Quantum fields out-of-equilibrium and information theory

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   - entanglement entropy
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2. Lecture
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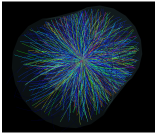
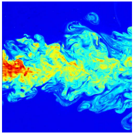
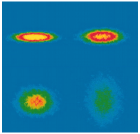
   - relative entropy and thermodynamics
   - local dissipation and entanglement
   - information geometry and quantum fields
   - entropic uncertainty relations

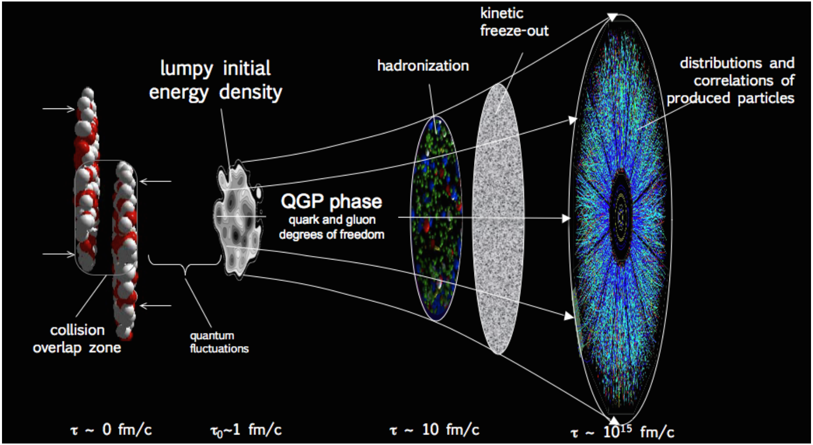
Fluid dynamics



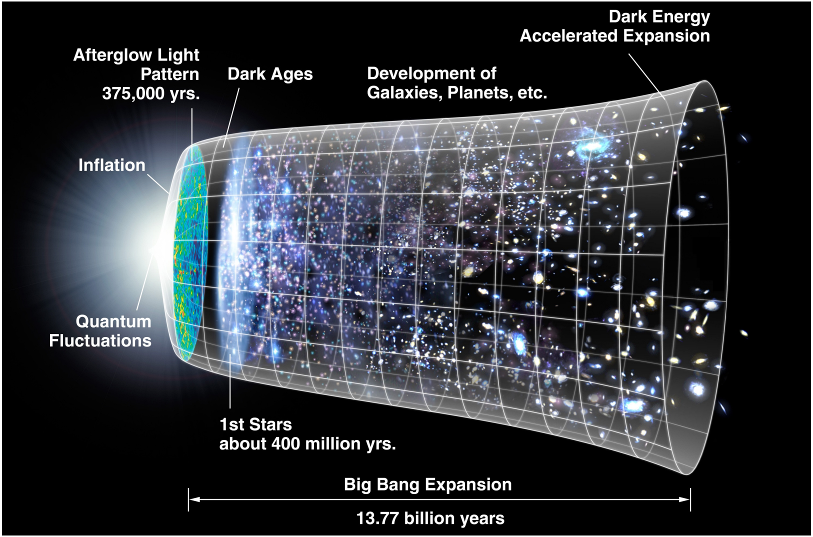




- long distances, long times or strong enough interactions
- quantum fields form a fluid!
- needs **macroscopic** fluid properties
  - thermodynamic equation of state $p(T, \mu)$
  - shear + bulk viscosity $\eta(T, \mu)$, $\zeta(T, \mu)$
  - heat conductivity $\kappa(T, \mu)$, ...
  - relaxation times, ...
  - electrical conductivity $\sigma(T, \mu)$
- fixed by **microscopic** properties encoded in Lagrangian $\mathscr{L}_{\text{QCD}}$
- old dream of condensed matter physics: understand the fluid properties!

High energy nuclear collisions: QCD fluid



The expanding Universe: cosmological fluid



Relativistic fluid dynamics

Energy-momentum tensor and conserved current

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + (p + \pi_{\text{bulk}})\Delta^{\mu\nu} + \pi^{\mu\nu}$$
$$N^\mu = n\, u^\mu + \nu^\mu$$

- tensor decomposition using fluid velocity u^μ, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$
- thermodynamic equation of state $p = p(T, \mu)$

Covariant **conservation laws** $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu N^\mu = 0$ imply

- equation for energy density ϵ
- equation for fluid velocity u^μ
- equation for particle number density n

Need in addition **constitutive relations** [e.g Israel & Stewart]

