Geometry, Noether currents and relativistic fluids

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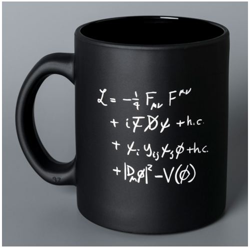
STRUCTURES CLUSTER OF EXCELLENCE

Understanding quantum field dynamics



- microscopic Lagrangian for many phenomena is known
- quantum field theories change with scale!
- need to understand quantum field dynamics away from simple limits
- important for condensed matter, optics, atomic physics, astrophysics, nuclear physics, cosmology, ...

What are the macroscopic evolution equations for quantum fields ?

One-particle irreducible or quantum effective action

- partition function $Z[J]$, Schwinger functional $W[J]$

$$Z[J] = \int D\chi \, e^{iS[\chi] - i\int_x \{J(x)\chi(x)\}}$$

- **quantum effective action** $\Gamma[\phi]$ defined by Legendre transform

$$\Gamma[\phi] = \int_x J(x)\phi(x) - W[J]$$

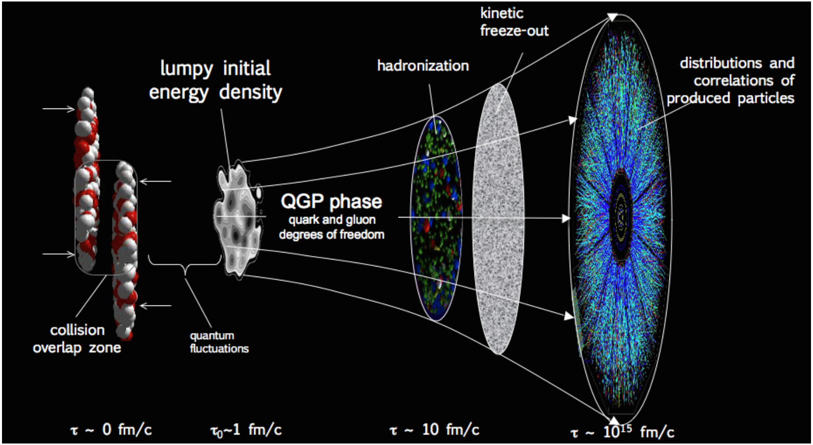
  with expectation values $\phi(x) = \delta W[J]/\delta J(x)$

- includes all quantum and statistical fluctuations !
- equation of motion for field expectation values

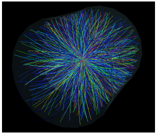
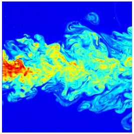
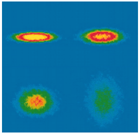
$$\frac{\delta}{\delta\phi(x)}\Gamma[\phi] = J(x)$$

- functional renormalization group: flow equation for $\Gamma[\phi]$
- can be used in and out of equilibrium
  [e. g. Floerchinger, JHEP 1205, 021 (2012); JHEP 1609, 099 (2016)]

High energy nuclear collisions: QCD fluid



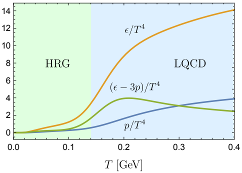
Fluid dynamics

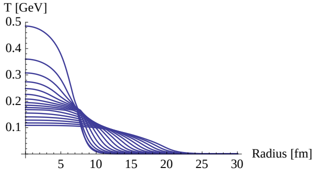
    

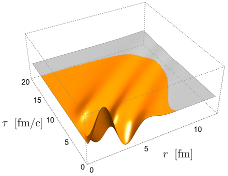
- long distances, long times or strong enough interactions
- quantum fields form a fluid!
- needs **macroscopic** fluid properties
  - thermodynamic equation of state $p(T, \mu)$
  - shear + bulk viscosity $\eta(T, \mu)$, $\zeta(T, \mu)$
  - heat conductivity $\kappa(T, \mu)$, ...
  - relaxation times, ...
  - electrical conductivity $\sigma(T, \mu)$
- fixed by **microscopic** properties encoded in Lagrangian $\mathscr{L}_{\text{QCD}}$
- old dream of condensed matter physics: understand the fluid properties!

