2.2 Quantization of gauge theories

The naive attempt to define the quantum field theory of gluodynamics,

$$Z[J] \overset{?}{=} \int \mathcal{D}A \; e^{-S_{YM}[A] + \int J^a_\mu A^a_\mu}, \qquad (2.10)$$

fails and generically leads to ill-defined quantities plagued by infinities. The reason is that the measure $\mathcal{D}A$ contains a huge redundancy, since many gauge-field configurations A^a_μ are physically equivalent.

In practice, the problems arise from the fact that already the free propagator following from (2.10) is ill-defined:

$$\begin{aligned} S_{YM} &= \frac{1}{4}\int d^D x \; F^a_{\mu\nu} F^a_{\mu\nu} \\ &= \frac{1}{2}\int d^D x \; A^a_\mu \left(-\partial_\lambda\partial_\lambda \delta_{\mu\nu} + \partial_\mu \partial_\nu \right) A^a_\nu + \mathcal{O}(A^3), \end{aligned} \qquad (2.11)$$

free propagator:

$$D_A \overset{?}{=} \left(-\partial^2 \mathbb{1} + \partial\otimes\partial\right)^{-1} \underset{\text{space}}{\overset{\text{momentum}}{\Longrightarrow}} \left(p^2 \mathbb{1} - p\otimes p\right)^{-1}. \qquad (2.12)$$

The RHS of (2.12) does not exist, since $p^2\mathbb{1} - p\otimes p$ has a zero eigenvalue:

with eigenvector $\sim p_\mu$:

$$(p^2 \delta_{\mu\nu} - p_\mu p_\nu)\, p_\nu = p^2 p_\mu - p_\mu p^2 = 0 \tag{2.13}$$

Defining the projector

$$(P_L)_{\mu\nu} = \partial_\mu \frac{1}{\partial^2} \partial_\nu , \tag{2.14}$$

the eigenvector with zero eigenvalue corresponds to the longitudinal component of the gauge field :

$$A^a_{L\,\mu} = P_{L\mu\nu} A^a_\nu$$

$$\Rightarrow \quad (-\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu)\, A^a_{L\nu} = 0 . \tag{2.15}$$

The generating functional can be made well-defined by removing the redundant degrees of freedom.

Ideally, we would like to remove this redundancy by picking one representative gauge field out of each set of gauge-equivalent potentials. The latter set is called a gauge-orbit :

$$[A^{\text{orbit}}_\mu] = \{ A^\omega_\mu \mid A_\mu = A^{\text{ref}}_\mu , \; U(\omega) \in SU(N_c) \} \tag{2.16}$$

where A^{ref}_μ is a reference gauge field which is representative for the orbit, and A^ω_μ denotes the gauge transform

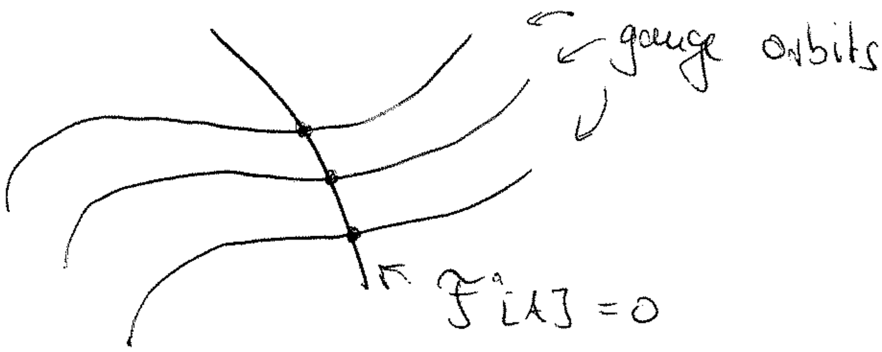
$$A_\mu^{\omega} = U A_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1} , \qquad (2.17)$$

with $\quad U = U(\omega) = e^{-i g \omega^a T^a} \quad , \omega^a = \omega^a(x) .$

For the quantum theory, we would like to have a measure $\mathcal{D}A$ which picks one representative gauge-field configuration out of each orbit. This is intended by choosing a <u>gauge-fixing</u> condition ($\mathcal{F} = \tau^a \mathcal{F}^a$)

$$\mathcal{F}^a[A] = 0 , \qquad (2.18)$$

for instance $\quad \mathcal{F}^a[A] = \partial_\mu A_\mu^a \quad$ (Lorenz gauge).