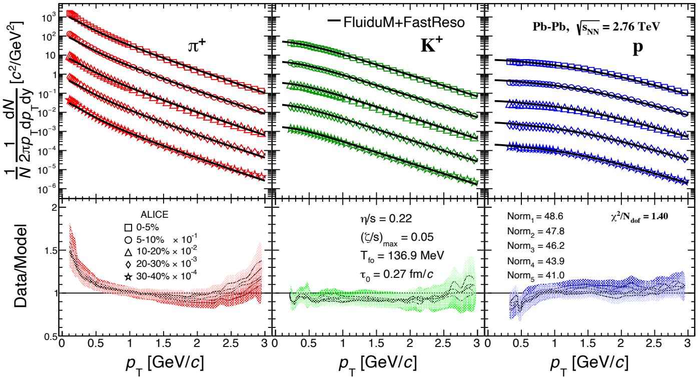
- equation for shear stress $\pi^{\mu\nu}$
- equation for bulk viscous pressure π_{bulk}

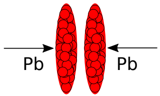
$$\tau_{\text{bulk}}\, u^\mu \partial_\mu \pi_{\text{bulk}} + \ldots + \pi_{\text{bulk}} = -\zeta\, \nabla_\mu u^\mu$$

- equation for diffusion current ν^μ

Particle production at the Large Hadron Collider

[Devetak, Dubla, Floerchinger, Grossi, Masciocchi, Mazeliauskas & Selyuzhenkov, JHEP 2020, 44]





- data are very precise now - high quality theory development needed!
- next step: include coherent fields / condensates

Entropy current, local dissipation and unitarity

- local dissipation = local entropy production

$$\nabla_\mu s^\mu(x) \geq 0$$

- e. g. from analytically continued quantum effective action
  [Floerchinger, JHEP 1609, 099 (2016)]
- fluid dynamics in Navier-Stokes approximation

$$\nabla_\mu s^\mu = \frac{1}{T}\left[2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\nabla_\rho u^\rho)^2\right] \geq 0$$

- unitary time evolution conserves von-Neumann entropy

$$S = -\text{Tr}\{\rho \ln \rho\} = -\text{Tr}\{(U\rho U^\dagger)\ln(U\rho U^\dagger)\} \qquad \Rightarrow \qquad \frac{d}{dt}S = 0$$

quantum information is globally conserved

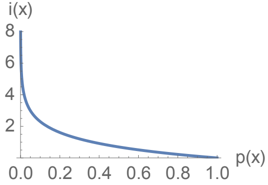
What is local dissipation in isolated quantum systems ?

Entropy and information

[Claude Shannon (1948), also Ludwig Boltzmann, Willard Gibbs (∼1875)]

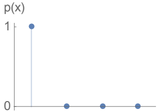
- consider a random variable x with probability distribution $p(x)$
- information content or "surprise" associated with outcome x

$$i(x) = -\ln p(x)$$

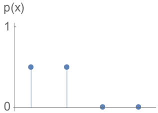


- entropy is expectation value of information content

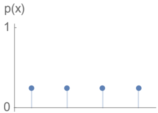
$$S(p) = \langle i(x) \rangle = -\sum_x p(x) \ln p(x)$$



$S = 0$



$S = \ln(2)$



$S = 2\ln(2)$

Thermodynamics

[..., Antoine Laurent de Lavoisier, Nicolas Léonard Sadi Carnot, Hermann von Helmholtz, Rudolf Clausius, Ludwig Boltzmann, James Clerk Maxwell, Max Planck, Walter Nernst, Willard Gibbs, ...]

- micro canonical ensemble: maximum entropy S for given conserved quantities E, N in given volume V
- starting point for development of thermodynamics ...

$$S(E, N, V), \qquad dS = \frac{1}{T} dE - \frac{\mu}{T} dN + \frac{p}{T} dV$$

- ... grand canonical ensemble with density operator ...

$$\rho = \frac{1}{Z} e^{-\frac{1}{T}(H - \mu N)}$$

- ... Einsteins probability for classical thermal fluctuations ...

$$dW \sim e^{S(\xi)} d\xi$$

Fluid dynamics

- uses thermodynamics *locally*

$$T(x), \qquad \mu(x), \qquad u^\mu(x), \ldots$$

- evolution from conservation laws

$$\nabla_\mu T^{\mu\nu}(x) = 0, \qquad \nabla_\mu N^\mu(x) = 0.$$

- local dissipation = local entropy production

$$\nabla_\mu s^\mu(x) = \partial_t s(x) + \vec{\nabla} \cdot \vec{s}(x) > 0$$

- in Navier-Stokes approximation with shear viscosity η, bulk viscosity ζ

$$\nabla_\mu s^\mu = \frac{1}{T} \left[2\eta \sigma_{\mu\nu} \sigma^{\mu\nu} + \zeta (\nabla_\rho u^\rho)^2 \right]$$

- how to understand this in quantum field theory?

