(More on) Phase transitions in Tensor Models

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Jena 2015

Introduction

Tensor models

Phase transition in the quartic model

Phase transitions in field theory

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Phase transition $\Leftrightarrow$ symmetry breaking

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Phase transition $\Leftrightarrow$ symmetry breaking

$$Z = \int [d\bar{\phi} d\phi] \; e^{-\left[\int \partial\bar{\phi}\partial\phi + m^2 \int \bar{\phi}\phi + \frac{\lambda}{2} \int (\bar{\phi}\phi)^2\right]}$$

invariant under complex rotations
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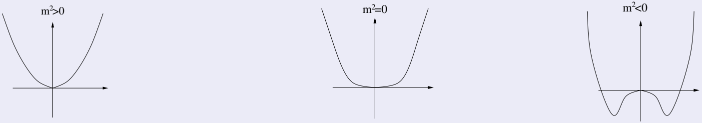

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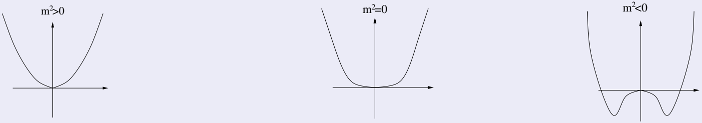

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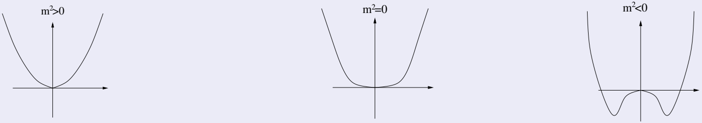

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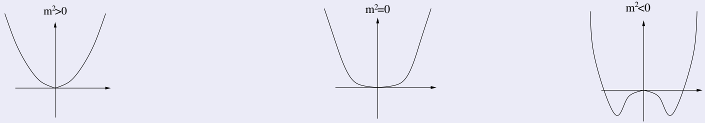

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Phase transition: zero eigenvalue of the "mass matrix"

$$\frac{\delta^2 S^{\text{notkinetic}}}{\delta\bar{\phi}\delta\phi}\Big|_{\bar{\phi}=\phi=0} = m^2 = 0$$

Continuum limit of Dynamical Triangulations

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“continuum limit” $\Leftrightarrow$ criticality

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DTs sum over random spaces:

$$f(g) = \sum_{\text{connected planar rooted quadrangulations}} (g^2)^{\# quadrangles} = \frac{(1-12g^2)^{\frac{3}{2}} - 1 + 18g^2}{54g^4}$$

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Continuum limit: send $g \nearrow g_{critical}$, $\sigma \searrow 0$ keeping the physical volume fixed.

DT continuum limit v.s. Field Theory phase transition

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Dynamical Triangulations are generated by matrix and tensor models:

$$\int [d\bar{T} dT] \ e^{-\bar{T}\cdot T - V_{\rm int}(\bar{T},T)}$$

with $V_{\rm int}(\bar{T}, T)$ invariant under conjugation by the unitary group

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Partition functions for field theories with no kinetic term

Tensor models:

- the continuum limit of the DT = phase transition
- breaking of the unitary invariance

Tensor invariants as Edge Colored Graphs

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Building blocks: tensors with no symmetry transforming as

$$T'_{b^1\dots b^D} = \sum U^{(1)}_{b^1 a^1} \dots U^{(D)}_{b^D a^D} \, T_{a^1\dots a^D} \ , \quad \bar{T}'_{p^1\dots p^D} = \sum \bar{U}^{(1)}_{p^1 q^1} \dots \bar{U}^{(D)}_{p^D q^D} \, \bar{T}_{q^1\dots q^D}$$

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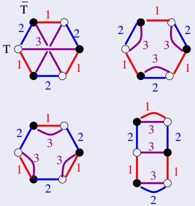
Invariants: colored graphs

$$\mathrm{Tr}_{\mathcal{B}}(T,\bar{T}) = \sum \prod_{v} T_{a^1_v \dots a^D_v} \prod_{\bar{v}} \bar{T}_{q^1_{\bar{v}} \dots q^D_{\bar{v}}} \prod_{c=1}^{D} \prod_{l^c=(w,\bar{w})} \delta_{a^c_w q^c_{\bar{w}}}$$



- White (black) vertices for T ($\bar{T}$).
- Edges for $\delta_{a^c q^c}$ colored by c, the position of the index.