Ideally, (2.18) should be satisfied by only one A_μ^a of each orbit. Unfortunately, this is actually impossible for standard smooth gauge-fixing conditions, owing to topological obstructions.

Some essence of this is captured by the following simplified example:

Consider the "action"

$$S[r] = -\frac{1}{2}\frac{r^2}{L^2} \tag{2.19}$$

with $r = \sqrt{x_1^2 + x_2^2}$, and a "gauge-invariance" corresponding to rotations about the origin

$$\vec{x}' = U(\omega)\,\vec{x} = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \tag{2.20}$$

and $\omega \in [0, 2\pi) \equiv I_{2\pi}$ "Gauge-invariant" observables $\mathcal{O}(r)$ have an expectation value

$$\langle \mathcal{O}(r) \rangle = \int dx_1 dx_2\, \mathcal{O}(r)\, e^{-S[r]} \tag{2.21}$$

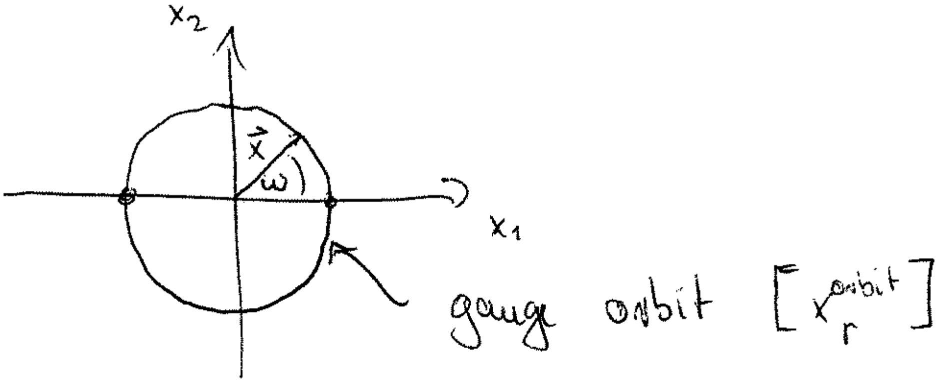
Of course, no problem arises here from the angular redundancy, and we could even decompose the measure into gauge-invariant and gauge-dependent degrees of freedom

$$dx\,dy = r\,dr\,d\omega \tag{2.22}$$

with $\omega = \arctan \frac{x_2}{x_1}$.

Since this is difficult in real gauge theories, let us try to solve this problem by "gauge fixing" fully formulated in terms of x_1, x_2; e.g.:

$$0 = \mathcal{F}[\vec{x}] = x_2(\omega) \qquad (2.23)$$



Owing to the topology of the gauge orbit $\sim S^1$, the gauge-fixing condition is satisfied by two points on the orbit.

Now consider the "Faddeev-Popov" determinant:

$$\Delta_{FP}^{-1}[x] := \int_{I_{2\pi}} d\omega \; \delta[\mathcal{F}[\vec{x}^{\omega}]] \, , \qquad (2.24)$$

which is gauge invariant:

$$\Delta_{FP}^{-1}[x^{\bar{\omega}}] = \int_{I_{2\pi}} d\omega \; \delta[\mathcal{F}[\vec{x}^{\bar{\omega}+\omega}]]$$

$$= \int_{I_{2\pi}} d(\omega+\bar{\omega}) \; \delta[\mathcal{F}[\vec{x}^{\bar{\omega}+\omega}]] \qquad (2.25)$$

$$\overset{\omega \to \omega - \bar{\omega}}{=} \int_{I_{2\pi}} d\omega \; \delta[\mathcal{F}[\vec{x}^{\omega}]] = \Delta_{FP}^{-1}[\vec{x}]$$

The Faddeev-Popov trick consists in inserting the following identity into the integral:

$$\langle \mathcal{O}(r)\rangle = \int dx_1 dx_2\, \mathcal{O}(r)\, e^{-S[r]} = \int dx_1 dx_2\, \mathcal{O}(r)\, \Delta_{FP}[x] \int_{I_{2\pi}} d\omega\, \delta[\mathcal{F}[x^\omega]] \cdot e^{-S[r]}$$

$$= \int_{I_{2\pi}} d\omega \int \underset{\updownarrow\, \text{g.i.}}{dx_1 dx_2\, \mathcal{O}(r)}\, \underset{\updownarrow\, \text{g.i.}}{\Delta_{FP}[x]}\, \delta[\mathcal{F}[x^\omega]]\, \underset{\updownarrow\, \text{g.i.}}{e^{-S[r]}}$$

$$= \int_{I_{2\pi}} d\omega \int dx_1^\omega dx_2^\omega\, \mathcal{O}(r)\, \Delta_{FP}[x^\omega]\, \delta[\mathcal{F}[x^\omega]]\, e^{-S[r^\omega]}$$

$$= \left(\int_{I_{2\pi}} d\omega\right) \int dx_1 dx_2\, \mathcal{O}(r)\, \Delta_{FP}[x]\, \delta[\mathcal{F}[x]]\, e^{-S[r]} \tag{2.26}$$