Flow and fluctuations in heavy ion collisions

FluiduM: Fluid dynamics of heavy ion collisions with Mode expansion
[Floerchinger & Wiedemann, PLB 728, 407 (2014), PRC 88, 044906 (2013), 89, 034914 (2014)]
[Floerchinger, Grossi & Lion, PRC 100, 014905 (2019)]



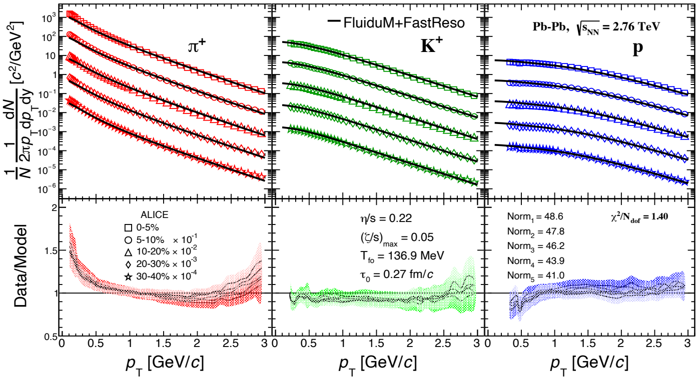


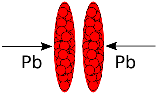


- background-fluctuation splitting + mode expansion
- analogous to cosmological perturbation theory
- substantially improved numerical performance (pseudospectral method)
- resonance decays included
  [Mazeliauskas, Floerchinger, Grossi & Teaney, EPJC 79, 284 (2019)]
- allows fast and precise comparison between theory and experiment

Particle production at the Large Hadron Collider

[Devetak, Dubla, Floerchinger, Grossi, Masciocchi, Mazeliauskas & Selyuzhenkov, JHEP 06 (2020) 044]





- data are very precise now - high quality theory development needed!
- next step: include coherent fields / condensates

Relativistic fluid dynamics

Energy-momentum tensor and conserved current

$$T^{\mu\nu} = \epsilon\, u^\mu u^\nu + (p + \pi_{\text{bulk}})\Delta^{\mu\nu} + \pi^{\mu\nu}$$
$$N^\mu = n\, u^\mu + \nu^\mu$$

- tensor decomposition using fluid velocity u^μ, $\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$
- thermodynamic equation of state $p = p(T, \mu)$

Covariant **conservation laws** $\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu N^\mu = 0$ imply

- equation for energy density ϵ
- equation for fluid velocity u^μ
- equation for particle number density n

Need **further evolution equations** [e.g Israel & Stewart]

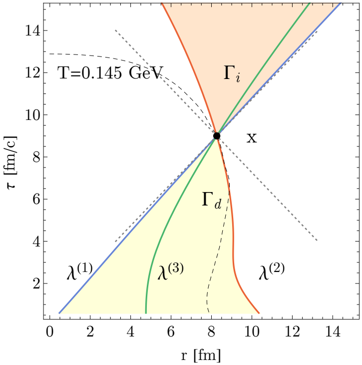
- equation for shear stress $\pi^{\mu\nu}$
- equation for bulk viscous pressure π_{bulk}

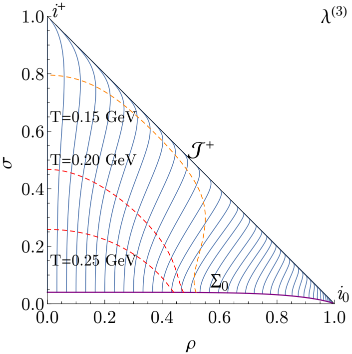
$$\tau_{\text{bulk}}\, u^\mu \partial_\mu \pi_{\text{bulk}} + \ldots + \pi_{\text{bulk}} = -\zeta\, \nabla_\mu u^\mu$$

- equation for diffusion current ν^μ
- non-hydrodynamic degrees of freedom are needed for relativistic causality!

Causality

[Floerchinger & Grossi, JHEP 08 (2018) 186]





- inequalities for relativistic causality
- dissipative fluid equations *can* be of hyperbolic type
- characteristic velocities depend on fluid fields
- need $|\lambda^{(j)}| < c$ for relativistic causality

Remarks

- fluid dynamics rather successful as non-equilibrium approximation in the macroscopic regime
- derivation from quantum effective action $\Gamma[\phi]$ wanted
  [Floerchinger, JHEP 1205, 021 (2012); JHEP 1609, 099 (2016)]
- expectation values *and* correlation functions of interest
- underlying principle: most excitations or modes relax quickly
  [Kadanoff & Martin (1963)]
- exception: conserved quantities like energy, momentum or particle density ("hydrodynamic modes")
- but: some non-hydrodynamic modes are needed for causality
- how to obtain additional equations of motion for them?