Invariant Actions for Tensor Models

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$$S(T,\bar{T}) = \sum T_{a^1\dots a^D}\,\bar{T}_{q^1\dots q^D}\prod_{c=1}^{D}\delta_{a^c q^c} - \sum_{\mathcal{B}} t_{\mathcal{B}}\mathrm{Tr}_{\mathcal{B}}(\bar{T},T)$$

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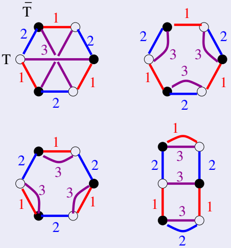
Feynman graphs: "effective vertices" $\mathcal{B}$.



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$$\mathrm{Tr}_{\mathcal{B}_1}(\bar{T},T)\,\mathrm{Tr}_{\mathcal{B}_2}(\bar{T},T)\dots$$

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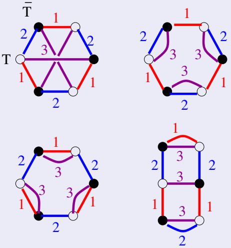

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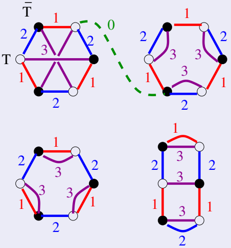

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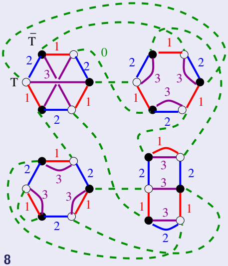

Graphs $\mathcal{G}$ with $D+1$ colors.

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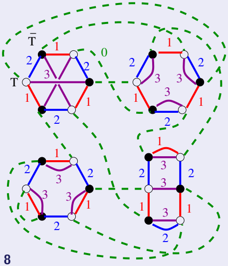

Graphs $\mathcal{G}$ with $D+1$ colors.

Represent triangulated D dimensional spaces.

Colored Graphs as gluings of colored simplices

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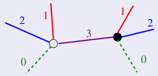
White and black $D + 1$ valent vertices connected by edges with colors $0, 1 \dots D$.



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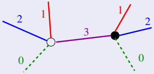
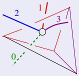
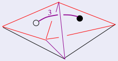


Vertex $\leftrightarrow$ colored D simplex .



Edges $\leftrightarrow$ gluings along $D-1$ simplices respecting all the colorings



The quartic tensor model

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The simplest quartic invariants correspond to "melonic" graphs with four vertices $\mathcal{B}^{(4),c}$

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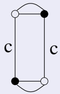

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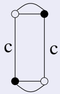

The simplest interacting theory: coupling constants $t_{\mathcal{B}} = \frac{g^2}{2}$ for some of the "melonic interactions" $\mathcal{B}^{(4),c}$, $c \in \mathcal{Q} = \{1,\dots Q\}$

Amplitudes and Dynamical Triangulations

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Expand in g (Feynman graphs):

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$$A^G(N) = g^{2\#Vertices} N^{D-\dots} = e^{\kappa_{D-2}(g,N)Q_{D-2} - \kappa_D(g,N)Q_D}$$

Discretized Einstein Hilbert action on the (dual) triangulation with Q_D equilateral D-simplices and Q_{D-2} $(D-2)$-simplices.

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$\ln Z = \sum_{q\geq 0} N^{(D-q)}(g_c - g)^{\nu_q}$, DT continuum limit: $g \nearrow g_c$

Introduction

Tensor models

Phase transition in the quartic model

What follows

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Q melonic interactions in D dimensions (tensors with D indices), g_c critical constant (continuum limit of DT)

- $Q \geq 2$
  - $g < g_c$ color and unitary symmetric vacuum
  - $g = g_c$ one mass eigenvalue becomes 0
  - $g > g_c$ vacuum state in the broken phase not yet found
- $Q = 1$
  - $g < g_c$ unitary symmetric vacuum
  - $g = g_c$ all mass eigenvalues become 0
  - $g > g_c$ vacuum states break the unitary symmetry