Entropy in quantum theory

[John von Neumann (1932)]

$$S = -\mathsf{Tr}\{\rho \ln \rho\}$$

- based on the quantum density operator ρ
- for pure states $\rho = |\psi\rangle\langle\psi|$ one has $S = 0$
- for diagonal mixed states $\rho = \sum_j p_j |j\rangle\langle j|$

$$S = -\sum_j p_j \ln p_j > 0$$

- unitary time evolution conserves entropy

$$-\mathsf{Tr}\{(U\rho U^\dagger)\ln(U\rho U^\dagger)\} = -\mathsf{Tr}\{\rho \ln \rho\} \qquad \rightarrow \qquad S = \text{const.}$$

- quantum information is globally conserved

What is an entropy current?

- *can not* be density of global von-Neumann entropy for closed system

$$\int_\Sigma d\Sigma_\mu \; s^\mu(x) \neq -\mathrm{Tr}\,\{\rho \ln \rho\}$$

- kinetic theory for weakly coupled (quasi-) particles [Boltzmann (1890)]

$$s^\mu(x) = -\int \frac{d^3p}{p^0}\,\{p^\mu f(x,p) \ln f(x,p)\}$$

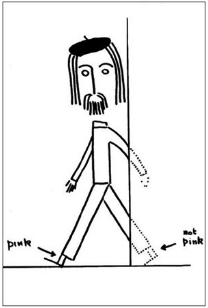
- molecular chaos: keep only single particle distribution $f(x,p)$
- how to go beyond weak coupling / quasiparticles?

Quantum entanglement

- *Can quantum-mechanical description of physical reality be considered complete?* [Einstein, Podolsky, Rosen (1935), Bohm (1951)]

$$\psi = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right)$$
$$= \frac{1}{\sqrt{2}} \left(|\rightarrow\rangle_A |\leftarrow\rangle_B - |\leftarrow\rangle_A |\rightarrow\rangle_B \right)$$

- *Bertlemann's socks and the nature of reality* [Bell (1980)]



Bell's inequalities and Bell tests

[John Stewart Bell (1966)]

- most popular version [Clauser, Horne, Shimony, Holt (1969)]

$$S = |E(a,b) - E(a,b') + E(a',b) + E(a',b')| \leq 2$$

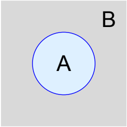
  holds for local hidden variable theories
- expectation value of product of two observables

$$E(a,b) = \langle A(a)B(b) \rangle$$

  with possible values $A = \pm 1$, $B = \pm 1$.
- depending on measurement settings a, a' and b, b' respectively
- quantum mechanical bound is $S \leq 2\sqrt{2}$
- experimental values $2 < S \leq 2\sqrt{2}$ rule out local hidden variables
- entanglement witness
- one measurement setting but at different times [Leggett, Garg (1985)]

Entropy and entanglement

- consider a split of a quantum system into two $A + B$



- reduced density operator for system A

$$\rho_A = \mathsf{Tr}_B\{\rho\}$$

- entropy associated with subsystem A

$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

- pure product state $\rho = \rho_A \otimes \rho_B$ leads to $S_A = 0$
- pure entangled state $\rho \neq \rho_A \otimes \rho_B$ leads to $S_A > 0$
- S_A is called entanglement entropy

Classical statistics

- consider system of two random variables x and y
- joint probability $p(x, y)$, joint entropy

$$S = -\sum_{x,y} p(x, y) \ln p(x, y)$$

- reduced or marginal probability $p(x) = \sum_y p(x, y)$
- reduced or marginal entropy

$$S_x = -\sum_{x} p(x) \ln p(x)$$

- one can prove: **joint entropy is greater than** or equal to **reduced entropy**

$$S \geq S_x$$

- **globally pure** state $S = 0$ is also **locally pure** $S_x = 0$

Quantum statistics

- consider system with two subsystems A and B
- combined state ρ , combined or full entropy

$$S = -\mathsf{Tr}\{\rho \ln \rho\}$$

- reduced density matrix $\rho_A = \mathsf{Tr}_B\{\rho\}$
- reduced or entanglement entropy

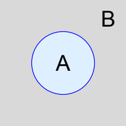
$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

- for quantum systems **entanglement makes a difference**

$$S \ngeq S_A$$

- **coherent information** $I_{B\rangle A} = S_A - S$ can be **positive**!
- **globally pure** state $S = 0$ can be **locally mixed** $S_A > 0$

Entanglement entropy in relativistic quantum field theory



- entanglement entropy of region A is a local notion of entropy

$$S_A = -\text{tr}_A \{\rho_A \ln \rho_A\} \qquad \rho_A = \text{tr}_B \{\rho\}$$

- for relativistic quantum field theories it is infinite already in vacuum state

$$S_A = \frac{\text{const}}{\epsilon^{d-2}} \int_{\partial A} d^{d-2}\sigma \sqrt{h} \; + \; \text{subleading divergences} \; + \; \text{finite}$$

