# *Evolution of dark matter velocity dispersion*

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# *References*

- based on: [Alaric Erschfeld & Stefan Floerchinger, *Evolution of dark matter velocity dispersion*, JCAP 06 (2019) 039]
- related: [P. McDonald, *How to generate a significant effective temperature for cold dark matter, from first principles*, JCAP 04 (2011) 032]

## *Why is velocity dispersion interesting*

- goes beyond ideal fluid / single stream approximation
- regularizes shell crossing singularities
- could help to describe cosmological fluid at smaller scales and later times

## *Vlasov-Poisson system*

- description of dark matter as classical particles on trajectories

$$\frac{d\mathbf{x}}{d\tau} = \frac{\mathbf{p}}{a\,m}\,, \qquad \frac{d\mathbf{p}}{d\tau} = -\,a\,m\,\nabla_{\mathbf{x}}\phi$$

- number of particles in phase space volume

$$f(\tau, \mathbf{x}, \mathbf{p})\, d^3x\, d^3p$$

- Vlasov equation (collision-less Boltzmann equation)

$$\partial_\tau f + \frac{\mathbf{p}}{a\,m} \cdot \nabla_{\mathbf{x}} f - a\,m\,\nabla_{\mathbf{x}}\phi \cdot \nabla_{\mathbf{p}} f = 0$$

- supplemented by FRW equation for scale factor $a(\tau)$
- Newtonian potential $\phi(\tau, \mathbf{x})$ from Poisson equation

$$\Delta_{\mathbf{x}}\phi = \frac{3}{2}\,\mathcal{H}^2\,\Omega_m\,\delta$$

# *Mass density field*

- mass density as integral or zero'th moment of distribution function

$$\rho(\tau, \mathbf{x}) = \frac{m}{a^3(\tau)} \int_{\mathbb{R}^3} d^3p \, f(\tau, \mathbf{x}, \mathbf{p})$$

- spatially homogeneous expectation value (determines $\Omega_m(\tau)$)

$$\bar{\rho}(\tau) = \langle \rho(\tau, \mathbf{x}) \rangle$$

- spatially varying density contrast field

$$\delta(\tau, \mathbf{x}) = \frac{\rho(\tau, \mathbf{x}) - \bar{\rho}(\tau)}{\bar{\rho}(\tau)}$$

# *Peculiar velocity or fluid velocity field*

- velocity field as first moment or first cumulant of distribution function

$$u_i(\tau, \mathbf{x}) = \frac{1}{\rho(\tau, \mathbf{x})} \frac{m}{a(\tau)^3} \int_{\mathbb{R}^3} d^3p \, \frac{p_i}{a(\tau)\, m} \, f(\tau, \mathbf{x}, \mathbf{p})$$

- expectation value excluded by symmetries

# *Velocity dispersion tensor*

- second cumulant of distribution function is *velocity dispersion tensor*

$$\sigma_{ij}(\tau, \mathbf{x}) = \frac{1}{\rho(\tau, \mathbf{x})} \frac{m}{a(\tau)^3} \int_{\mathbb{R}^3} d^3p \, \frac{p_i p_j}{a(\tau)^2 \, m^2} \, f(\tau, \mathbf{x}, \mathbf{p}) - u_i(\tau, \mathbf{x}) \, u_j(\tau, \mathbf{x})$$

- quantifies deviation of particle velocity from velocity field $u_i(\tau, \mathbf{x})$
- symmetries allow expectation value

$$\langle \sigma_{ij}(\tau, \mathbf{x}) \rangle = \delta_{ij} \, \bar{\sigma}(\tau)$$

- expectation value positive semi-definite $\bar{\sigma}(\tau) \geq 0$
- $\bar{\sigma} = 0$ in single stream approximation
- deviation from expectation values defines fluctuation field

$$\varsigma_{ij}(\tau, \mathbf{x}) = \sigma_{ij}(\tau, \mathbf{x}) - \delta_{ij} \, \bar{\sigma}(\tau)$$

# *Generating functions 1*

- moment generating function

$$M(\tau,\mathbf{x};\mathbf{l}) = \frac{m}{a(\tau)^3\bar{\rho}(\tau)}\int_{\mathbb{R}^3} d^3p\ \exp\left(\frac{\mathbf{l}\cdot\mathbf{p}}{a(\tau)m}\right) f(\tau,\mathbf{x},\mathbf{p})$$

  such that moments of distribution are

$$m^{(n)}_{i_1\dots i_n}(\tau,\mathbf{x}) = \left[\prod_{j=1}^{n}\frac{\partial}{\partial l_{i_j}}\right] M(\tau,\mathbf{x};\mathbf{l})\Big|_{\mathbf{l}=0}$$

