## 2.4 Perturbative Effective Action

Let us compute the perturbative effective action for gauge theories, using the background field gauge. For a general field theory, we found in Eq. (2.48)

$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\ln\det\frac{S^{(2)}[\phi]}{S^{(2)}[0]} + \dots \tag{2.61}$$

for the 1-loop approximation of $\Gamma[\phi]$. Repeating the argument for gauge theories, we obtain in the background gauge:

$$e^{-\Gamma[\bar{A}]} \underset{Q_{cl}=0}{=} \int \mathcal{D}Q \; e^{-S[\bar{A}+Q] - S_{gf}[\bar{A},Q]} \Delta_{FP}[\bar{A},Q]$$

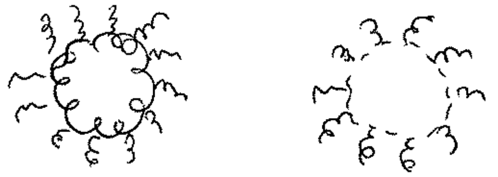
$$= e^{-S[\bar{A}]} \int \mathcal{D}Q \; e^{-\frac{1}{2}\int Q^a_\mu (S+S_{gf})^{(2)ab}_{\mu\nu}[\bar{A}]\, Q^a_\nu + \mathcal{O}(Q^3)}$$

$$\cdot \; \Delta_{FP}[\bar{A}, Q=0] \quad + \text{ higher loops}$$

↑ already one ghost loop

$$= \mathcal{N}\cdot \det{}^{-1/2} M[\bar{A}] \cdot \Delta_{FP}[\bar{A}]\, e^{-S[\bar{A}]} + \text{higher} \tag{2.62}$$

where
$$M^{ab}_{\mu\nu}[\bar{A}] = \frac{\delta^2 (S+S_{gf})[\bar{A}]}{\delta Q^a_\mu \,\delta Q^b_\nu} = -\bar{D}^{ac}_\alpha \bar{D}^{cb}_\alpha \delta_{\mu\nu} + 2g f^{abc} \bar{F}^c_{\mu\nu} + \left(1-\frac{1}{\alpha}\right)\bar{D}^{ac}_\mu \bar{D}^{cb}_\nu$$



$$\Rightarrow \Gamma[\bar{A}] = S[\bar{A}] + \frac{1}{2}\ln\det M[\bar{A}] - \ln\det(-\bar{D})^2$$

$$+ \text{ normalization } + \text{higher loops} \qquad (2.63)$$

where we have used Eq. (2.54) for $\Delta_{FP}$ in the background gauge.

( If we include quarks, we will get another determinant from the fermionic path integral:

$$-\ln\det\left(-i\not{D}[\bar{A}] + m\right). \qquad (2.64)$$ )

The evaluation of these functional determinants for arbitrary $\bar{A}$ is difficult. The resulting action will contain nonlinearities and nonlocalities. In the following, we will be satisfied with the exploration of the nonlinearities; the nonlocalities will, for instance, be small for slowly varying fields with

$$\frac{D_\alpha F^{\mu\nu}}{|F|^{3/2}} \to 0 \quad (\text{in a smooth gauge})$$

The following further assumptions simplify the calculation:

Consider a

- pseudo abelian field:

$$F^a_{\mu\nu} = \underset{\uparrow \atop \text{const.}}{n^a} \; \underset{\uparrow \atop \text{const}}{F_{\mu\nu}} \quad , \quad n^a n^a = 1$$

(2.65)

- pure magnetic field

$$F_{12} = B = -F_{21} = \text{const.} \qquad F_{\mu\nu} = 0 \text{ otherwise}$$

- Feynman gauge, $\alpha = 1$

( for covariant constant fields, $D_\alpha F_{\mu\nu} = 0$, $\Gamma[A]$ is actually independent of $\alpha$; beyond the Landau gauge $\alpha = 0$, the gauge-fixing condition $\delta[\mathcal{F}^a]$ is not implemented exactly, but in a "smeared" fashion )

Here the gluonic fluctuation operator simplifies to

$$M^{ab}_{\mu\nu}[\bar{A}]\Big|_{\alpha=1} \overset{(2.62)}{=} -D^{2\,ab}\delta_{\mu\nu} + 2g f^{acb} \bar{F}^c_{\mu\nu} \quad , \qquad (2.66)$$

the first term being the covariant Laplacian, and the second term $2g f^{acb} \bar{F}^c_{\mu\nu} = -2ig (\mathbf{F}^c)^{ab} \bar{F}^c_{\mu\nu}$ is a gluon-spin-field coupling ( c.f. $\sim \vec{\mu} \cdot \vec{B}$ )