- UV divergence proportional to surface area
- relativistic quantum fields are very strongly entangled already in vacuum
- Theorem [Helmut Reeh & Siegfried Schlieder (1961)]: local operators in region A can create all (non-local) particle states

Entanglement entropy in non-relativistic quantum field theory

[Natalia Sanchez-Kuntz & Stefan Floerchinger, PRA 103, 043327 (2021)]

- non-relativistic quantum field theory for Bose gas

$$S = \int dt d^{d-1}x \left\{ \varphi^* \left[i\partial_t + \frac{\vec{\nabla}^2}{2m} + \mu \right] \varphi - \frac{\lambda}{2} \varphi^{*2} \varphi^2 \right\}$$

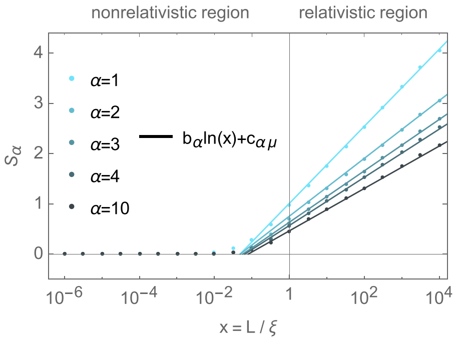
- Bogoliubov dispersion relation

$$\omega = \sqrt{\frac{\vec{p}^2}{2M}\left(\frac{\vec{p}^2}{2M} + 2\lambda\rho\right)} \approx \begin{cases} c_s|\vec{p}| & \text{for} \quad p \ll \sqrt{2M\lambda\rho} \quad \text{(phonons)} \\ \frac{\vec{p}^2}{2M} & \text{for} \quad p \gg \sqrt{2M\lambda\rho} \quad \text{(particles)} \end{cases}$$

- low momentum regime like theory of massless relativistic scalar particles
- high momentum regime non-relativistic
- what atre the entanglement properties?
- for $\rho = 0$ the entanglement entropy vanishes

Entanglement entropy in Bose-Einstein condensates

[Natalia Sanchez-Kuntz & Stefan Floerchinger, PRA 103, 043327 (2021)]



- one-dimensional Bose-Einstein condensate with subregion A of length L
- reduced density matrix $\rho_A = \text{Tr}_B\{\rho\}$
- Rényi entanglement entropy

$$S_\alpha = -\frac{1}{\alpha - 1} \ln \text{Tr}\{\rho_A^\alpha\}$$

- inverse healing length $1/\xi = \sqrt{2M\lambda\rho}$ acts like UV regulator
- at large $L \gg \xi$ we confirm CFT behaviour with $b_\alpha = c\frac{\alpha+1}{6\alpha}$

Relative entropy

- classical relative entropy or Kullback-Leibler divergence

$$S(p\|q) = \sum_j p_j \ln(p_j/q_j)$$

- not symmetric distance measure, but a *divergence*

$$S(p\|q) \geq 0 \qquad \text{and} \qquad S(p\|q) = 0 \quad \Leftrightarrow \quad p = q$$

- quantum relative entropy of two density matrices (also a *divergence*)

$$S(\rho\|\sigma) = \mathsf{Tr}\,\{\rho\,(\ln\rho - \ln\sigma)\}$$

- signals how well state ρ can be distinguished from a model σ
- Gibbs inequality: $S(\rho\|\sigma) \geq 0$
- $S(\rho\|\sigma) = 0$ if and only if $\rho = \sigma$

Significance of Kullback-Leibler divergence

Uncertainty deficit

- true distribution p_j and model distribution q_j
- *uncertainty deficit* is expected surprise $\langle -\ln q_j \rangle = -\sum_j p_j \ln q_j$ minus real information content $-\sum_j p_j \ln p_j$

$$S(p\|q) = -\sum_j p_j \ln q_j - \left(-\sum_j p_j \ln p_j \right)$$

Asymptotic frequencies

- true distribution q_j and frequency after N drawings $p_j = \frac{N(x_j)}{N}$
- probability to find frequencies p_j for large N goes like

$$e^{-NS(p\|q)}$$

- probability for fluctuation around expectation value $\langle p_j \rangle = q_j$ tends to zero for large N and when divergence $S(p\|q)$ is large

Advantages of relative entropy

Continuum limit $\quad p_j \to f(x)dx \quad q_j \to g(x)dx$

- not well defined for entropy

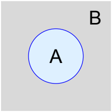
$$S = -\sum p_j \ln p_j \overset{\lightning}{\to} -\int dx f(x) \left[\ln f(x) + \ln dx\right]$$

- relative entropy remains well defined

$$S(p\|q) \to S(f\|g) = \int dx\, f(x) \ln(f(x)/g(x))$$

Local quantum field theory

- entanglement entropy $S(\rho_A)$ for spatial region divergent in relativistic QFT
- relative entanglement entropy $S(\rho_A\|\sigma_A)$ well defined
- rigorous definition in terms of Tomita–Takesaki theory of modular automorphisms on von-Neumann algebras [Huzihiro Araki (1976)]