Covariant energy-momentum conservation

- quantum effective action $\Gamma[\phi, g]$ at stationary matter fields

$$\frac{\delta}{\delta\phi(x)}\Gamma[\phi, g] = 0$$

- diffeomorphism is gauge transformation of metric

$$g_{\mu\nu}(x) \to g_{\mu\nu}(x) + \nabla_\mu \varepsilon_\nu(x) + \nabla_\nu \varepsilon_\mu(x)$$

- energy-momentum tensor defined by

$$\delta\Gamma[\phi, g] = \frac{1}{2}\int d^d x \sqrt{g}\, T^{\mu\nu}(x)\delta g_{\mu\nu}(x)$$

- from invariance of $\Gamma[\phi, g]$ under diffeomorphisms

$$\nabla_\mu T^{\mu\nu}(x) = 0$$

- worked here in Riemann geometry with Levi-Civita connection

$$\delta\Gamma_\mu{}^\rho{}_\nu = \frac{1}{2}g^{\rho\lambda}\left(\nabla_\mu \delta g_{\nu\lambda} + \nabla_\nu \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\nu}\right)$$

Hypermomentum current

[von der Heyde, Kerlick & Hehl (1976)] [Floerchinger & Grossi, arXiv:2102.11098]

- connection can be varied independent of the metric

$$\delta\Gamma = \int d^dx\sqrt{g}\left\{\frac{1}{2}\mathscr{U}^{\mu\nu}(x)\delta g_{\mu\nu}(x) - \frac{1}{2}\mathscr{S}^{\mu}{}_{\rho}{}^{\sigma}(x)\delta\Gamma_{\mu}{}^{\rho}{}_{\sigma}(x)\right\}$$

  with new symmetric tensor $\mathscr{U}^{\mu\nu}$ and *hypermomentum* current $\mathscr{S}^{\mu}{}_{\rho}{}^{\sigma}$
- hypermomentum current can be decomposed further

$$\mathscr{S}^{\mu}{}_{\rho}{}^{\sigma} = Q^{\mu}{}_{\rho}{}^{\sigma} + W^{\mu}\,\delta_{\rho}{}^{\sigma} + S^{\mu}{}_{\rho}{}^{\sigma} + S^{\sigma\mu}{}_{\rho} + S_{\rho}{}^{\mu\sigma}$$

  with
  - spin current $S^{\mu\rho\sigma} = -S^{\mu\sigma\rho}$
  - dilatation current W^{μ}
  - shear current $Q^{\mu\rho\sigma} = Q^{\mu\sigma\rho}$, $Q^{\mu\rho}{}_{\rho} = 0$

Equations of motion for dilatation and shear current

[Floerchinger & Grossi, arXiv:2102.11098]

- variation of connection contains Levi-Civita part and non-Riemannian part

$$\delta\Gamma_{\mu}{}^{\rho}{}_{\sigma} = \frac{1}{2} g^{\rho\lambda} \left(\nabla_\mu \delta g_{\sigma\lambda} + \nabla_\sigma \delta g_{\mu\lambda} - \nabla_\lambda \delta g_{\mu\sigma} \right) + \delta C_{\mu}{}^{\rho}{}_{\sigma} + \delta D_{\mu}{}^{\rho}{}_{\sigma}$$

- variation at $\delta C_{\mu}{}^{\rho}{}_{\sigma} = \delta D_{\mu}{}^{\rho}{}_{\sigma} = 0$ gives energy-momentum tensor

$$T^{\mu\nu} = \mathscr{U}^{\mu\nu} + \frac{1}{2} \nabla_\rho \left(Q^{\rho\mu\nu} + W^\rho g^{\mu\nu} \right)$$

- new equation of motion for dilatation or Weyl current

$$\nabla_\rho W^\rho = \frac{2}{d} (T^{\mu}{}_{\mu} - \mathscr{U}^{\mu}{}_{\mu})$$