Let us perform the calculation within a few steps:

(1) Diagonalization in color space
all color-space dependence comes in the form of

$$-i f^{abc} n^c \equiv (T^c)^{ab} n^c$$

which is hermitean and can be diagonalized
with eigenvalues

$$\nu_\ell \;,\quad \ell = 1 \dots N_c^2 - 1 \;,\quad \text{being real}, \qquad (2.67)$$

and corresponding to "charges" of the gluon components with respect to the $n^a$ color axis.

E.g. $\quad SU(2): \quad \nu_\ell = -1, 0, 1$

$\qquad\quad SU(3): \quad \nu_\ell = -1, -\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2}, 1 \quad$ for $n^a = \delta^{a3}$

In the following, we will only need the identity

$$\sum |\nu_\ell|^2 = \operatorname{tr} n^c T^c n^b T^b = N_c . \qquad (2.68)$$

We find $(\bar{A} = A)$

$$\begin{aligned}
\Gamma^1[A] &= \frac{1}{2} \ln \det M[A] - \ln \det(-D^2) \\
&= \frac{1}{2} \operatorname{Tr} \ln M[A] - \operatorname{Tr} \ln(-D^2) \\
&= \sum_{\ell=1}^{N_c^2-1} \Big\{ \frac{1}{2} \operatorname{Tr} \ln \left( -\hat{D}^2 \delta_{\mu\nu} + 2ig\, \nu_\ell F_{\mu\nu} \right) \\
&\qquad - \operatorname{Tr} \ln \left( -\hat{D}^2 \right) \Big\} \qquad (2.69)
\end{aligned}$$

where we defined

$$\hat{D}_\mu = \partial_\mu - i g \nu_e A_\mu . \tag{2.70}$$

(2) Spectrum of the covariant Laplacian:

Finding the spectrum of $-\hat{D}^2$ is indeed identical to the quantum mechanical particle in a const. magnetic field with Hamiltonian $-\hat{D}^2 = 2H$. The solution comes in the form of Landau levels

$$\text{Spect.}\left\{-\hat{D}^2\right\} = p_0^2 + p_z^2 + g|\nu_e| B (2n+1), \quad n=0,1,\dots \tag{2.71}$$

(3) Spectrum of the gluon-spin - field coupling

Since $F_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & -B & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, we find

$$\text{Spect.}\left\{2ig F_{\mu\nu} T^c n^c\right\} = \begin{cases} 0 & \text{multiplicity } 2 \quad \text{longitudinal \& timelike DOF} \\ 2g\nu_e B & \\ -2g\nu_e B & \end{cases} \quad \text{transverse DOF with respect to } B \text{ field} \tag{2.72}$$

$$\overset{(2.69)}{\Longrightarrow} \quad \Gamma^1[A] = \sum_{e=1}^{N_c^2-1} \Big\{ \tfrac{1}{2}\mathrm{Tr}\ln\left(-\hat{D}^2 + 2g\nu_e B\right) + \tfrac{1}{2}\mathrm{Tr}\ln\left(-\hat{D}^2 - 2g\nu_e B\right)$$
$$+ \cancel{2\cdot\tfrac{1}{2}\mathrm{Tr}\ln\left(-\hat{D}^2\right) - \mathrm{Tr}\ln\left(-\hat{D}^2\right)} \Big\}$$

The ghosts cancel the longitudinal and timelike gluon contributions! This removes the overabundant gauge degrees of freedom.

(4) transversal gluon modes

$$\mathrm{Spect}\left\{-\hat{D}^2 \pm 2g\nu_e B\right\} = \begin{cases} p_0^2 + p_z^2 + g|\nu_e|B(2n+3) \\ p_0 + p_z^2 + g|\nu_e|B(2n-1) \end{cases}, \quad n = 0, 1, \ldots \tag{273}$$

We observe that the spectrum contains negative modes for $(p_0^2 + p_z^2) < g|\nu_e|B$, for $n = 0$. This "tachyonic" fluctuation is also called the Nielsen-Olesen unstable mode.