Monotonicity of relative entropy

[Göran Lindblad (1975)]

- monotonicity of relative entropy

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) \leq S(\rho|\sigma)$$

  with $\mathcal{N}$ completely positive, trace-preserving map

- $\mathcal{N}$ unitary time evolution

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) = S(\rho|\sigma)$$

- $\mathcal{N}$ open system evolution with generation of entanglement to environment

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) < S(\rho|\sigma)$$

- basis for many proofs in quantum information theory
- leads naturally to second-law type relations

Thermodynamics from relative entropy

[Stefan Floerchinger & Tobias Haas, PRE 102, 052117 (2020)]

- relative entropy has very nice properties
- but can thermodynamics be derived from it ?
- can *entropy* be replaced by *relative entropy* ?
- first step to understand fluid dynamics

Principle of maximum entropy

[Edwin Thompson Jaynes (1963)]

- take macroscopic state characteristics as fixed, e. g.

  energy E, particle number N, momentum $\vec{p}$,

- principle of maximum entropy: among all possible microstates σ (or distributions q) the one with *maximum entropy* S is preferred

$$S(\sigma_{\text{thermal}}) = \text{max}$$

- why? assume $S(\sigma) < \text{max}$, than σ would contain additional information not determined by macroscopic variables, which is not available
- maximum entropy = minimal information

Principle of minimum expected relative entropy

[Stefan Floerchinger & Tobias Haas, PRE 102, 052117 (2020)]

- take macroscopic state characteristics as fixed, e. g.

  energy E, particle number N, momentum $\vec{p}$,

- principle of minimum expected relative entropy: preferred is the model σ from which allowed states ρ are least distinguishable on average

$$\langle S(\rho\|\sigma_{\text{thermal}})\rangle = \int D\rho \; S(\rho\|\sigma_{\text{thermal}}) = \text{min}$$

- similarly for classical probability distributions

$$\langle S(p\|q)\rangle = \int Dp \; S(p\|q) = \text{min}$$

- need to define *measures* Dp and $D\rho$ on spaces of probability distributions p and density matrices ρ, respectively

Measure on space of probability distributions

- consider set of normalized probability distributions p in agreement with macroscopic constraints
- manifold with local coordinates $\xi = \{\xi^1, \dots, \xi^m\}$
- integration in terms of coordinates

$$\int Dp = \int d\xi^1 \cdots d\xi^m \, \mu(\xi^1, \dots, \xi^m)$$

- want this to be invariant under coordinate changes $\xi \to \xi'(\xi)$
- possible choice is *Jeffreys prior* as integral measure [Harold Jeffreys (1946)]

$$\mu(\xi) = \text{const} \times \sqrt{\text{det } g_{\alpha\beta}(\xi)}$$

- uses Riemannian metric $g_{\alpha\beta}(\xi)$ on space of probability distributions: *Fisher information metric* [Ronald Aylmer Fisher (1925)]

$$g_{\alpha\beta}(\xi) = \sum_j p_j(\xi) \frac{\partial \ln p_j(\xi)}{\partial \xi^\alpha} \frac{\partial \ln p_j(\xi)}{\partial \xi^\beta}$$

Permutation invariance

- can now integrate functions of p

$$\int Dp\, f(p) = \int d^m\xi\, \mu(\xi)\, f(p(\xi))$$

- consider maps $\{p_1, \ldots p_{\mathcal{N}}\} \to \{p_{\Pi(1)}, \ldots p_{\Pi(\mathcal{N})}\}$ where $j \to \Pi(j)$ is a permutation, abbreviated $p \to \Pi(p)$
- want to show $Dp = D\Pi(p)$ such that

$$\int Dp\, f(p) = \int Dp\, f(\Pi(p))$$

- convenient to choose coordinates

$$p_j = \begin{cases} (\xi^j)^2 & \text{for } j = 1, \ldots, \mathcal{N}-1, \\ 1 - (\xi^1)^2 - \ldots - (\xi^{\mathcal{N}-1})^2 & \text{for } j = \mathcal{N}. \end{cases}$$

wich allows to write

$$\int Dp = \frac{1}{\Omega_{\mathcal{N}}} \int_{-1}^{1} d\xi^1 \cdots d\xi^{\mathcal{N}}\, \delta\left(1 - \sqrt{\sum_{\alpha=1}^{\mathcal{N}} (\xi^\alpha)^2}\right) = \int D\Pi(p)$$

Minimizing expected relative entropy

- consider now the functional

$$B(q,\lambda) = \int Dp \left[S(p\|q) + \lambda \left(\sum_i q_i - 1 \right) \right]$$

