# *Fluctuations around Bjorken flow and the onset of turbulent phenomena*

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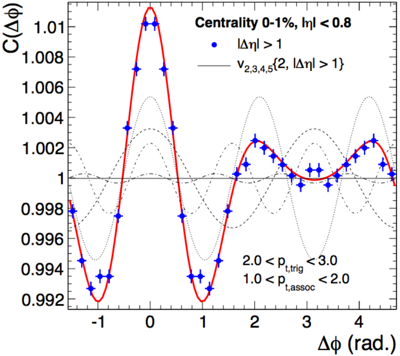
# *A puzzle: $v_3$ and $v_5$*




(ALICE, arXiv:1105.3865, similar pictures also from CMS, ATLAS, Phenix)

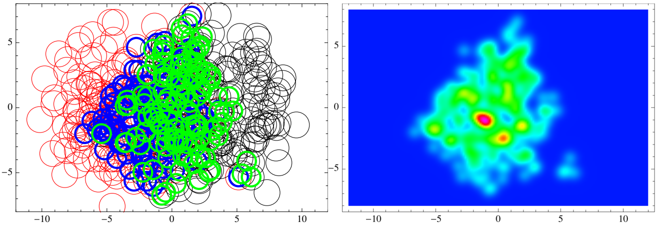
- quite generally, one can expand

$$C(\Delta\phi) \sim 1 + \sum_{n=2}^{\infty} 2\, v_n\, \cos(n\,\Delta\phi)$$

- from symmetry reasons one expects naively $v_3 = v_5 = \ldots = 0$

# *Event-by-event fluctuations*

- argument for $v_3 = v_5 = 0$ is based on smooth energy density distribution
- there can be deviations from this due to event-by-event fluctuations
- for example using a Glauber model




- this leads to sizeable $v_3$ and $v_5$

# *Generalized Glauber model*

- Fluctuations due to nucleon positions: used so far

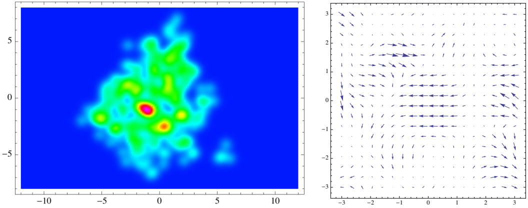
$$\epsilon(\tau, \mathbf{x}, y) = \sum_{i=1}^{N_{\text{part}}} \epsilon_w(\tau, \mathbf{x} - \mathbf{x}_i, y), \qquad u^\mu = (1, 0, 0, 0)$$

- can be generalized to include also velocity fluctuations

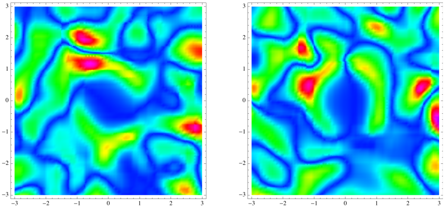
$$T^{\mu\nu}(\tau, \mathbf{x}, y) = \sum_{i=1}^{N_{\text{part}}} T_w^{\mu\nu}(\tau, \mathbf{x} - \mathbf{x}_i, y)$$

- More generally describe primordial fluid fields by
  - expectation values $\langle \epsilon(\tau_0, \mathbf{x}, y) \rangle, \langle u^\mu(\tau_0, \mathbf{x}, y) \rangle, \langle n_B(\tau_0, \mathbf{x}, y) \rangle$
  - correlation functions $\langle \epsilon(\tau_0, \mathbf{x}, y)\, \epsilon(\tau_0, \mathbf{x}', y') \rangle$, etc.
- Origin of this fluctuations is initial state physics and early-time, non-equilibrium dynamics.

# *Velocity fluctuations*



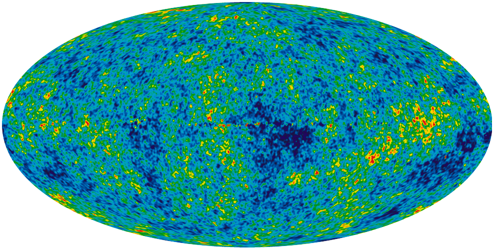

- take here a simple model where transverse velocity for every participant is Gaussian distributed with width $0.1c$
- vorticity $|\partial_1 u^2 - \partial_2 u^1|$ and divergence $|\partial_1 u^1 + \partial_2 u^2|$