Its existence implies that fluctuations with long wavelength, i.e., $p_0^2 + p_z^2$ small, do not cost any action if the gluon-spin has a suitable orientation with respect to the field. The covariant constant field is thus not a minimum, but a saddlepoint of the action:

$\Rightarrow$ this vacuum $F \overset{\mathrm{cov.}}{=} \mathrm{const}$ (Savvidy vacuum) is unstable, "false vacuum"

Still, we can view $F = \text{const}$ as a technical assumption to do the calculation. $F = \text{const}$ may still be a reasonable approximation for slowly varying fields.

(5) trace computation: $(L^4 = \Omega \text{ spacetime vol.})$

$$\Gamma^1_{[A]} = \underbrace{\sum_{\ell=1}^{N_c^2-1} \frac{1}{2} \underbrace{\frac{g|\nu_\ell| B L^2}{2\pi}}_{\substack{\text{density of} \\ \text{Landau levels}}} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \frac{dp_0 dp_z}{\left(\frac{2\pi}{L}\right)^2} \sum_{\lambda=+3,-1}}_{\equiv \mathrm{Tr}} \ln\left(p_0^2 + p_z^2 + g|\nu_\ell| B(2n+\lambda)\right) \tag{2.74}$$

Eq. (2.74) contains divergencies of different type:

- for small momenta, the integral runs into the unstable mode. This can be dealt with by analytic continuation of the log, giving rise to an imaginary part of $\Gamma^1$; for the present purpose, this part is less relevant.
- for large momenta, the integral diverges logarithmically since it receives contributions from fluctuations on all scales; these div's will be responsible for the running of the coupling.

Let us deal with both types of div's simultaneously with the aid of a Zeta-function / propertime regularization. For this, we write the log as

$$\ln x = \lim_{\varepsilon \to 0} \left( \frac{1}{\varepsilon} - \frac{i^{\varepsilon} \mu^{2\varepsilon}}{\varepsilon \Gamma(\varepsilon)} \lim_{\delta \to 0} \int_0^\infty dT\, T^{\varepsilon - 1} e^{-iTx - T\delta} \right),$$

for $x \in \mathbb{R}$

(2.75)

with a scale $\mu$ to render the RHS dimensionless, $[x] = 2$, $[T] = -2$, $[\mu^{2\varepsilon} T^{\varepsilon}] = 0$.

The $p_0, p_z$-Integration and $n$-Summation can straightforwardly be carried out:

$$\Gamma^1_{[A]} = -\lim_{\varepsilon \to 0} \frac{\Omega}{16\pi^2} \sum_{\ell=1}^{N_c^2-1} (g|\nu_\ell|B)^2 \left( \frac{\mu^2}{g|\nu_\ell|B} \right)^{\varepsilon} \frac{1}{\varepsilon \Gamma(\varepsilon)}$$

$$\cdot \left\{ \int_0^\infty \frac{dT\, T^{\varepsilon-2}}{\sinh T} + i^{\varepsilon} \int_0^\infty dT\, T^{\varepsilon - 2} \sin T \right\}$$

$$= -\Omega \sum_{\ell=1}^{N_c^2-1} \frac{(g|\nu_\ell|B)^2}{16\pi^2} \left[ \frac{11}{6\varepsilon} - \frac{11}{6} \ln \frac{gB}{\mu^2} + \text{const} \right]$$

(2.76)

(The constant carries also an imaginary part.)

Using $\sum_{\ell=1}^{N_c^2-1} |\nu_\ell|^2 = N_c$, the total one-loop action yields

$$\begin{aligned}\Gamma[A] &= S_{YM}[A] + \hat{\Gamma}[A] \\ &= \int d^4x \left[ \frac{1}{2} B^2 - \left( \frac{11}{6\varepsilon} \frac{g^2 N_c}{16\pi^2} + \text{const} \right) B^2 \right. \\ &\qquad \left. + \frac{11}{6} N_c \frac{(gB)^2}{16\pi^2} \ln \frac{gB}{\mu^2} \right], \qquad (2.77)\end{aligned}$$

which is expressed in terms of unrenormalized fields and coupling (i.e. fields & coupling at a microscopic scale). The physical parameters are fixed in terms of a renormalization condition for the renormalized quantities. Here, we use the Coleman-Weinberg renormalization condition

$$\left. \frac{\partial \mathcal{L}}{\partial \left( \frac{1}{2} g_R^2 B_R^2 \right)} \right|_{g_R B_R = \mu^2} = 1/g_R^2 \,, \quad \Gamma = \int d^4x \, \mathcal{L} \qquad (2.78)$$

which fixes the residue of the gluon propagator at $\mu$ to be $= 1$.