- variation with respect to q_j

$$0 \stackrel{!}{=} \delta B = \sum_j \int Dp \left[-\frac{p_j}{q_j} + \lambda \right] \delta q_j$$

leads by permutation invariance to the uniform distribution

$$q_j = \langle p_j \rangle = \frac{1}{\mathcal{N}}$$

- microcanonical distribution has minimum expected relative entropy!
- least distinguishable within the set of allowed distributions

Measure on space of density matrices

- measure on space of density matrices $D\rho$ can be defined similarly in terms of coordinates ξ but using now *quantum Fisher information metric*

$$g_{\alpha\beta}(\xi) = \mathsf{Tr}\left\{ \frac{\partial \rho(\xi)}{\partial \xi^\alpha} \frac{\partial \ln \rho(\xi)}{\partial \xi^\beta} \right\}$$

- definition uses symmetric logarithmic derivative such that

$$\frac{1}{2}\rho(d\ln\rho) + \frac{1}{2}(d\ln\rho)\rho = d\rho$$

- appears also as limit of relative entropy for states that approach each other

$$S(\rho(\xi + d\xi)\|\rho(\xi)) = \frac{1}{2} g_{\alpha\beta}(\xi)\, d\xi^\alpha\, d\xi^\beta + \ldots$$

Unitary transformations as isometries

- consider unitary map

$$\rho(\xi) \to \rho'(\xi) = U\rho(\xi)\,U^\dagger = \rho(\xi')$$

- again normalized density matrix but at coordinate point ξ'
- induced map on coordinates $\xi \to \xi'(\xi)$ is an *isometry*

$$g_{\alpha\beta}(\xi)\,d\xi^\alpha\,d\xi^\beta = g_{\alpha\beta}(\xi')\,d\xi'^\alpha\,d\xi'^\beta$$

- can be used to show invariance of measure such that

$$\int D\rho\, f(\rho) = \int D\rho\, f(U\rho U^\dagger)$$

Minimizing expected relative entropy on density matrices

- consider now the functional

$$B = \int D\rho\, S(\rho\|\sigma) = \int d^m\xi\, \mu(\xi)\, S(\rho(\xi)\|\sigma)$$

- minimization $0 \stackrel{!}{=} \delta B$ leads to microcanonical density matrix

$$\sigma_{\mathsf{m}} = \frac{1}{\mathcal{N}} \mathbb{1}$$

  on space allowed by macroscopic constraints
- anyway only possibility for unique minimum $\sigma_{\mathsf{m}} = U \sigma_{\mathsf{m}} U^\dagger$

Microcanonical ensemble

- microcanonical ensemble

$$\sigma_{\mathsf{m}} = \frac{1}{Z_{\mathsf{m}}} \delta(H - E(\sigma_{\mathsf{m}}))\delta(N - N(\sigma_{\mathsf{m}}))$$

- relative entropy of arbitrary state ρ to microcanonical state

$$S(\rho \| \sigma_{\mathsf{m}}) = \begin{cases} -S(\rho) + S(\sigma_{\mathsf{m}}) & \text{for } E(\rho) \equiv E(\sigma_{\mathsf{m}}) \\ & \text{and } N(\rho) \equiv N(\sigma_{\mathsf{m}}) \\ +\infty & \text{else} \end{cases}$$

- differential for $dE(\rho) \equiv dE(\sigma_{\mathsf{m}})$ and $dN(\rho) \equiv dN(\sigma_{\mathsf{m}})$

$$\begin{aligned} dS(\rho \| \sigma_{\mathsf{m}}) &= -\, dS(\rho) + dS(\sigma_{\mathsf{m}}) \\ &= -\, dS(\rho) + \beta\, dE(\rho) - \beta\mu\, dN(\rho) \end{aligned}$$

- gives an alternative definition of temperature

$$\beta = \frac{1}{T}$$

Canonical and grand-canonical ensemble

- transition to canonical and grand-canonical ensembles follows the usual construction

$$\sigma_{\mathrm{gc}} = \frac{1}{Z} e^{-\beta(H-\mu N)}$$

- relative entropy of arbitrary state ρ to grand-canonical state σ_{gc}

$$\begin{aligned} S(\rho\|\sigma_{\mathrm{gc}}) = &- S(\rho) + S(\sigma_{\mathrm{gc}}) + \beta\,(E(\rho) - E(\sigma_{\mathrm{gc}})) \\ &- \beta\mu\,(N(\rho) - N(\sigma_{\mathrm{gc}})). \end{aligned}$$

- differential

$$\begin{aligned} dS(\rho\|\sigma_{\mathrm{gc}}) = &- dS(\rho) + \beta\, dE(\rho) - \beta\mu\, dN(\rho) \\ &+ (E(\rho) - E(\sigma_{\mathrm{gc}}))\, d\beta \\ &- (N(\rho) - N(\sigma_{\mathrm{gc}}))\, d(\beta\mu), \end{aligned}$$