- first few moments

$$m^{(0)}(\tau,\mathbf{x}) = 1+\delta(\tau,\mathbf{x})$$
$$m^{(1)}_i(\tau,\mathbf{x}) = [1+\delta(\tau,\mathbf{x})]u_i(\tau,\mathbf{x})$$
$$m^{(2)}_{ij}(\tau,\mathbf{x}) = [1+\delta(\tau,\mathbf{x})]\left[u_i(\tau,\mathbf{x})u_j(\tau,\mathbf{x})+\sigma_{ij}(\tau,\mathbf{x})\right]$$

## *Generating functions 2*

- cumulant generating function

$$C(\tau,\mathbf{x};\mathbf{l}) = \ln(M(\tau,\mathbf{x};\mathbf{l}))$$

  such that cumulants are given by

$$c^{(n)}_{i_1\ldots i_n}(\tau,\mathbf{x}) = \left[\prod_{j=1}^{n}\frac{\partial}{\partial l_{i_j}}\right] C(\tau,\mathbf{x};\mathbf{l})\Big|_{\mathbf{l}=0}$$

- first few cumulants

$$c^{(0)}(\tau,\mathbf{x}) = \ln(1+\delta(\tau,\mathbf{x}))$$
$$c^{(1)}_i(\tau,\mathbf{x}) = u_i(\tau,\mathbf{x})$$
$$c^{(2)}_{ij}(\tau,\mathbf{x}) = \sigma_{ij}(\tau,\mathbf{x})$$

- Vlasov equation for cumulant generating function

$$\partial_\tau C + \mathcal{H}\,\mathbf{l}\cdot\nabla_{\mathbf{l}}C + \nabla_{\mathbf{x}}C\cdot\nabla_{\mathbf{l}}C + \nabla_{\mathbf{x}}\cdot\nabla_{\mathbf{l}}C + \mathbf{l}\cdot\nabla_{\mathbf{x}}\phi = 0$$

## *Truncations*

**single stream / ideal cold fluid** approximation

- cumulant generating function

$$C(\tau, \mathbf{x}; \mathbf{l}) = c^{(0)}(\tau, \mathbf{x}) + l_i c_i^{(1)}(\tau, \mathbf{x})$$

- distribution function

$$f(\tau, \mathbf{x}, \mathbf{p}) = \frac{a(\tau)^3}{m} \rho(\tau, \mathbf{x})\, \delta^{(3)}(\mathbf{p} - a(\tau) m \mathbf{u}(\tau, \mathbf{x}))$$

**Gaussian** approximation

- cumulant generating function

$$C(\tau, \mathbf{x}; \mathbf{l}) = c^{(0)}(\tau, \mathbf{x}) + l_i c_i^{(1)}(\tau, \mathbf{x}) + \frac{1}{2} l_i l_j c_{ij}^{(2)}(\tau, \mathbf{x})$$

- distribution function

$$f(\tau, \mathbf{x}, \mathbf{p}) = \frac{\rho(\tau, \mathbf{x})}{\sqrt{(2\pi)^3 \det(\boldsymbol{\sigma})}} \exp\left[-\frac{1}{2} (p_i - a m u_i) \frac{\sigma_{ij}^{-1}}{a^2 m^2} (p_j - a m u_j)\right]$$

# *Equations of motion for cumulants*

- for density contrast

$$\dot{\delta} + u_i\,\delta_{,i} + (1+\delta)\,u_{i,i} = 0$$

- for fluid velocity

$$\dot{u}_i + \mathcal{H}\,u_i + u_j\,u_{i,j} + \sigma_{ij,j} + \sigma_{ij}\,\ln(1+\delta)_{,j} + \phi_{,i} = 0$$

- for velocity dispersion tensor

$$\dot{\sigma}_{ij} + 2\mathcal{H}\,\sigma_{ij} + u_k\,\sigma_{ij,k} + \sigma_{jk}\,u_{i,k} + \sigma_{ik}\,u_{j,k} = -\pi_{ijk,k} - \pi_{ijk}\,\ln(1+\delta)_{,k}$$

- single stream / ideal fluid approximation

$$\sigma_{ij} = 0$$

- *apparent* self-consistency of single stream: $\sigma_{ij} = 0$ is fixed point
- but could in fact be *unstable* fixed point