- new equation of motion for shear current

$$\nabla_\rho Q^{\rho\mu\nu} = 2 \left[T^{\mu\nu} - \mathscr{U}^{\mu\nu} - \frac{g^{\mu\nu}}{d} (T^{\sigma}{}_{\sigma} - \mathscr{U}^{\sigma}{}_{\sigma}) \right]$$

- non-conserved Noether currents

Spin current

[Floerchinger & Grossi, arXiv:2102.11098]

- tetrad formalism: vary tetrad ${V_\mu}^A$ and spin connection ${\Omega_\mu}^{AB}$

$$\delta\Gamma = \int d^dx\sqrt{g}\left\{ {\mathscr{T}^\mu}_A(x)\delta {V_\mu}^A(x) - \frac{1}{2}{S^\mu}_{AB}(x)\delta{\Omega_\mu}^{AB}(x)\right\}$$

  with
  - canonical energy-momentum tensor ${\mathscr{T}^\mu}_A$
  - spin current ${S^\mu}_{AB}$
- symmetric energy-momentum tensor in Belinfante-Rosenfeld form

$$T^{\mu\nu}(x) = \mathscr{T}^{\mu\nu}(x) + \frac{1}{2}\nabla_\rho\left[S^{\rho\mu\nu}(x) + S^{\mu\nu\rho}(x) + S^{\nu\mu\rho}(x)\right]$$

- equation of motion for spin current

$$\nabla_\mu S^{\mu\rho\sigma} = \mathscr{T}^{\sigma\rho} - \mathscr{T}^{\rho\sigma}$$

- non-conserved Noether current

Implications for relativistic fluid dynamics

- dilatation current, shear current and spin current provide additional information about quantum field theory out-of-equilibrium and relativistic fluids
- their contribution to energy-momentum tensor vanishes in equilibrium or for ideal fluids
- new equations of motion could allow formulation of extended and causal relativistic fluid dynamics with "non-hydrodynamic" modes
- expectation values and correlation functions obtained from coupling quantum fields to non-Riemannian geometry

Non-Riemannian geometry

[Floerchinger & Grossi, arXiv:2102.11098]

- general connection

$$\begin{aligned}\Gamma_\mu{}^\rho{}_\sigma = & \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\sigma\lambda} + \partial_\sigma g_{\mu\lambda} - \partial_\lambda g_{\mu\sigma} \right) + C_\mu{}^\rho{}_\sigma \\ & + \hat{B}_\mu{}^\rho{}_\sigma + \hat{B}_{\sigma\mu}{}^\rho - \hat{B}^\rho{}_{\mu\sigma} + B_\mu \delta^\rho{}_\sigma + B_\sigma \delta_\mu{}^\rho - B^\rho g_{\mu\sigma}.\end{aligned}$$

- contorsion $C_\mu{}^\rho{}_\sigma$ = gauge field for local Lorentz transformations
- Weyl gauge field B_μ = gauge field for local dilatations
- proper non-metricity $\hat{B}_\mu{}^\rho{}_\sigma$ = gauge field for local shear transformations
- local Lorentz transformations, dilatations and shear transformations together form the group $\mathrm{GL}(d)$ of basis changes in tangent space / the frame bundle
- all these transformations are *extended symmetries*: they change the quantum effective action Γ but in a specific way
- *extended symmetries* $\Rightarrow$ *non-conserved* Noether currents

Extended symmetries 1

[Floerchinger & Grossi, arXiv:2102.11098]

- consider transformation of fields

$$\phi(x) \to \phi(x) + id\xi^j(x)\, T_j \phi(x)$$

- might be non-Abelian with structure constants

$$[T_k, T_l] = i f_{kl}{}^j T_j$$

- introduce external gauge field and covariant derivative

$$D_\mu \phi(x) = \left(\nabla_\mu - i A_\mu^j(x) T_j \right) \phi(x)$$

- gauge field transforms as usual

$$A_\mu^j(x) \to A_\mu^j(x) + f_{kl}{}^j A_\mu^k(x) d\xi^l(x) + \nabla_\mu d\xi^j(x)$$

Extended symmetries 2

[Floerchinger & Grossi, arXiv:2102.11098]