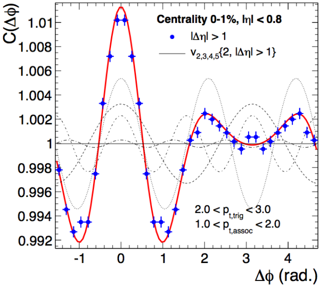


# *Why are fluctuations interesting?*

- **Hydrodynamic fluctuations**: Local and event-by-event perturbations around the average of hydrodynamical fields:
  - energy density $\epsilon$
  - fluid velocity $u^\mu$
  - more general also: baryon number density $n_B$, ...
- measure for deviations from equilibrium
- contain interesting information from early times
- an be used to constrain thermodynamic and transport properties
- might affect other phenomena, e.g. jet quenching

# *Similarities to cosmic microwave background*






- fluctuation spectrum contains info from early times
- many numbers can be measured and compared to theory
- detailed understanding of evolution needed
- could trigger precision era in heavy ion physics

## *Setup for treating fluctuations*

- ensemble average over many events with fixed impact parameter $b$ is described by smooth hydrodynamical fields

$$\bar{\epsilon} = \langle \epsilon \rangle$$
$$\bar{u}^\mu = \langle u^\mu \rangle$$

- fluctuations are added on top

$$\epsilon = \bar{\epsilon} + \delta\epsilon$$
$$u^\mu = \bar{u}^\mu + \delta u^\mu$$

- here we use Bjorkens simplified model (infinite extend in transverse plane)

$$\bar{\epsilon} = \bar{\epsilon}(\tau)$$
$$\bar{u}^\mu = (1, 0, 0, 0)$$
$$u^\mu = \bar{u}^\mu + \left(\delta u^\tau, u^1, u^2, u^y\right)$$

# *Linearized equations for fluctuations*

- consider only terms linear in $\delta\epsilon,\ (u^1, u^2, u^y)$
- decompose velocity field into
  - gradient term, described by divergence

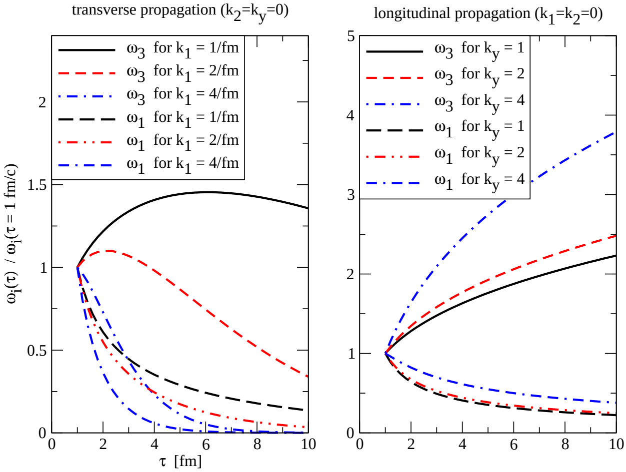
$$\theta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y$$

  - rotation term, described by vorticity

$$\begin{aligned} \omega_1 &= \tau\, \partial_2 u^y - \frac{1}{\tau} \partial_y u^2 \\ \omega_2 &= \frac{1}{\tau} \partial_y u^1 - \tau\, \partial_1 u^y \\ \omega_3 &= \partial_1 u^2 - \partial_2 u^1 \end{aligned}$$

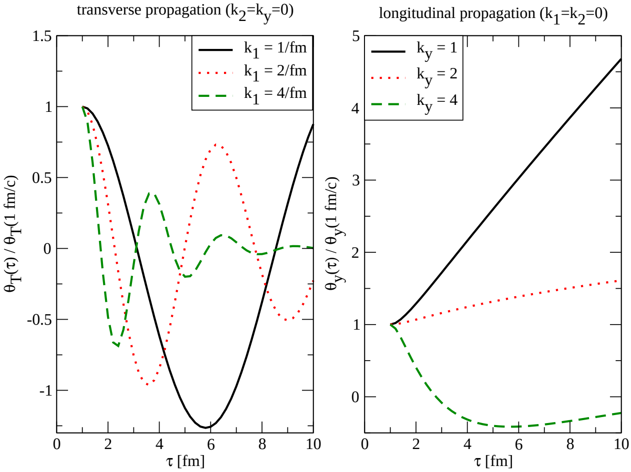
- $\theta$ and $\delta\epsilon$ are coupled: density or sound waves
- vorticity modes decouple from $\theta$ and $\delta\epsilon$

# *Vorticity modes*




- solve equations in Fourier space $\omega_j = \omega_j(\tau, k_1, k_2, k_y)$
- short wavelength modes get damped by viscosity
- some modes can grow, however!