## *Evolution of background fields*

- from continuity equation or covariant energy conservation

$$\dot{\bar{\rho}} + 3\mathcal{H}\,\bar{\rho} = 0$$

- depends on expectation values or background fields only
- solution simple dilution $\bar{\rho}(\tau) \sim 1/a(\tau)^3$

- from second moment of Vlasov equation

$$\dot{\bar{\sigma}}(\tau) + 2\mathcal{H}(\tau)\bar{\sigma}(\tau) + \frac{1}{3}\,\langle \varsigma_{ii,j}(\tau,\mathbf{x})\,u_j(\tau,\mathbf{x})\rangle + \frac{2}{3}\,\langle \varsigma_{ij}(\tau,\mathbf{x})\,u_{i,j}(\tau,\mathbf{x})\rangle = 0$$

- depends on two-point correlations or integrals of equal time spectra of fluctuation fields ("backreaction")
- decrease of $\bar{\sigma}(\tau) \sim 1/a(\tau)^2$ if Hubble rate $\mathcal{H}(\tau)$ dominates
- fluctuation fields could dominate at late time and modify this

# *Scalar, vector, tensor decomposition 1*

- go to Fourier space as usual

$$\Psi(\tau, \mathbf{x}) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\, \Psi(\tau, \mathbf{k})$$

- decomposition of velocity field into irreducible representations of SO(3)

$$u_j(\tau, \mathbf{k}) = -\frac{ik_j}{k^2}\, \theta(\tau, \mathbf{k}) + \varepsilon_{jkl}\, \frac{ik_k}{k^2}\, \omega_l(\tau, \mathbf{k})$$

with *velocity divergence* (scalar field)

$$\theta(\tau, \mathbf{k}) \equiv ik_j\, u_j(\tau, \mathbf{k})$$

and *vorticity* (solenoidal vector field)

$$\omega_j(\tau, \mathbf{k}) \equiv \varepsilon_{jkl}\, ik_k\, u_l(\tau, \mathbf{k})$$

## *Scalar, vector, tensor decomposition 2*

- fluctuating part of velocity divergence can be decomposed into irreducible representations of SO(3) like

$$\varsigma_{ij}(\tau,\mathbf{k}) = \delta_{ij}\,\varsigma(\tau,\mathbf{k}) + \frac{3}{2}\left(\frac{k_i k_j}{k^2} - \frac{\delta_{ij}}{3}\right)\vartheta(\tau,\mathbf{k}) - \frac{(\varepsilon_{ikl}\,k_j + \varepsilon_{jkl}\,k_i)\,k_k}{k^2}\,\vartheta_l(\tau,\mathbf{k}) + \vartheta_{ij}(\tau,\mathbf{k})$$

- two scalar fields

$$\varsigma(\tau,\mathbf{k}), \qquad \vartheta(\tau,\mathbf{k})$$

- one solenoidal vector field

$$\vartheta_j(\tau,\mathbf{k})$$

- one symmetric, transverse and traceless tensor field

$$\vartheta_{ij}(\tau,\mathbf{k})$$

# *Power spectra and backreation to velocity dispersion*

- introduce the mixed equal time power spectra

$$(2\pi)^3\,\delta_{\mathsf{D}}^{(3)}(\mathbf{k}+\mathbf{k}')\,P_{\varsigma\theta}(\tau,k) = \big\langle \varsigma(\tau,\mathbf{k})\;\theta(\tau,\mathbf{k}')\big\rangle$$
$$(2\pi)^3\,\delta_{\mathsf{D}}^{(3)}(\mathbf{k}+\mathbf{k}')\,P_{\vartheta\theta}(\tau,k) = \big\langle \vartheta(\tau,\mathbf{k})\;\theta(\tau,\mathbf{k}')\big\rangle$$
$$(2\pi)^3\,\delta_{\mathsf{D}}^{(3)}(\mathbf{k}+\mathbf{k}')\,\Delta_{ij}(\mathbf{k})\,P_{\vartheta\omega}(\tau,k) = \big\langle \vartheta_i(\tau,\mathbf{k})\;\omega_j(\tau,\mathbf{k}')\big\rangle$$

- allows to write evolution equation for velocity dispersion background

$$\dot{\bar{\sigma}}(\tau) + 2\mathcal{H}(\tau)\bar{\sigma}(\tau) - Q(\tau) = 0$$