- change of effective action $\Gamma[\phi, A]$

$$\Gamma[\phi + id\xi^j T_j \phi, A_\mu^j + f_{kl}{}^j A_\mu^k d\xi^l + \nabla_\mu d\xi^j] = \Gamma[\phi] + \int d^d x \sqrt{g} \left\{ \mathcal{I}_j(x)\, d\xi^j(x) \right\}$$

- define current through

$$\mathscr{J}_j^{\,\mu}(x) = \frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta A_\mu^j(x)}$$

- obtain conservation-type relation (for $\delta\Gamma/\delta\phi = 0$)

$$D_\mu \mathscr{J}_j^{\,\mu}(x) = \nabla_\mu \mathscr{J}_j^{\,\mu}(x) + f_{jk}{}^l A_\mu^k(x) \mathscr{J}_l^{\,\mu}(x) = -\mathcal{I}_j(x)$$

- global symmetry $\mathcal{I}_j(x) = 0 \Rightarrow$ conserved Noether current
- extended symmetry $\mathcal{I}_j(x) \neq 0$ but known at macroscopic level $\Rightarrow$ non-conserved Noether current

Entropy current, local dissipation and unitarity

- local dissipation = local entropy production

$$\nabla_\mu s^\mu(x) \geq 0$$

- e. g. from analytically continued quantum effective action
  [Floerchinger, JHEP 1609, 099 (2016)]
- fluid dynamics in Navier-Stokes approximation

$$\nabla_\mu s^\mu = \frac{1}{T}\left[2\eta\sigma_{\mu\nu}\sigma^{\mu\nu} + \zeta(\nabla_\rho u^\rho)^2\right] \geq 0$$

- unitary time evolution conserves von-Neumann entropy

$$S = -\mathsf{Tr}\{\rho \ln \rho\} = -\mathsf{Tr}\{(U\rho U^\dagger)\ln(U\rho U^\dagger)\} \qquad \Rightarrow \qquad \frac{d}{dt}S = 0$$

  quantum information is globally conserved

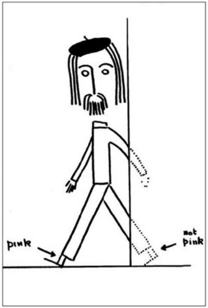
What is local dissipation in isolated quantum systems ?

Quantum entanglement

- *Can quantum-mechanical description of physical reality be considered complete?* [Einstein, Podolsky, Rosen (1935), Bohm (1951)]

$$\psi = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B \right)$$
$$= \frac{1}{\sqrt{2}} \left(|\rightarrow\rangle_A |\leftarrow\rangle_B - |\leftarrow\rangle_A |\rightarrow\rangle_B \right)$$

- *Bertlemann's socks and the nature of reality* [Bell (1980)]



Classical statistics

- consider system of two random variables x and y
- joint probability $p(x,y)$, joint entropy

$$S = -\sum_{x,y} p(x,y) \ln p(x,y)$$

- reduced or marginal probability $p(x) = \sum_y p(x,y)$
- reduced or marginal entropy

$$S_x = -\sum_{x} p(x) \ln p(x)$$

- one can prove: **joint entropy is greater than** or equal to **reduced entropy**

$$S \geq S_x$$

- **globally pure** state $S = 0$ is also **locally pure** $S_x = 0$

Quantum statistics

- consider system with two subsystems A and B
- combined state ρ , combined or full entropy

$$S = -\mathsf{Tr}\{\rho \ln \rho\}$$

- reduced density matrix $\rho_A = \mathsf{Tr}_B\{\rho\}$
- reduced or **entanglement entropy**

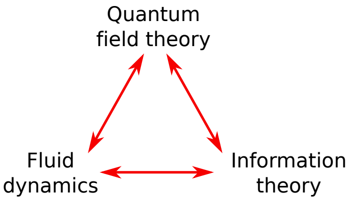
$$S_A = -\mathsf{Tr}_A\{\rho_A \ln \rho_A\}$$

- pure **product** state $\rho = \rho_A \otimes \rho_B$ leads to $S_A = 0$
- pure **entangled** state $\rho \neq \rho_A \otimes \rho_B$ leads to $S_A > 0$
- for quantum systems **entanglement makes a difference**

$$S \not\geq S_A$$

- **coherent information** $I_{B\rangle A} = S_A - S$ can be **positive**!
- **globally pure** state $S = 0$ can be **locally mixed** $S_A > 0$