From the gauge invariance of the background-field effective action, we can derive an important observation. Upon the transition from unrenormalized to renormalized quantities, gauge covariant objects have to stay gauge covariant, such as

$$\underbrace{\underbrace{D_\mu^{ab} = \partial_\mu \delta^{ab}}_{\text{RG invariant}} + g f^{acb} A_\mu^c}_{\text{gauge covariant}} \overset{!}{\equiv} \partial_\mu \delta^{ab} + g_R f^{acb} A_{R\mu}^c . \qquad (2.79)$$

Hence the product $g \cdot A_\mu = g_R A_{R\mu}$ must be RG invariant (in the background-field gauge!).

The RG rescalings of $g$ and $B$ in (2.77) thus must be of the form

$$B_R^2 = B^2 Z_F^{-1} \quad , g_R^2 = g^2 Z_F \qquad (2.80)$$

with a "wave function renormalization" $Z_F$.

From (2.77), we observe that (2.78) is satisfied, if

$$Z_F^{-1} = 1 - 2\left(\frac{11}{6\varepsilon}\frac{g^2 N_c}{16\pi^2}\right) + \text{const}! \qquad (2.81)$$

The action then reads

$$\Gamma[A] = \int d^4x \left[ \frac{1}{2} B_R^2 + \frac{1}{4} b_0 \, \frac{1}{2} (g_R B_R)^2 \ln \frac{(g_R B_R)^2}{e \mu^4} \right] \tag{2.82}$$

where $$b_0 = \frac{11}{3} N_c \cdot \frac{1}{8\pi^2} .$$

Now, the RG scale $\mu$ is arbitrary. Changing $\mu$ together with an adjustment of $g_R$ and $B_R$ should leave the physics invariant:

$$\Gamma[A] = \int d^4x \left[ \frac{1}{2} B_R^2 \left( 1 + b_0 g_R^2 \ln \frac{\mu'}{\mu} \right) + \frac{1}{4} b_0 \frac{1}{2} (g_R B_R)^2 \ln \frac{(g_R B_R)^2}{e \mu'^4} \right]$$

$$\equiv \int d^4x \left[ \frac{1}{2} B_R'^2 + \frac{1}{4} b_0 \frac{1}{2} (g_R' B_R')^2 \ln \frac{(g_R' B_R')^2}{e \mu'^4} \right] \tag{2.83}$$

where we introduced

$$B_R'^2(\mu') = B_R^2(\mu) \left( 1 + b_0 \, g_R^2(\mu) \ln \frac{\mu'}{\mu} \right)$$

$$g_R^{2\prime}(\mu') = \frac{g_R^2(\mu)}{1 + b_0 \, g_R^2(\mu) \ln \frac{\mu'}{\mu}} . \tag{2.84}$$

(2.83) is indeed formally identical to (2.82). However, since $\mu$ and $\mu'$ are considered to be physically different scales, Eq. (2.84) tells us how the coupling changes upon a change of scales $\Rightarrow$ "running coupling".

For fixed $g_R^2(\mu)$ for a given $\mu$, $g_R^2(\mu') \to 0$ for $\mu'/\mu \to \infty$ : the coupling becomes asymptotically free in the UV (high-momentum region).

This is characterized by the negative sign of the $\beta$ function:

$$\beta_{g^2} := \mu \partial_\mu g_R^2(\mu) = -b_0 g_R^4 < 0$$

$$(1\text{-loop order}) \qquad (2.85)$$

The effective action can be brought into an RG invariant form ( from now on, we drop the subscript "R" ) :

$$\Gamma[A] = \frac{1}{4} \int d^4_x \, b_0 \, g^2 \mathcal{F} \ln \frac{2 g^2 |\mathcal{F}|}{e \kappa^2}$$

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where $\mathcal{F} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \overset{E=0}{=} \frac{1}{2} B^2$ (2-86)

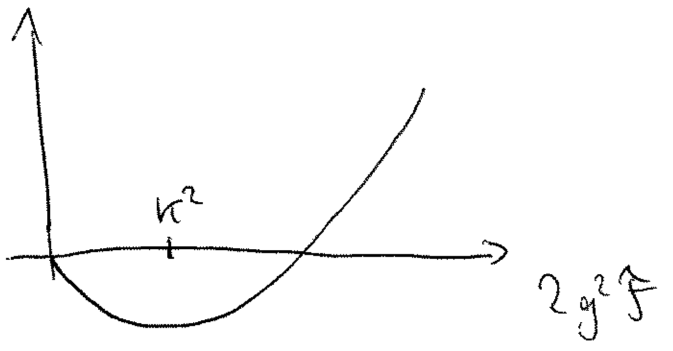
$\left( = \frac{1}{2} (B^2 - E^2) \right)$

and $\kappa^2 = \mu^4 \, e^{-\frac{4}{b_0 g^2}}$ .

