2.3 Background gauge & perturbation theory

In practice, it is difficult to make use out of the defining generating functional (2.35). In order to get acquainted with gauge-fixed quantization, let us first confine to perturbation theory where the problems related to the Gribov ambiguity are absent; we will come back to these in the course of this Lecture.

Let us first analyze the lowest-order (nontrivial lowest-order) perturbation theory, starting from Eq. (8) on p. 18, i.e., ignoring the complications from gauge fixing for a second,

$$e^{-\Gamma[\Phi]} = \int_{\Lambda} \mathcal{D}\varphi \, e^{-S[\Phi+\varphi] + \int \frac{\delta \Gamma[\Phi]}{\delta \Phi} \varphi} \qquad (2.45)$$

Perturbation theory corresponds to a steepest-descent / saddle-point approximation of the integral, for which we need

$$S[\phi+\varphi] - \int \frac{\delta\Gamma[\phi]}{\delta\phi}\varphi$$

$$= S[\phi] + \int \left(\frac{\delta S[\phi]}{\delta\varphi} - \frac{\delta\Gamma[\phi]}{\delta\phi}\right)\varphi$$

$$+ \frac{1}{2}\int \varphi \underbrace{\frac{\delta^2 S[\phi]}{\delta\varphi\,\delta\varphi}}_{=:\, S^{(2)}_{[\phi]}} \varphi + \mathcal{O}(\varphi^3) \qquad (2.46)$$

At the saddle point $\phi = \phi_{sp}$, the linear term vanishes. Truncating the series at quadratic order leaves us with a Gaussian integral:

$$e^{-\Gamma[\phi]} \simeq \int_\Lambda \mathcal{D}\varphi \; e^{-S[\phi] - \frac{1}{2}\int \varphi\, S^{(2)}_{[\phi]}\,\varphi} + \dots$$

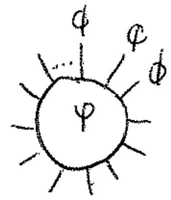
$$= e^{-S[\phi]} \cdot \mathcal{N}\, \det_\Lambda^{-1/2} S^{(2)}_{[\phi]} + \dots \qquad (2.47)$$

The normalization of the correlator $\langle 1 \rangle = 1$ on p. 16 implies $\Gamma[0] = 0$, $\Rightarrow \mathcal{N} = \left(\det_\Lambda^{-1/2} S^{(2)}_{[0]}\right)^{-1}$ such that

$$\Gamma[\phi] = S[\phi] + \frac{1}{2}\ln\det \frac{S^{(2)}_{[\phi]}}{S^{(2)}_{[0]}} + \dots \qquad (2.48)$$

As can be seen upon expansion of the ln det in powers of ϕ, the ln det term corresponds to a sum of all possible 1-loop terms with any number of external ϕ legs:

$$\ln\det \frac{S^{(2)}_{[\phi]}}{S^{(2)}_{[0]}} \sim \sum_{n=1}^{\infty} \frac{1}{n} \qquad (2.49)$$



The ellipsis in (2.48) denotes higher-loop terms, which we will neglect in the following. Since the loop expansion corresponds to a coupling expansion, we expect this expansion to hold at weak coupling.

Now we could try to do a saddle-point approximation of the gauge-fixed generating functional (2.35); but here we encounter a conceptual problem:

On the one hand, we expect that a properly quantized gauge theory results in a gauge-invariant effective action $\Gamma[A] \equiv \Gamma[A^\omega]$.

On the other hand, gauge fixing is necessary for integrating over the fluctuations.

This seeming paradox can be resolved with the aid of the background-field gauge.

In this gauge, we decompose the gauge field A into a background field $\bar{A}$ and a fluctuation field Q,

$$A = \bar{A} + Q . \tag{2.50}$$

From a "quantum-field viewpoint", $\bar{A}$ is just an external parameter; gauge symmetry on the quantum level is carried by Q.

This is expressed by the quantum-field transformation (<u>QFTf</u>):

$$\begin{aligned} \bar{A}' &= \bar{A} \\ \bar{Q}' &= U\,(\bar{A}+Q)\,U^{-1} - \frac{i}{g}\,(\partial_\mu U)\,U^{-1} - \bar{A} \end{aligned} \qquad (2.51)$$

with $(\bar{A}+Q)$ transforming "as usual".

This is the symmetry which we have to gauge fix for being able to do the functional integral.