- choices for $\beta = 1/T$ and μ such that $E(\rho) = E(\sigma_{\mathrm{gc}})$ and $N(\rho) = N(\sigma_{\mathrm{gc}})$ extremize relative entropy $S(\rho\|\sigma_{\mathrm{gc}})$

Thermal fluctuations and relative entropy

- "mesoscopic" quantities ξ fluctuate in thermal equilibrium, for example energy in a subvolume
- traditional theory goes back to Einsteins work on critical opalescence [Albert Einstein (1910)]

$$dW \sim e^{S(\xi)} d\xi$$

- entropy can be replaced by relative entropy between state $\rho(\xi)$ (where ξ is sharp) and thermal state σ (where it ξ is fluctuating)

$$dW = \frac{1}{Z} e^{-S(\rho(\xi)\|\sigma)} \sqrt{\det g_{\alpha\beta}(\xi)}\, d^m\xi$$

- resembles closely probability for fluctuations in frequencies $p_j = \frac{N(x_j)}{N}$

$$\sim e^{-NS(p\|q)}$$

Third law of thermodynamics

[Walter Nernst (1905)]

- many equivalent formulations available already
- [Max Planck (1911)]: entropy S approaches a constant for $T \to 0$ that is independent of other thermodynamic parameters

$$\lim_{T \to 0} S(\sigma) = S_0 = \text{const}$$

- new formulation with relative entropy: relative entropy $S(\rho_0 \| \sigma)$ between ground state ρ_0 and a thermodynamic model state σ approaches zero for $T \to 0$

$$\lim_{T \to 0} S(\rho_0 \| \sigma) = 0$$

- second law can also be formulated with relative entropy

Local thermal equilibrium in a quantum field theory

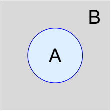
- consider non-equilibrium situation with
  - true density matrix ρ
  - local equilibrium approximation

$$\sigma = \frac{1}{Z} e^{-\int d\Sigma_\mu \{\beta_\nu(x) T^{\mu\nu} + \alpha(x) N^\mu\}}$$

- reduced density matrices $\rho_A = \text{Tr}_B\{\rho\}$ and $\sigma_A = \text{Tr}_B\{\sigma\}$
- σ is very good model for ρ in region A when

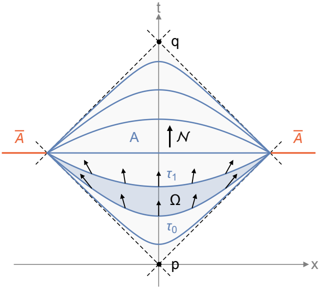
$$S_A = \text{Tr}_A\{\rho_A(\ln \rho_A - \ln \sigma_A)\} \to 0$$

- does *not* imply that globally $\rho = \sigma$



Local form of second law for open systems 1

[Neil Dowling, Stefan Floerchinger & Tobias Haas, PRD 102, 105002 (2020)]



- local description of quantum field theories in space-time regions bounded by two light cones [e. g. Rudolf Haag (1992), Huzihiro Araki (1992)]
- unitary evolution for isolated systems, more generally CPTP map

$$\rho(\tau_0) \to \mathcal{N}(\rho(\tau_0)) = \rho(\tau_1)$$

Local form of second law for open systems 2

[Neil Dowling, Stefan Floerchinger & Tobias Haas, PRD 102, 105002 (2020)]

- compare to global equilibrium state

$$\sigma = \frac{1}{Z} \exp\left[- \int_{\Sigma(\tau)} d\Sigma_\mu \left\{ \beta_\nu T^{\mu\nu} + \alpha N^\mu \right\} \right]$$

with entropy current

$$s^\mu = -\beta_\nu T^{\mu\nu} - \alpha N^\mu + p\beta^\mu$$

- relative entropy

$$S(\rho||\sigma) = \text{Tr}\left\{ \rho \big(\ln(\rho) - \ln(\sigma) \big) \right\}$$

$$= -S(\rho) + \ln(Z) + \text{Tr}\Big\{ \rho \int d\Sigma_\mu \big(\beta_\nu T^{\mu\nu} + \alpha N^\mu \big) \Big\}$$

$$= -S(\rho) + \int d\Sigma_\mu \Big\{ -s^\mu(\sigma) + \beta_\nu \left[T^{\mu\nu}(\rho) - T^{\mu\nu}(\sigma) \right] + \alpha \left[N^\mu(\rho) - N^\mu(\sigma) \right] \Big\}$$

- monotonicity of relative entropy

$$\Delta S(\rho\|\sigma) = S(\rho(\tau_1)\|\sigma(\tau_1)) - S(\rho(\tau_0)\|\sigma(\tau_0)) \leq 0$$