# *Sound modes*




- oscillation for long times
- short wavelength modes get damped by viscosity
- also here some modes can grow!

# *Linear vs. non-linear evolution*

- for linearized theory one can easily determine two-point correlation function at late times from the one at early times
- more difficult for non-linear evolution of fluctuations
- evolution equation for $n$-point functions couple: *"Closure problem"* in fluid dynamics literature
- needs more elaborate tools: functional techniques, numerical simulation, ...
- ... but here we do something else: we map the problem to another one!

# *Limits of linearized theory*

linear approximation works for:

- energy density

$$\frac{\delta\epsilon}{\bar{\epsilon}} \ll 1$$

- velocity field

$$\mathsf{Re} \ll 1$$

large Reynolds number $\mathsf{Re} \gg 1$ leads to turbulence!

typical numbers: $T = 0.3\,\mathsf{Gev}$, $l = 5\,\mathsf{fm}$, $u_T = 0.1c$

$$\Rightarrow \quad \mathsf{Re} \approx \frac{1}{\eta/s} \approx \mathcal{O}(10)$$

# *Small Mach number*

$$\mathsf{Ma} = \frac{\sqrt{u_1 u^1 + u_2 u^2 + u_y u^y}}{c_S} \ll 1$$

- turbulent motion can be described as “compression-less”

$$\theta = \partial_1 u^1 + \partial_2 u^2 + \partial_y u^y = 0$$

- sound modes decouple from vorticity, they are much faster
- this does not mean that there are no sound waves present

# *Change of variables*

- kinematic viscosity, essentially independent of time

$$\nu_0 = \frac{\eta}{s\,T_{\mathsf{Bj}}(\tau_0)}$$

- new time variable (*not* laboratory time)

$$t = \frac{3}{4\tau_0^{1/3}}\,\tau^{4/3} \qquad\qquad \partial_t = \left(\frac{\tau_0}{\tau}\right)^{1/3}\partial_\tau$$

- rescaled velocity field

$$v_j = \left(\frac{\tau_0}{\tau}\right)^{1/3} u_j$$

- temperature field

$$d = \left(\frac{\tau_0}{\tau}\right)^{2/3} \ln\left(\frac{T}{T_{\mathsf{Bj}}(\tau)}\right)$$

# *Compression-less flow*

this leads to

$$\partial_t v_j + \sum_{m=1}^{2} v_m \partial_m v_j + \frac{1}{\tau^2} v_y \partial_y v_j + \partial_j d$$
$$-\nu_0 \left( \partial_1^2 + \partial_2^2 + \frac{1}{\tau^2} \partial_y^2 \right) v_j = 0.$$

- index $j = 1, 2, y$
- solenoidal constraint

$$\partial_1 v_1 + \partial_2 v_2 + \frac{1}{\tau^2} \partial_y v_y = 0$$

- for large times $\tau$ effectively *two-dimensional Navier-Stokes*!

# *Reynolds numbers*

assume

- typical velocity in transverse direction $v_T$
- typical velocity in rapidity direction $v_y$
- typical transverse length scale $l$
- typical rapidity difference $\Delta y$
- kinematic viscosity $\nu_0 = \frac{\eta}{s\,T}$

define

$$\mathrm{Re}^{(T)} = \frac{v_T l}{\nu_0}, \qquad \mathrm{Re}^{(y)} = \frac{v_y\, l^2}{\nu_0\, \Delta y}\frac{1}{\tau^2} \qquad \text{for} \quad \frac{l}{\tau \Delta y} \ll 1$$

and

$$\mathrm{Re}^{(T)} = \frac{v_T\, \tau^2\, \Delta y^2}{\nu_0\, l}, \qquad \mathrm{Re}^{(y)} = \frac{v_y\, \Delta y}{\nu_0} \qquad \text{for} \quad \frac{l}{\tau \Delta y} \gg 1$$

# *Reynolds numbers 2*

- depending on $\mathrm{Re}^{(T)}$ and $\mathrm{Re}^{(y)}$ there are different regimes
- for many initial conditions one has $\mathrm{Re}^{(T)} \gg 1$ at large $\tau$
- this implies two-dimensional turbulent behavior!