In (2.26), the integral over the gauge group $\sim S^1$ has been factorized and can be absorbed into the normalization; the remaining part is a gauge-fixed integral.

Let us now compute the Faddeev-Popov determinant for our particular gauge fixing (2.23):

$$\Delta_{FP}^{-1}[x] = \int_{I_{2\pi}} d\omega\, \delta[\mathcal{F}[\vec{x}^\omega]]$$

$$= \int_{I_{2\pi}} d\omega\, \frac{1}{\left|\det \frac{\delta \mathcal{F}[x^\omega]}{\delta\omega}\right|} \sum_i \delta(\omega - \omega_i) \tag{2.27}$$

redundant in the present ordinary example

Here, we have used a functional notation in order to make contact with the field-theoretic setting; but, of course, in the present problem, all derivatives are ordinary derivatives. The ω_i's are solutions to

$$\mathcal{F}[x^{\omega}]\big|_{\omega=\omega_i} = 0 \qquad (2.28)$$

We get

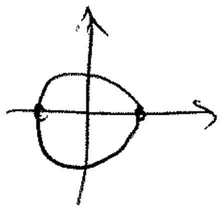
$$\frac{\delta \mathcal{F}[x^{\omega}]}{\delta \omega} = \frac{\delta}{\delta \omega}\left(x_1 \sin\omega + x_2 \cos\omega \right)$$

$$= x_1 \cos\omega - x_2 \sin\omega$$

$$= \cos\omega \left(x_1 - x_2 \tan\omega \right)$$

$$= \cos\omega \left(x_1 - \frac{x_2^2}{x_1} \right) \qquad (2.29)$$

$$\Rightarrow \quad \omega_1 = 0 \quad , \quad \omega_2 = \pi \quad , \quad x_2 = 0$$



$$\Rightarrow \Delta_{FP}^{-1}[x] = \int_{I_{2\pi}} d\omega \, \frac{1}{|x_1 \cos\omega|} \left(\delta(\omega) + \delta(\omega-\pi) \right)$$

$$= \frac{1}{|x_1|} + \frac{1}{|-x_1|} = 2 \, \frac{1}{|x_1|}$$

$$\Rightarrow \quad \Delta_{FP}[x] = \frac{1}{2} \, |x_1| \qquad (2.30)$$

In the full system, we get contributions from both gauge copies at $\omega = 0$ and $\omega = \pi$.

Imagine, we would only be interested in "perturbation theory" near $\theta \sim 0$; then Δ_{FP} would reduce to

$$\Delta_{FP}^{-1}[x]\Big|_{\text{pert}} \simeq \int_{-\varepsilon}^{\varepsilon} d\omega \ \delta(x_2(\omega)) = \frac{1}{\cdot x_1} \qquad (2.31)$$

Since $x_1 > 0$ near $\theta \simeq 0$, we can drop the absolute-value prescription. In this case, we can represent $\Delta_{FP}[x]$ by a Grassmann integral

$$\Delta_{FP}[x]\Big|_{\text{pert}} = x_1 \mathrel{\hat{=}} \det \frac{\delta \mathcal{F}}{\delta \omega}\Big|_{\omega=0} = \int d\bar{c}\, dc\ e^{-\bar{c} \frac{\delta \mathcal{F}}{\delta \omega}\big|_{\omega=0} c} \qquad (2.32)$$

such that Δ_{FP} can be written in terms of a QFT contribution with a <u>local</u> action

$$S_{gh} = \bar{c} \frac{\delta \mathcal{F}}{\delta \omega} c \qquad \text{(ghost action)}. \qquad (2.33)$$