- allows to formulate local forms of the second law for fluids

Local form of second law for open systems 3

[Neil Dowling, Stefan Floerchinger & Tobias Haas, PRD 102, 105002 (2020)]

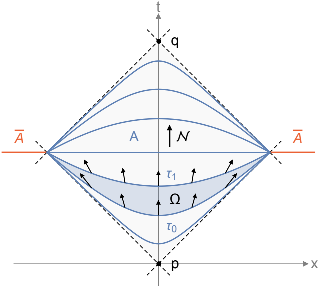
- assume now that one can write

$$\Delta S(\rho) = S(\rho(\tau_1)) - S(\rho(\tau_0)) = \int_\Omega d^d x \sqrt{g}\, \mathfrak{s}(\rho)(x)$$

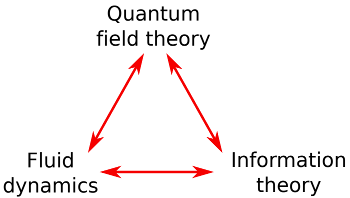
- find from monotonicity of relative entropy a local form of the second law

$$\mathfrak{s}(\rho) + \beta_\nu \nabla_\mu T^{\mu\nu}(\rho) + \alpha \nabla_\mu N^\mu(\rho) \geq 0$$

- next step: time evolution for isolated fluids



Quantum field dynamics



- new hypothesis

local dissipation = *quantum entanglement generation*

- quantum information is spread
- locally, quantum state approaches mixed state form
- full loss of *local* quantum information = *local* thermalization

Information geometry

- explore concepts like relative entropy, Fisher metric, Amari-Chentsov connections, dual affine structure
- Information geometry can be applied to thermodynamics
- Information geometry for classical statistical field theory
- Information geometry for quantum field theory in real time

Exponential families and the effective action 1

- class of probability distributions

$$p(X, J)\,dX = \exp(-I(X) + J^{\alpha}\varphi_{\alpha}(X) - W(J))\,dX$$

with partition function

$$Z(J) = \exp(W(J)) = \int dX \exp(-I(X) + J^{\alpha}\varphi_{\alpha}(X))$$

- change of variables

$$p(\varphi, J)\,d^m\varphi = \exp(-S(\varphi) + J^{\alpha}\varphi_{\alpha} - W(J))\,d^m\varphi$$

- introduce a measure

$$d\mu(\varphi) = \exp(-S(\varphi))\,d^m\varphi$$

- partition function becomes

$$Z(J) = \exp(W(J)) = \int d\mu(\varphi) \exp(J\varphi)$$

Exponential families and the effective action 2

- can do affine transformations

$$J^\alpha \to J'^\alpha = M^\alpha{}_\beta J^\beta + c^\alpha$$

- Legendre transform

$$\Gamma(\phi) = \sup_J \{J^\alpha \phi_\alpha - W(J)\}$$

with expectation value

$$\phi_\alpha = \langle \varphi_\alpha \rangle = \frac{\partial}{\partial J^\alpha} W(J) = \frac{1}{Z(J)} \int d\mu(\varphi) \exp(J\varphi) \varphi_\alpha$$

- can also do affine transformations

$$\phi_\alpha \to \phi'_\alpha = N_\alpha{}^\beta \phi_\beta + d_\alpha$$

Exponential families and the effective action 3

- relative entropy between distributions at diffrent sources

$$S(p(J)||p(J')) = \int d\mu(\varphi) \exp(J^\alpha \varphi_\alpha - W(J))$$
$$\times \ln\left(\exp(J\varphi - W(J))/\exp(J'\varphi - W(J'))\right)$$
$$= (J - J')^\alpha \phi_\alpha - W(J) + W(J')$$
$$= \Gamma(\phi) - W(J') - J'^\alpha \phi_\alpha$$

where expectation value ϕ is with respect to the distribution at source J

- Fisher information metric

$$g_{\alpha\beta}(J) = \frac{\partial^2}{\partial J^\alpha \partial J^\beta} S(p(J)||p(J'))\big|_{J=J'} = \frac{\partial}{\partial J^\alpha}\phi_\beta$$
$$= \frac{\partial^2}{\partial J^\alpha \partial J^\beta} W(J) = \langle(\varphi_\alpha - \phi_\alpha)(\varphi_\beta - \phi_\beta)\rangle$$

- inverse Fisher metric

$$g^{\alpha\beta}(\phi) = \frac{\partial^2}{\partial \phi_\alpha \partial \phi_\beta}\Gamma(\phi)$$

Conclusions

- local thermalization and fluid dynamics can be formulted in terms of relative entropy
- local dissipation = entanglement generation (?)
- quantum information theory for non-equilibrium dynamics
- thermodynamics can be developed in terms of relative entropies
- second law for isolated situations needs further investigations
- outlook: information geometry for quantum fields in and out of equilibrium