# *Describing turbulence*

- turbulence is best described statistically
- hydrodynamic fluctuations in general have probability distribution

$$p_\tau[u^\mu(\tau, x_1, x_2, y), \epsilon(\tau, x_1, x_2, y)]$$

- assume this probability distribution to have the symmetries:
  - translational and rotational symmetries in transverse plane
  - Bjorken boost invariance
- implies for expectation values

$$\langle u^\mu(\tau, x_1, x_2, y)\rangle = (1, 0, 0, 0), \quad \langle \epsilon(\tau, x_1, x_2, y)\rangle = \bar{\epsilon}(\tau)$$

and for correlation functions

$$\langle u^i(\tau, x_1, x_2, y)\, u^j(\tau, x_1', x_2', y')\rangle = G_u^{ij}(\tau, |\mathbf{x} - \mathbf{x}'|, y - y')$$

## *Velocity spectrum*

in two-dimensional situation

$$\langle v_m(t,\mathbf{x})v_n(t,\mathbf{x}')\rangle = \int \frac{d^2k}{(2\pi)^2} e^{i\mathbf{k}(\mathbf{x}-\mathbf{x}')} \left(\delta_{mn} - \frac{k_m k_n}{k_1^2 + k_2^2}\right) \frac{2\pi}{k} E(t,k)$$

the function $E(t,k)$ describes how kinetic energy is distributed over different length scales

# *Turbulence in $d = 3$*

fully developed turbulence

$$\text{Re} \to \infty$$

dissipated energy per unit time

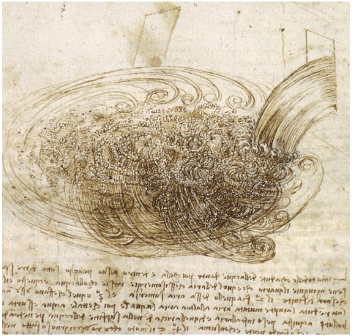
$$\frac{d}{dt}\langle \vec{v}^2 \rangle = -\nu_0 \left\langle (\vec{\nabla} \times \vec{v})^2 \right\rangle = -\varepsilon$$

RICHARDSON (1922):

*Big whorls have little whorls,*
*Which feed on their velocity;*
*And little whorls have lesser whorls,*
*And so on to viscosity.*

KOLMOGOROV (1941):

$$E(k) \sim \varepsilon^{2/3} k^{-5/3}$$



L. DA VINCI (CA. 1500)

# *Turbulence in $d = 2$*

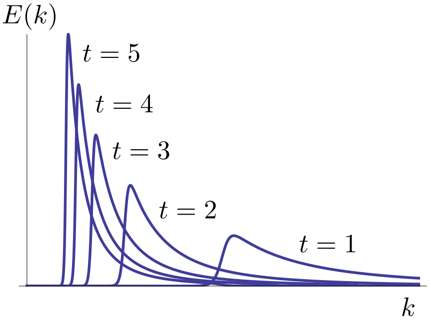
KRAICHNAN (1967):

- vorticity is conserved for $\nu_0 \to 0$
- scaling theory of forced turbulence in $d = 2$
- inverse cascade of energy to small wave numbers !
- cascade of vorticity to large wave numbers

$$E(k) \sim k^{-3}$$

- qualitatively different to $d = 3$, emerges here dynamically

# *Decaying turbulence in $d = 2$*




BATCHELOR (1969):

- scaling theory of decaying turbulence in $d = 2$

$$E(t,k) = \lambda^3\, t\; f(k\,\lambda\,t) \qquad \text{with} \qquad \lambda^2 = \langle \vec{v}^2 \rangle = \text{const.}$$

- turbulent motion goes to smaller and smaller wave numbers

# *Hydrodynamic fluctuations and freeze-out*

- for *one* configuration of hydrodynamic fields one can use traditional freeze-out prescription, e.g. Cooper-Frye

$$E\frac{dN}{d^3p} = \int \frac{p_\mu d\Sigma^\mu}{(2\pi)^3} f(x,p)$$

where for ideal Boltzmann gas

$$f(x,p) = d\, e^{\frac{p_\mu u^\mu(x)}{T(x)}}$$

- the experimental measurement corresponds to an ensemble average of this, however

$$f(x,p) \to \langle\, f(x,p)\,\rangle = d \left\langle e^{\frac{p_\mu u^\mu(x)}{T(x)}} \right\rangle$$

where the averaging is over configurations of $u^\mu(x)$, $T(x)$.