- with integral over wave numbers

$$Q(\tau) = \frac{1}{3}\int_{\mathbf{q}}\Big[P_{\varsigma\theta}(\tau,q) - 2\,P_{\varsigma\theta}(\tau,q) - 4\,P_{\vartheta\omega}(\tau,q)\Big]$$

- integral typically UV dominated
- need to evolve the additional power spectra in time

## *New time variable and rescaled fields*

- new time variable

$$\eta(\tau) = \ln\left(\frac{D_+(\tau)}{D_+(\tau_{\text{in}})}\right) \qquad f(\tau) = \frac{\partial \ln(D_+(\tau))}{\partial \ln(a(\tau))}$$

with linear growth factor in single stream approximation $D_+(\tau)$

- combined and rescaled field vector

$$\Psi(\eta, \mathbf{k}) \equiv \begin{pmatrix} \delta(\eta, \mathbf{k}) \\ -\theta(\eta, \mathbf{k}) / \left(f(\eta)\, \mathcal{H}(\eta)\right) \\ \varsigma(\eta, \mathbf{k}) / \left(f(\eta)\, \mathcal{H}(\eta)\right)^2 \\ \vartheta(\eta, \mathbf{k}) / \left(f(\eta)\, \mathcal{H}(\eta)\right)^2 \\ \omega_i(\eta, \mathbf{k}) / \left(f(\eta)\, \mathcal{H}(\eta)\right) \\ \vartheta_i(\eta, \mathbf{k}) / \left(f(\eta)\, \mathcal{H}(\eta)\right)^2 \\ \vartheta_{ij}(\eta, \mathbf{k}) / \left(f(\eta)\, \mathcal{H}(\eta)\right)^2 \end{pmatrix}$$

- need to rescale also velocity dispersion background field

$$\hat{\sigma}(\eta) = \frac{\bar{\sigma}(\eta)}{f^2(\eta)\, \mathcal{H}^2(\eta)}$$

## *Evolution equations*

- time evolution

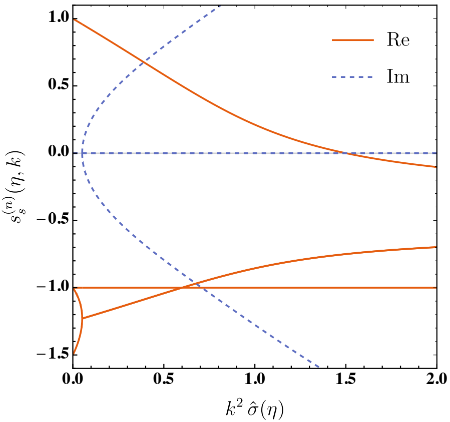
$$\partial_\eta \Psi_a(\eta, \mathbf{k}) + \Omega_{ab}(\eta, k)\, \Psi_b(\eta, \mathbf{k}) + I_a(\eta, \mathbf{k}) + J_a(\eta, \mathbf{k}) = 0$$

- linear evolution matrix for scalars (similar for vectors and tensors)

$$\Omega_{ab}(\eta, k) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -\frac{3}{2}\frac{\Omega_m(\eta)}{f^2(\eta)} + k^2\hat{\sigma}(\eta) & \frac{3}{2}\frac{\Omega_m(\eta)}{f^2(\eta)} - 1 & k^2 & k^2 \\ 0 & -\frac{2}{3}\hat{\sigma}(\eta) & 3\frac{\Omega_m(\eta)}{f^2(\eta)} - 2 & 0 \\ 0 & -\frac{4}{3}\hat{\sigma}(\eta) & 0 & 3\frac{\Omega_m(\eta)}{f^2(\eta)} - 2 \end{pmatrix}$$

- new scalar modes $\varsigma$ and $\vartheta$ generated from $\delta$ and $\theta$ by linear mixing
- eigenvalues of $\Omega_{ab}(\eta, k)$ determine growth / decay for given $\eta$ and $k$
- quadratic and cubic mode coupling terms $I_a$ and $J_a$

## *Linear scalar growth factors*




- eigenvalues of $\Omega_{ab}(\eta, k)$
- for $k = 0$ one growing mode and three decaying modes
- for large $k$ or at small scales imaginary parts develop (oscillations)
- velocity dispersion background sets a new scale $k_{\text{fs}} \sim 1/\sqrt{\hat{\sigma}}$