Quantum field dynamics

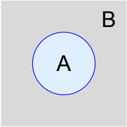


- new hypothesis

local dissipation = *quantum entanglement generation*

- quantum information is spread
- locally, quantum state approaches mixed state form
- full loss of *local* quantum information = *local* thermalization

Entanglement entropy in quantum field theory



- entanglement entropy of region A is a local notion of entropy

$$S_A = -\text{tr}_A \{\rho_A \ln \rho_A\} \qquad \rho_A = \text{tr}_B \{\rho\}$$

- however, it is infinite already in vacuum state

$$S_A = \frac{\text{const}}{\epsilon^{d-2}} \int_{\partial A} d^{d-2}\sigma \sqrt{h} \; + \; \text{subleading divergences} \; + \; \text{finite}$$

- UV divergence proportional to entangling surface
- quantum fields are very strongly entangled already in vacuum
- Theorem [Reeh & Schlieder (1961)]: local operators in region A can create all particle states

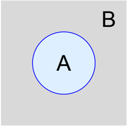
Relative entropy

- **relative entropy** of two density matrices

$$S(\rho|\sigma) = \mathsf{tr}\,\{\rho\,(\ln\rho - \ln\sigma)\}$$

- measures how well state ρ can be distinguished from a model σ
- Gibbs inequality: $S(\rho|\sigma) \geq 0$
- $S(\rho|\sigma) = 0$ if and only if $\rho = \sigma$
- quantum generalization of Kullback-Leibler divergence

Relative entanglement entropy



- consider now reduced density matrices

$$\rho_A = \mathsf{Tr}_B\{\rho\}, \qquad \sigma_A = \mathsf{Tr}_B\{\sigma\}$$

- define **relative entanglement entropy**

$$S_A(\rho|\sigma) = \mathsf{Tr}\left\{\rho_A\left(\ln\rho_A - \ln\sigma_A\right)\right\} = -\mathsf{Tr}\left\{\rho_A \ln\Delta_A\right\}$$

  with relative modular operator Δ_A
- measures how well ρ is represented by σ locally in region A
- UV divergences cancel: contains real physics information
- well defined in algebraic quantum field theory [Araki (1977)]
  [see also works by Casini, Myers, Lashkari, Witten, Liu, ...]

An approximate local description

[Dowling, Floerchinger & Haas, PRD 102 (2020) 10, 105002]

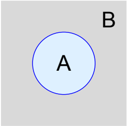
- consider non-equilibrium situation with
  - true density matrix ρ
  - local equilibrium approximation

$$\sigma = \frac{1}{Z} e^{-\int d\Sigma_\mu \{\beta_\nu(x) T^{\mu\nu} + \alpha(x) N^\mu\}}$$

- reduced density matrices $\rho_A = \text{Tr}_B\{\rho\}$ and $\sigma_A = \text{Tr}_B\{\sigma\}$
- σ is very good model for ρ in region A when

$$S_A = \text{Tr}_A\{\rho_A(\ln \rho_A - \ln \sigma_A)\} \to 0$$

- does *not* imply that globally $\rho = \sigma$



Monotonicity of relative entropy

- monotonicity of relative entropy [Lindblad (1975)]

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) \leq S(\rho|\sigma)$$

  with $\mathcal{N}$ completely positive, trace-preserving map

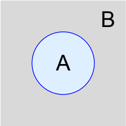
- $\mathcal{N}$ unitary evolution

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) = S(\rho|\sigma)$$

- $\mathcal{N}$ open system evolution with generation of entanglement to environment

$$S(\mathcal{N}(\rho)|\mathcal{N}(\sigma)) < S(\rho|\sigma)$$

- leads to local, second law type relation
  [Dowling, Floerchinger & Haas, PRD 102 (2020) 10, 105002]



Conclusions

- new divergence-type equations of motion
- dilatation current, shear current and spin current
- quantum field theory in non-Riemannian geometry
- extended symmetries $\Rightarrow$ non-conserved Noether currents
- relativistic fluid dynamics has a foundation in quantum information theory
- description of local thermalization in terms of relative entropy
- quantum field theoretic description with two density matrices:
  - true density matrix ρ evolves unitary
  - fluid model σ agrees locally but evolves non-unitary