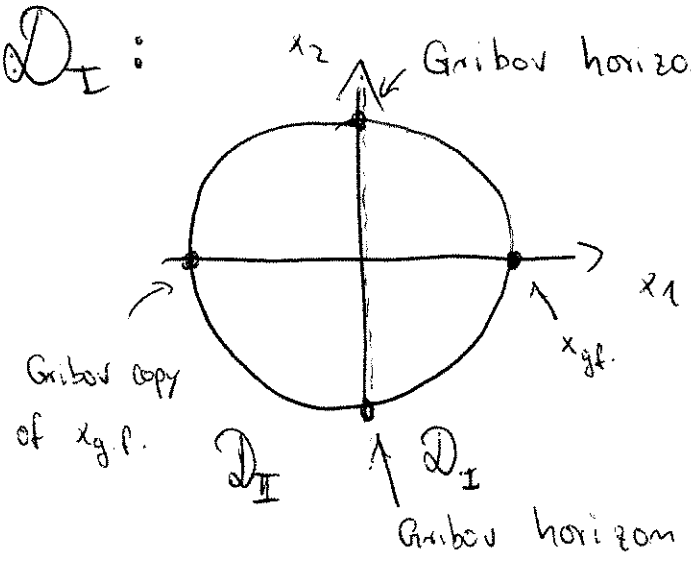
Note that this construction does not hold beyond perturbation theory:

$$\Delta_{FP}^{-1}[x] \overset{?}{=} \int_{I_{2\pi}} d\omega \frac{1}{\det \frac{\delta F}{\delta \omega}\Big|_{\omega_i}} \sum_i \delta(\omega-\omega_i)$$

$$= \frac{1}{x_1} + \frac{1}{-x_1} = 0 \quad ↯$$

This would correspond to an insertion of ∞ into the integral instead of an identity in (7.26).

In order to maintain the local ghost-action form, but still arrive at a nonperturbatively valid definition of the "theory", we can confine the integral to the "first Gribov region" $\mathcal{D}_I$:



The first Gribov horizon includes the perturbative origin $x_1 > 0$, $x_2 = 0$ and is bounded by the Gribov horizon which is defined by those configurations for which $\det \frac{\delta F}{\delta \omega}$ vanishes:

$$\langle \Theta(x) \rangle := \frac{\left(\int dx_1 dx_2 \, \Theta(x) \, \Delta_{FP}[x] \, \delta[F[\omega]] \, e^{-S[x]} \right)}{\int_{\Omega_I} dx_1 dx_2 \, \Theta(x) \, \det \frac{\delta F}{\delta \omega} \, \delta[F[x]] \, e^{-S[x]}} \tag{2.34}$$

This is a non-perturbative definition which - except for a normalization - corresponds to the full original integral.

The same type of reasoning can be applied to the Faddeev - Popov quantization of gauge theories (most formulas given above in functional notation can literally be translated to the gauge-theory case). We obtain the generating functional

$$Z[J] = \int_{\Omega_I} \mathcal{D}A \; \frac{1}{M[A]} \det \frac{\delta F}{\delta \omega} \; \delta[F[A]] \, e^{-S_{YM} + \int J_\mu^a A_\mu^a} \tag{2.35}$$

The factor $\frac{1}{M[A]}$ takes care of the fact that there might by $M[A]$ gauge-equivalent configurations to a reference field A even within the first Gribov region Ω_I.

Eq. (2.35) represents a generating functional for gauge theories with a well-defined gauge-fixing procedure.

For a quantitative treatment of (2.35), it is useful to write as many terms as possible in the form of local contributions to the action.

E.g.

$$\delta[\mathcal{F}[A]] \sim e^{-\frac{1}{2\alpha}\int(\mathcal{F}[A])^2}\Big|_{\alpha\to 0} \tag{2.36}$$

(representation of the δ functional)

$$\Rightarrow e^{-S_{gf}}, \quad S_{gf} = \frac{1}{2\alpha}\int(\mathcal{F}[A])^2$$

and the determinant can be exponentiated by means of Grassmann-valued (anti-commuting) fields:

$$\det \frac{\delta\mathcal{F}[A]}{\delta\omega} = \int \mathcal{D}\bar{c}\,\mathcal{D}c \; e^{-\int \bar{c}\frac{\delta\mathcal{F}}{\delta\omega}c} . \tag{2.37}$$

These "ghost" fields live in the adjoint representation of the algebra: $\quad c = c^a\tau^a \quad , \bar{c}^a = \bar{c}^a\tau^a$

$$\Rightarrow \int \bar{c}\,\frac{\delta\mathcal{F}}{\delta\omega}\,c \equiv \int \bar{c}^a\,\frac{\delta\mathcal{F}^a}{\delta\omega^b}\,c^b$$

inheriting the transformation $\bar{c}' = U\bar{c}\,U^+$ (invariance of the measure).