The intermediate field representation

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Hubbard Stratanovich transformation

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integrate out $\bar{\phi}, \phi$ (Gaussian) to get an effective theory for h

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tensors: $e^{\frac{g^2}{2}N^{D-1}\sum\left(T_{a^1\dots a^D}\bar{T}_{q^1\dots q^D}\prod_{c'\neq c}\delta_{a^{c'}q^{c'}}\right)\delta_{a^c p^c}\delta_{b^c q^c}\left(T_{b^1\dots b^D}\bar{T}_{p^1\dots p^D}\prod_{c'\neq c}\delta_{b^{c'}p^{c'}}\right)}$

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The intermediate field representation

Hubbard Stratanovich transformation

$$\int d\bar{\phi}d\phi \;\; e^{-\bar{\phi}\phi+\frac{g^2}{2}(\bar{\phi}\phi)^2} = \int d\bar{\phi}d\phi \;\; e^{-\bar{\phi}\phi} \int dh \;\; e^{-\frac{1}{2}h^2+g\bar{\phi}h\phi}$$

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$$Z(g) = \int \left(\prod_{c\in\mathcal{Q}} [dH^c] \right) \; e^{-\frac{1}{2}\sum_{c\in\mathcal{Q}} N^{D-1}\mathrm{Tr}_c[H^c H^c]+\mathrm{Tr}_{\mathcal{D}}[\ln R(H)]} \;,$$

$$R(H) = \frac{1}{\mathbf{1}^{\otimes\mathcal{D}} - g\sum_{c\in\mathcal{Q}} H^c \otimes \mathbf{1}^{\otimes(\mathcal{D}\setminus c)}}$$

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Unitary invariant, color symmetric solution $H^c = a\mathbf{1}$ with

$$a = \frac{g}{1-gQa} \Rightarrow a_{\mp} = \frac{1\mp\sqrt{1-4Qg^2}}{2Qg}$$

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$$Z = e^{-N^D\left(\frac{Qa_{\mp}^2}{2} + \ln(1-gQa_{\mp})\right)} \times \int [dM^c] e^{-\frac{1}{2}N^{D-1}(1-a_{\mp}^2)\sum_{c\in\mathcal{Q}} \mathrm{Tr}_c[M^c M^c] + \frac{1}{2}N^{D-2}\frac{Q-1}{Q}a_{\mp}^2\left(\sum_{c\in\mathcal{Q}}\mathrm{Tr}_c[M^c]\right)^2 + Q(M)}$$

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- the (diagonalized) mass matrix:

$$\frac{\delta^2 S}{\delta M^c_{\alpha\beta}\delta M^{c'}_{\gamma\delta}}\Big|_{M=0} = N^{D-1}(1 - a_{\mp}^2)\left(\delta^{cc'}\delta_{\alpha\delta}\delta_{\beta\gamma} - \frac{1}{QN}\delta_{\alpha\beta}\delta_{\gamma\delta}\right) + N^{D-1}(1 - Qa_{\mp}^2)\left(\frac{1}{QN}\delta_{\alpha\beta}\delta_{\gamma\delta}\right)$$

small $g \Rightarrow a_+ \nearrow \infty$, $a_- \searrow 0$ hence only $H^c = a_- \mathbf{1}$ is stable

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$Q = 1$: all mass eigenvalues are equal and become zero at criticality!

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- at $g < g_c$ only $a_- \mathbf{1}$ is stable
- at $g = g_c = \frac{1}{2}$ all mass eigenvalues are zero
- at $g > g_c$ all the vacua have broken unitary symmetry and are stable

Conclusion

Q melonic interactions in D dimensions (tensors with D indices), g_c critical constant (continuum limit of DT)

- $Q \geq 2$
  - $g < g_c$ color and unitary symmetric vacuum
  - $g = g_c$ one mass eigenvalue becomes 0
  - $g > g_c$ how does the color and unitary symmetry gets broken? one can show that it can not be that only the color symmetry gets broken
- $Q = 1$
  - $g < g_c$ unitary symmetric vacuum
  - $g = g_c$ all mass eigenvalue become 0
  - $g > g_c$ explicit vacua with broken unitary symmetry