Excerpt from Lecture Notes: Holger Gies, "Introduction to the Functional RG and Applications to Gauge Theories", Ed. A. Schwenk et al., Springer LNP XXX, (2007)

1.1 Basics of QFT

In quantum field theory (QFT), all physical information is stored in correlation functions. For instance, consider a collider experiment with two incident beams and $(n-2)$ scattering products. All information about this process can be obtained from the *n-point function*, a correlator of n quantum fields. In QFT, we obtain this correlator by definition from the product of n field operators at different spacetime points $\varphi(x_n)$ averaged over all possible field configurations (quantum fluctuations).

In Euclidean QFT, the field configurations are weighted with an exponential of the action $S[\varphi]$,

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle := \mathcal{N} \int \mathcal{D}\varphi \, \varphi(x_1) \dots \varphi(x_n) \, e^{-S[\varphi]}, \tag{1}$$

where we fix the normalization $\mathcal{N}$ by demanding that $\langle 1 \rangle = 1$. We assume that Minkowski-valued correlators can be defined from the Euclidean ones by analytic continuation. We also assume that a proper regularized definition of the measure can be given (for instance, using a spacetime lattice discretization), which we formally write as $\int \mathcal{D}\varphi \to \int_\Lambda \mathcal{D}\varphi$; here, Λ denotes an ultraviolet (UV) cutoff. This regularized measure should also preserve the symmetries of the theory: for a symmetry transformation U which acts on the fields, $\varphi \to \varphi^U$, and leaves the action invariant, $S[\varphi] \to S[\varphi^U] \equiv S[\varphi]$, the invariance of the measure implies

$$\int_\Lambda \mathcal{D}\varphi \to \int_\Lambda \mathcal{D}\varphi^U \equiv \int_\Lambda \mathcal{D}\varphi. \tag{2}$$

For simplicity, let φ denote a real scalar field; the following discussion also holds for other fields such as fermions with minor modifications. All n-point correlators are summarized by the generating functional $Z[J]$,

$$Z[J] \equiv e^{W[J]} = \int \mathcal{D}\varphi \, e^{-S[\varphi]+\int J\varphi}, \tag{3}$$

with source term $\int J\varphi = \int d^D x \, J(x)\varphi(x)$. All n-point functions are obtained by functional differentiation:

$$\langle \varphi(x_1) \dots \varphi(x_n) \rangle = \frac{1}{Z[0]} \left(\frac{\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \right)_{J=0}. \tag{4}$$

Once the generating functional is computed, the theory is solved.

In Eq. (3), we have also introduced the generating functional of *connected correlators*[1], $W[J] = \ln Z[J]$, which, loosely speaking, is a more efficient way to store the physical information. An even more efficient information storage is obtained by a Legendre transform of $W[J]$: the *effective action* Γ:

$$\Gamma[\phi] = \sup_J \left(\int J\phi - W[J] \right). \tag{5}$$

For any given ϕ, a special $J \equiv J_{\text{sup}} = J[\phi]$ is singled out for which $\int J\phi - W[J]$ approaches its supremum. Note that this definition of Γ automatically guarantees that Γ is convex. At $J = J_{\text{sup}}$, we get

$$0 \stackrel{!}{=} \frac{\delta}{\delta J(x)} \left(\int J\phi - W[J] \right)$$
$$\Rightarrow \quad \phi = \frac{\delta W[J]}{\delta J} = \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J} = \langle \varphi \rangle_J. \tag{6}$$

[1] In this short introduction, we use but make no attempt at fully explaining the standard QFT nomenclature; for the latter, we refer the reader to any standard QFT textbook, such as [1, 2].

This implies that ϕ corresponds to the expectation value of φ in the presence of the source J. The meaning of Γ becomes clear by studying its derivative at $J = J_{\text{sup}}$

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = -\int_y \frac{\delta W[J]}{\delta J(y)}\frac{\delta J(y)}{\delta\phi(x)} + \int_y \frac{\delta J(y)}{\delta\phi(x)}\phi(y) + J(x) \stackrel{(6)}{=} J(x). \quad (7)$$

This is the *quantum equation of motion* by which the effective action $\Gamma[\phi]$ governs the dynamics of the field expectation value, taking the effects of all quantum fluctuations into account.

From the definition of the generating functional, we can straightforwardly derive an equation for the effective action:

$$e^{-\Gamma[\phi]} = \int_\Lambda \mathcal{D}\varphi \exp\left(-S[\phi+\varphi] + \int \frac{\delta\Gamma[\phi]}{\delta\phi}\varphi\right). \quad (8)$$

Here, we have performed a shift of the integration variable, $\varphi \to \varphi + \phi$. We observe that the effective action is determined by a nonlinear first-order functional differential equation, the structure of which is itself a result of a functional integral. An exact determination of $\Gamma[\phi]$ and thus an exact solution has so far been found only for rare, special cases.

As a first example of a functional technique, a solution of Eq. (8) can be attempted by a *vertex expansion* of $\Gamma[\phi]$,

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!}\int d^Dx_1 \dots d^Dx_n\, \Gamma^{(n)}(x_1, \dots, x_n)\, \phi(x_1)\dots\phi(x_n), \quad (9)$$

where the expansion coefficients $\Gamma^{(n)}$ correspond to the *one-particle irreducible (1PI) proper vertices*. Inserting Eq. (9) into Eq. (8) and comparing the coefficients of the field monomials results in an infinite tower of coupled integro-differential equations for the $\Gamma^{(n)}$:

the Dyson-Schwinger equations. This functional method of constructing approximate solutions to the theory via truncated Dyson-Schwinger equations, i.e., via a finite truncation of the series Eq. (9) has its own merits and advantages; their application to gauge theories is well developed; see, e.g., [3, 4, 5, 6]. Here, we proceed by amending the RG idea to functional techniques in QFT.

References

[1] M.E. Peskin and D.V. Schröder, "An Introduction To Quantum Field Theory," Addison-Wesley (1995).

[2] S. Pokorski, "Gauge field theories," Cambridge (1987).

[3] R. Alkofer and L. von Smekal, Phys. Rept. **353**, 281 (2001).

[4] C. D. Roberts and S. M. Schmidt, Prog. Part. Nucl. Phys. **45**, S1 (2000).

[5] P. Maris and C. D. Roberts, Int. J. Mod. Phys. E **12**, 297 (2003).

[6] C. S. Fischer, J. Phys. G **32**, R253 (2006).