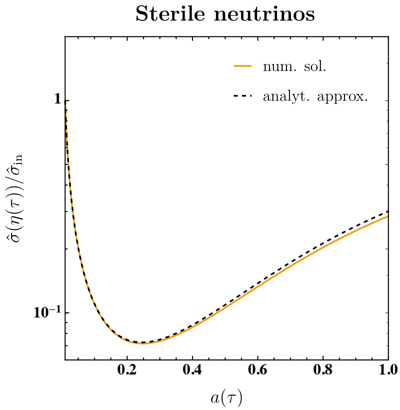
## *Non-linear background but linear perturbations around it*

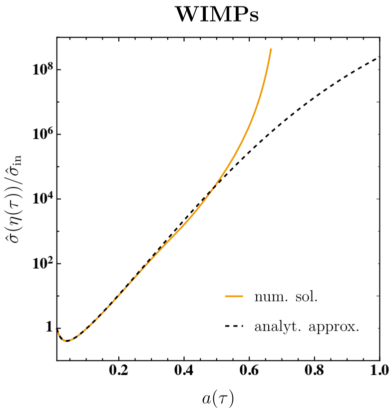
- evolution equations for background $\hat{\sigma}(\eta)$ and fluctuation fields $\Psi(\eta, \mathbf{k})$ are coupled through non-linear terms
- also fluctuation fields evolve non-linearly
- for first study: neglect non-linear interactions among fluctuating fields

# *Two dark matter models: sterile neutrinos and WIMPS*

- compare two dark matter candidates
- sterile neutrinos
  - mass $m \approx 1$ keV
  - free-streaming wave number at matter-rad. eq. $k_{\mathsf{fs,eq}} \approx 10\ h/\mathsf{Mpc}$
  - velocity dispersion background at matter-rad. eq. $\hat{\sigma}_{\mathsf{eq}} \approx 10^{-2}\ \mathsf{Mpc}^2/h^2$
- weakly interacting massive particles
  - mass $m \approx 100$ GeV
  - free-streaming wave number at matter-rad. eq. $k_{\mathsf{fs,eq}} \approx 10^7\ h/\mathsf{Mpc}$
  - velocity dispersion background at matter-rad. eq. $\hat{\sigma}_{\mathsf{eq}} \approx 10^{-15}\ \mathsf{Mpc}^2/h^2$
- initial power spectrum for WIMPS extends further into the UV
- will lead to substantial difference for evolution of velocity dispersion

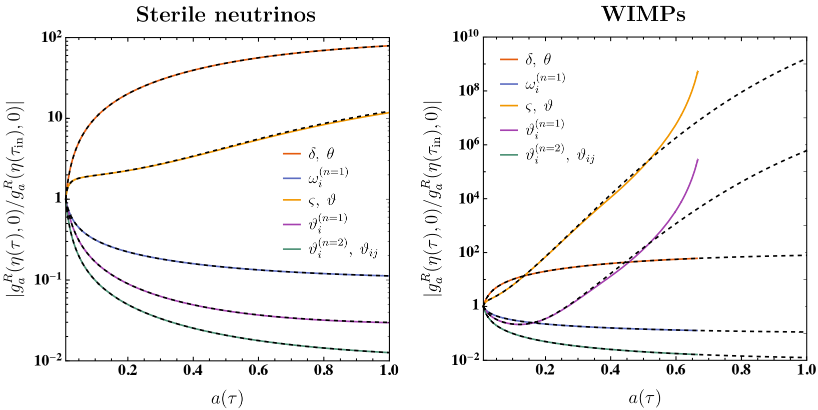
# *Velocity dispersion background evolution*



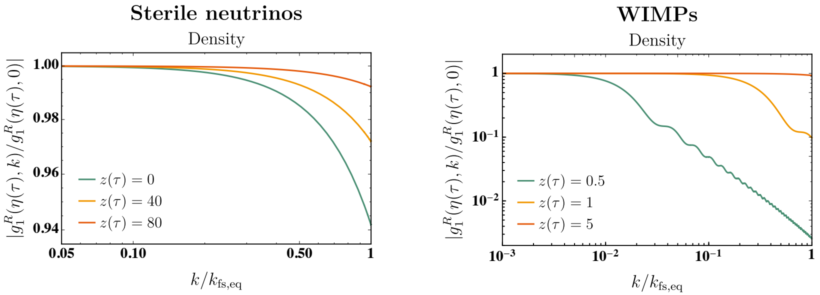

- at early times $\hat{\sigma}$ decreases due to cosmological expansion
- at later times increase due to backreaction
- analytic approximation works very well for sterile neutrinos
- double exponential growth of $\hat{\sigma}$ observed at late times for WIMPs (difficult to resolve numerically)