Unfortunately, no local description for $\mu[A]$ is known.

Example: Lorenz Landau gauge:

$$\mathcal{F}^a[A] = \partial_\mu A^a_\mu . \tag{2.38}$$

$$\Rightarrow \delta[\mathcal{F}[A]] \rightarrow e^{-S_{gf}} \Big|_{\alpha \to 0}$$

$$S_{gf} = \frac{1}{2\alpha} \int (\partial_\mu A^a_\mu)^2 = -\frac{1}{2\alpha} \int A^a_\mu \partial_\mu \partial_\nu A^a_\nu$$

$$= \frac{1}{2\alpha} \int A^a_\mu \, P_{L\,\mu\nu}(-\partial^2) \, A^a_\nu \tag{2.39}$$

$$= \frac{1}{2\alpha} \int A^a_{L\mu} \, (-\partial^2) \, A^a_{L\mu}$$

The gauge-fixing action involves the longitudinal projector P_L. In the Landau-gauge limit $\alpha \to 0$, all contributions from the A_L's are suppressed in the functional integral and thus decouple completely from physical amplitudes.

For the Faddeev-Popov operator it is useful to study <u>infinitesimal</u> gauge transformations first:

$$\mathcal{U}(\omega) = e^{-ig\omega^a \tau^a} \simeq 1 - ig\omega^a \tau^a + \mathcal{O}(\omega^2) \tag{2.40}$$

$$\Rightarrow \quad A_\mu^\omega = U A_\mu U^{-1} - \frac{i}{g}(\partial U) U^{-1}$$

$$= (1 - ig\omega) A_\mu (1 + ig\omega) - (\partial_\mu \omega)(1 + ig\omega) + \mathcal{O}(\omega^2)$$

$$= A_\mu - ig \underbrace{[\omega, A_\mu]}_{if^{abc}\omega^a A_\mu^b \tau^c} - \partial_\mu \omega + \mathcal{O}(\omega^2)$$

$$= A_\mu^a \tau^a + g f^{abc} \omega^a A_\mu^b \tau^c - \partial_\mu \omega^a \tau^a + \mathcal{O}(\omega^2)$$

$$\Rightarrow \quad A_\mu^\omega = A_\mu + \delta A_\mu + \mathcal{O}(\omega^2)$$

$$\text{with} \quad \delta A_\mu^a = -\partial_\mu \omega^a + g f^{abc} \omega^b A_\mu^c \tag{2.41}$$

In terms of the adjoint covariant derivative$^{\text{Eq}(1.25)}$, this reads :

$$\delta A_\mu^a = -D_\mu^{ab}[A]\, \omega^b \tag{2.42}$$

From here, we can immediately compute the Faddeev-Popov operator

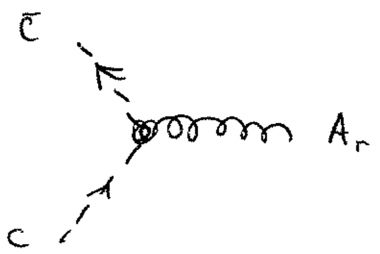
$$\frac{\delta \mathcal{F}^a[A]}{\delta \omega^b} = \frac{\delta\, \partial_\mu A_\mu^a}{\delta \omega^b} = \partial_\mu \frac{\delta A_\mu^a}{\delta \omega^b} = -\partial_\mu D_\mu^{ab}[A] \tag{2.43}$$

In the Landau gauge, the ghost action hence reads

$$\underline{\underline{S_{gh}}} := \int \bar{c}\, \frac{\delta \mathcal{F}}{\delta \omega}\, c = -\int \bar{c}^a \partial_\mu D_\mu^{ab} c^b = \underline{\underline{\int \partial_\mu \bar{c}^a D_\mu^{ab} c^b}} \tag{2.44}$$

$$= \int (\partial_\mu \bar{c}^a)(\partial_\mu c^a) + g \int (\partial_\mu \bar{c}^a)\, f^{acb} A_\mu^c c^b .$$

The last term obviously corresponds to a ghost-gluon interaction (in abelian gauge theories where $f^{abc} \to 0$ this coupling is absent):



Expanding around $A = 0$ in perturbation theory, the Faddeev-Popov operator reduces to $-\partial^2$ which is a positive operator, e.g., on L_2.
(it has only trivial zero modes with constant functions as eigenfunctions which play no role.)
Hence, the Gribov ambiguity is irrelevant in perturbation theory.