## ... and similar for the two-particle spectrum

$$E_A E_B \frac{dN}{d^3p_A d^3p_B} = \int (p_A)_\mu d\Sigma^\mu \; (p_B)_\nu d\Sigma'^\nu \left\langle f(x, p_A) \; f(x', p_B) \right\rangle$$
$$+ s_{B/F} \int \tfrac{1}{2}(p_A + p_B)_\mu d\Sigma^\mu \; \tfrac{1}{2}(p_A + p_B)_\nu d\Sigma'^\nu$$
$$\times e^{i(p_A - p_B)_\mu (x - x')^\mu} \left\langle f(x, \tfrac{p_A + p_B}{2}) \; f(x', \tfrac{p_A + p_B}{2}) \right\rangle$$

- with $s_{B/F} = \pm 1$ for identical bosons/fermions and $s_{B/F} = 0$ otherwise
- There are now also cross-correlations between hydro-fluctuations at $x$ and $x'$ playing a role
- These lead to deviations from factorization!!

## *One-particle spectra*

- fluctuations in fluid fields modify the one-particle spectra
- up to quadratic order in fluctuations

$$\frac{dN}{d^3p} = \frac{dN_0}{d^3p} + \frac{d\delta N_1}{d^3p}\langle(u^1)^2\rangle + \frac{d\delta N_2}{d^3p}\langle(u^y)^2\rangle + \frac{d\delta N_3}{d^3p}\langle(T - T_{\mathsf{fo}})^2\rangle$$

- explicit expressions can be derived
- modifies also $v_n(p_T)$ and HBT radii
- effect qualitatively similar to viscosity correction to spectrum
- must be analyzed more quantitatively for phenomenology
- depends only on a few numbers

# *Two-particle spectra*

- correlation function of particles with momenta $\vec{p}_A$ and $\vec{p}_B$

$$C(\vec{p}_A, \vec{p}_B) = \frac{\frac{dN}{d^3p_A d^3p_B}}{\frac{dN}{d^3p_A}\frac{dN}{d^3p_B}}.$$

- particular interesting are *identical particles*
- $C(\vec{p}_A, \vec{p}_B)$ depends on *correlation functions* of hydrodynamic fields at different space-time points, e.g. $\langle T(x)T(x')\rangle$
- characteristic power-law decay with $|\vec{p}_A - \vec{p}_B|$ in turbulent situation
- allows in principle to test Kraichnans law $E(k) \sim k^{-3}$

# *Effects on macroscopic motion of fluid*

- turbulent fluctuations might affect macroscopic motion
  - modified equation of state
  - modified transport properties
- anomalous, turbulent or eddy viscosity
  - proposed by Asakawa, Bass, Müller (2006) for plasma turbulence and Romatschke (2007) for fluid turbulence
  - could become negative in $d = 2$ (Kraichnan (1976))
  - depends on detailed state of turbulence – not universal
  - gradient expansion needs separation of scales
- more work needed

# *Summary*

- Fluid fluctuations contain interesting information about:
  - Early time dynamics
  - Equation of state
  - Transport properties
- All kinds of fluid fluctuations should be propagated:
  - Density fluctuations
  - Velocity fluctuations
  - Baryon number fluctuations, Electric charge fluctuations,...
  - Deviations from smooth geometry in transverse plane
  - Deviations from Bjorken boost invariance
- Here we investigated mainly velocity fluctuations:
  - Vorticity modes can grow
  - New physics phenomenon: Onset of fluid turbulence
  - Interesting effects on one- and two-particle spectra
- Contribution of hydrodynamic fluctuations need not factorize in two-particle spectrum

# Backup

# *Little Bang vs. Big Bang*

**Heavy Ions**

Bjorken model

$$x^\mu = (\tau, x^1, x^2, y)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \tau^2 \end{pmatrix}$$

$$\epsilon_0(\tau), \quad u_0^\mu = (1,0,0,0)$$

+ hydrodyn. fluctuations

$$\epsilon = \epsilon_0(\tau) + \epsilon_1(\tau, x^1, x^2, y)$$
$$u^\mu = u_0^\mu + u_1^\mu(\tau, x^1, x^2, y)$$

**Cosmology**

Friedmann-Robertson-Walker

$$x^\mu = (t, x^1, x^2, x^3)$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a(t) & & \\ & & a(t) & \\ & & & a(t) \end{pmatrix}$$

$$\epsilon_0(t), \quad u_0^\mu = (1,0,0,0)$$

+ hydrodyn. fluctuations

$$\epsilon = \epsilon_0(t) + \epsilon_1(t, x^1, x^2, x^3)$$
$$u^\mu = u_0^\mu + u_1^\mu(t, x^1, x^2, x^3)$$

+ gravity fluctuations