Here $\kappa^2$ is RG invariant : $\mu \partial_\mu \kappa^2 = 0$ .

The total result is rather surprising: We started off with YM theory which is scale-free. Upon quantization, the coupling turns into a scale-dependent object. Choosing a certain value of $g^2$ really means fixing $g^2$ at a certain scale. So $g^2$ is traded for a dimensionful scale $\kappa^2$ : "dimensional transmutation" .

Plot of $\Gamma[A]$:




We observe a minimum of $\Gamma$ at

$$2g^2\mathcal{F}_{min} = \kappa^2 , \qquad (2.87)$$

hence, the (perturbative 1-loop) vacuum prefers a non-vanishing gluon field strength, a "gluon condensate".

If we had included $N_f$ massless quarks, we would have obtained a similar result with

$$b_0 = \frac{1}{\cancel{8\pi}2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right) . \qquad (2.88)$$

There are, however, a number of reasons why this vacuum state is not satisfactory:

1.: This "Savvidy vacuum" is unstable
"false vacuum"

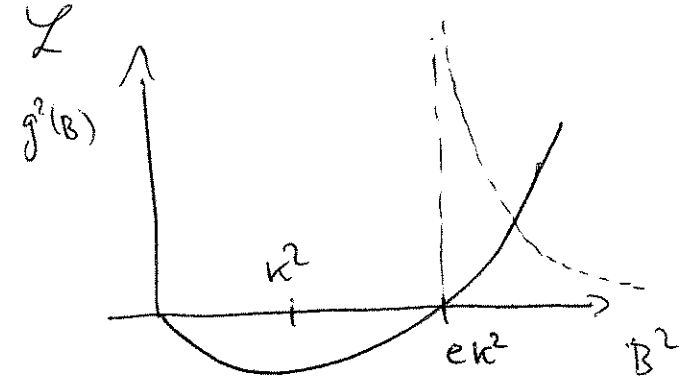
2.: cov. constant fields distinguish a special space direction $\sim \vec{B}$, i.e. Lorentz invariance is broken (contrary to experiment)

These two problems may be circumvented by a $\vec{B}$-field which is constant within domains of length $L \lesssim \sqrt{gB}^{-1}$, such that the unstable mode's momenta are cut off.
If the domains are randomly oriented (cf. Weiss domains in a ferromagnet), Lorentz-invariance will be restored on scales $> L$.
Together with the conservation of magnetic flux this leads to the Kopenhagen vacuum or "spaghetti vacuum" which has been difficult

to handle. But there are more points of critique:

3. The 1-loop calculation is only reliable for $g^2/_{4\pi} \ll 1$. Let, for instance, $\frac{g_0^2}{4\pi} \ll 1$ for $g_0^2 = g^2(\mu_0^2 = B_0)$ at a reference scale $\mu_0^2 = B_0 >> \kappa$

$$\Rightarrow \quad g^2(B) = \frac{g_0^2}{1 + \frac{1}{2} b_0 g_0^2 \ln \frac{B}{\mu_0^2}} = \frac{g_0^2}{1/_2\, b_0 g_0^2 \left( \ln \frac{B}{\kappa} - \frac{1}{2} \right)} \tag{2.89}$$




↑ Landau pole

$\Rightarrow \Gamma^{1\text{-loop}}$ is no longer valid at the minimum, since the coupling diverges already at $e\kappa^2$.

4. The assumption that the vacuum is dominated by slowly varying fields is unjustified.

5. Chiral condensate

$$\langle \bar\Psi \Psi \rangle = \mathcal{N} \int \underset{\substack{\uparrow \\ \text{all fields}}}{\mathcal{D}\Phi} \; \bar\Psi \Psi \, e^{-\int \ldots - m\bar\Psi\Psi \ldots} = -\frac{\partial}{\partial m} \mathcal{L}_{(gB,m)} \xrightarrow{m \to 0} 0 \qquad (2.90)$$

$\Rightarrow$ no chiral symmetry breaking to one-loop order.