# *Growths functions for $k = 0$*



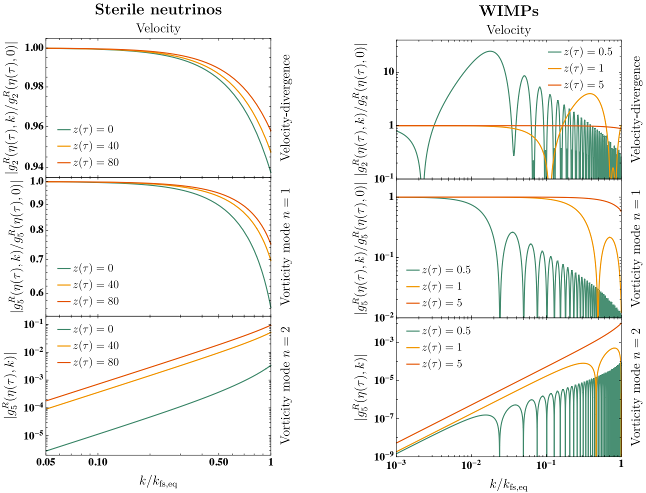

- Growth function in the long wavelength limit $k \to 0$
- analytic approximation works well for sterile neutrinos
- velocity dispersion perturbations partly grow

# *Transfer functions for density contrast*




- k-dependence of transfer functions for density contrast
- suppression for large $k$ due to velocity dispersion
- oscillations for large $k$ and at late times

# *Transfer functions for velocity*




- k-dependence of transfer functions for velocity divergence and vorticity
- suppression for large $k$ due to velocity dispersion
- oscillations for large $k$ and at late times (large contributions to integrals)

## *Conclusions & Outlook*

- velocity dispersion background / expectation value $\bar{\sigma}(\tau)$ evolves
- decays at early times but grows strongly due to non-linear backreaction at later times
- statistical field theory description of cosmological structure formation can include velocity dispersion to address small scales
- velocity dispersion might allow to distinguish between dark matter models
- vector and tensor perturbations get generated by velocity dispersion + non-linear terms
- 1-PI effective action + renormalization group approach can now be extended to smaller scales and later times

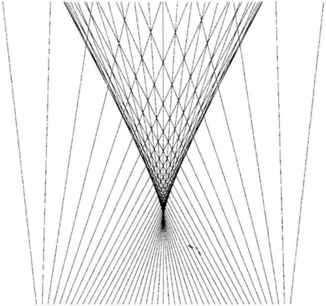
*Backup slides*

## *Non-linear corrections*

- beyond our current implementation, non-linear modifications for fluctuation fields must be taken into account
- leads to suppression of propagator
- suppressions cancels again approximately for equal time power spectrum
- backreaction effect on $\hat{\sigma}(\eta)$ could be even larger

## *Shell crossing*

- matter streams can cross and they typically will at late times



- particle velocity becomes locally multiple-valued
- caustic-like singularities
- velocity dispersion jumps discontinuously to

$$\sigma_{ij} \neq 0, \qquad \sum_j \sigma_{jj} > 0$$

- singular behavior gets regulated when one has

$$\bar{\sigma} = \frac{1}{3} \sum_j \langle \sigma_{jj} \rangle > 0$$

## *Analytic approximation*

- free-streaming wave number $k_{\mathsf{fs}}$ cuts the power spectrum in the UV
- for small enough $k_{\mathsf{fs}}$ an analytic approximation becomes available
- transfer functions can be approximated by $k = 0$ behavior
- density contrast and velocity divergence have standard growth functions

$$\tilde{D}_1(\eta) = e^{\eta - \eta_{\mathsf{in}}} \,, \qquad \tilde{D}_2(\eta) = e^{\eta - \eta_{\mathsf{in}}}$$

- scalar velocity dispersion modes have

$$\tilde{D}_3(\eta) = \frac{2}{3\,C_1}\,\tanh(H_1(\eta)) \,, \qquad \tilde{D}_4(\eta) = \frac{4}{3\,C_1}\,\tanh(H_1(\eta))$$

  with

$$H_1(\eta) \equiv C_1\,(e^{\eta - \eta_{\mathsf{in}}} - 1) + \mathsf{artanh}(C_2)$$

  and

$$C_1 \equiv \sqrt{\frac{2\,\sigma_d^2}{3}} \,, \qquad C_2 \equiv \frac{\sqrt{24\,\sigma_d^2}}{3 + \sqrt{9 + 24\,\sigma_d^2}}$$

- allows to find also $\hat{\sigma}(\eta)$