For this, we choose the gauge-fixing condition:

$$\begin{aligned} \bar{\mathcal{F}}^a[\bar{A},Q] &= D_\mu^{ab}[\bar{A}]\,Q_\mu^b \equiv \bar{D}_\mu^{ab}\,Q_\mu^b \\ \Rightarrow \quad S_{gf}[\bar{A},Q] &= \frac{1}{2}\int (\bar{D}_\mu^{ab}\,Q_\mu^b)^2 . \end{aligned} \qquad (2.52)$$

The <u>important</u> observation now is that S_{gf} – even though not gauge invariant under QFTf – is invariant under an additional symmetry, the background field transformation (BFTf):

$$\begin{aligned} \bar{A}' &= U\,\bar{A}\,U^{+} - \frac{i}{g}\,(\partial_\mu U)\,U^{+} \\ Q' &= U\,Q\,U^{+}, \end{aligned} \qquad (2.53)$$

(with $(\bar{A}+Q)$ transforming as usual again).

The invariance of (2.52) is obvious, since $\bar{D}$ as well as Q transform homogeneously under the BFTf.

The integration measure $\mathcal{D}A \to \mathcal{D}Q$ is also invariant under BFTf.

For the Faddeev-Popov term, we need the infinitesimal QFTf: $\left(e^{-ig\omega^a \tau^a}\right)$

$$Q'^a_\mu = Q^a_\mu + gf^{abc}\,\omega^b\left(Q^c_\mu + \bar{A}^c_\mu\right) - \partial_\mu \omega^a$$

$$= Q^a_\mu - D^{ac}_\mu[\bar{A}+Q]\,\omega^c$$

$$\Rightarrow \quad \frac{\delta \overline{\tilde{f}^a}[\bar{A},Q]}{\delta \omega^b} = - D^{ac}_\mu[\bar{A}]\,D^{cb}_\mu[\bar{A}+Q]$$

$$\Rightarrow \quad \Delta_{FP}[\bar{A},Q] = \det\left(-D^{ac}_\mu[\bar{A}]\,D^{cb}_\mu[\bar{A}+Q]\right) \tag{2.54}$$

Aiming at the quantum correlations, the source term in the generating functional is coupled to Q only: $e^{\int jA} \to e^{\int jQ}$

We end up with the generating functional in background gauge:

$$\bar{Z}[J,\bar{A}] = \int \mathcal{D}Q \, e^{-S_{YM}[\bar{A}+Q] - S_{gf}[\bar{A},Q]} \, \Delta_{FP}[\bar{A},Q] \, e^{\int JQ} \tag{2.55}$$

Note that $\bar{Z}[J,A]$ is manifestly invariant under BFTF if the source J transforms homogeneously. (Even a ghost-field representation of Δ_{FP} is invariant under BFTF, if the ghosts also transform homogeneously.)

Manifest BFTF invariance also holds for the effective action

$$\bar{\Gamma}[\bar{A},Q_{cl}] = \sup_J \left[\int J Q_{cl} - \ln \bar{Z}[J,\bar{A}] \right] . \tag{2.56}$$

But <u>how</u> is $\bar{\Gamma}[\bar{A},Q_{cl}]$ related to the original effective action $\Gamma[A_{cl}]$ which we are most interested in?

For this, let us shift the integration variable $Q \to Q - \bar{A}$:

$$\bar{Z}[J,\bar{A}] = Z[J] \cdot e^{-\int J \cdot \bar{A}} , \tag{2.57}$$

where $Z[J]$ is the standard generating functional with an unusual gauge fixing

$$\mathcal{F}[Q] \equiv \bar{\mathcal{F}}[\bar{A}, Q-\bar{A}] . \tag{2.58}$$

Legendre trafo yields:

$$\bar{\Gamma}[\bar{A}, Q_{cl}] = \sup_{J}\left[\int J(\bar{A}+Q) - \ln Z[J]\right]$$

$$\equiv \Gamma[\bar{A}+Q_{cl}]. \qquad (2.59)$$

Here we rediscover the standard effective action with an unusual argument.

From the BFTF invariance of $\bar{\Gamma}$, we now conclude for $A = \bar{A}$, i.e. $Q_{cl}=0$, that

$$\Gamma[A] = \bar{\Gamma}[\bar{A}=A, Q_{cl}=0] \qquad (2.60)$$

is manifestly invariant under

$$A \to A' = U A U^{-1} - \frac{i}{g}(\partial U) U^{-1}.$$

$\Rightarrow$ $\Gamma[A]$ in the background-field gauge inherits manifest gauge invariance from the Background invariance

$\Rightarrow$ Q_{cl} - Vacuum diagrams are sufficient to analyze the structure of the theory
($\bar{A}$ only on external lines, Q on internal lines)
$\Rightarrow$ reduction of Feynman diagrams