$$\frac{1}{2}\int {}^{C}j_r \frac{1}{(-\partial^2)} {}^{C}j_r = -\frac{e^2}{2}\frac{1}{4\pi^2}\oint_C dx^1_r \oint_C dx^2_r \frac{1}{|(x^1)-(x^2)|^2} . \tag{3.26}$$

Now $dx^1_\mu dx^2_r$ is only $\neq 0$ if $dx^1_r \parallel dx^2_r$ for our C, i.e. if x^1_r and x^2_r are on the same or on opposite sides. Representing the photon exchange by , there are 4 types of contributions:



The latter two describe the electromagnetic self-interactions of a particle, contributing to the (mainly divergent) self-energy. For the interactions between the e^+e^- pair, these terms are irrelevant and we drop them.
The remaining integrals can straightforwardly be performed and we obtain:

$$\frac{1}{2}\int {}^{c}j_r \frac{1}{(-\partial^2)} {}^{c}j_r \Big|_{\substack{\text{without} \\ \text{selfenergies}}} = -\frac{e^2}{2\pi^2}\left(-\frac{T}{R}\arctan\frac{T}{R} + \frac{1}{2}\ln\left(1+\frac{T^2}{R^2}\right)\right) \tag{3.27}$$

This implies for the static potential (3.18)

$$\underline{\underline{V(R)}} = -\lim_{T\to\infty}\frac{1}{T}\ln e^{-\frac{e^2}{2\pi^2}\left(-\frac{T}{R}\underbrace{\arctan\frac{T}{R}}_{\to\frac{\pi}{2}} + \frac{1}{2}\ln\left(1+\frac{T^2}{R^2}\right)\right)}$$

$$= -\lim_{T\to\infty}\frac{1}{T}\frac{e^2}{2\pi^2}\frac{T}{R}\frac{\pi}{2} = \underline{\underline{-\frac{e^2}{4\pi R}}}, \tag{3.28}$$

which exactly corresponds to the Coulomb potential !