On Recent Low Energy Approximations To Pure Gauge Theories

Andreas Wipf, TPI Jena

Tübingen, September 06, 2001

in collaboration with:

Thomas Heinzl (Jena)
Leander Dittmann (Jena)
Pierre van Baal (Leiden)

OUTLINE:

- Cho connection and Abelian Projections
- Low Energy Approximations
- Toward calculating $S_{\rm eff}$ by Inverse MC

Cho connection and Abelian Projections

Starting point: CHO CONNECTION:

$SU(2)$: given $n(x) = \vec{n}(x)\vec{\sigma}, \quad \vec{n} \cdot \vec{n} = 1$

$$\hat{A} = (A,n)n + i[n,dn] = \hat{A}(A,n)$$

PROPERTIES:

- $\hat{D}n = 0$ covariantly constant
- $(A,n) = (\hat{A},n)$
- For constant n : $\hat{A} = (A,n)n$

Conversely:

$$Dn = 0 \Longrightarrow A_\mu = C_\mu n + i[n, \partial_\mu n] \quad \text{reducible}$$

GAUGE TRANSFORMATIONS:

$$\begin{aligned} A &\rightarrow V(A+id)V^{-1} = {}^V\!A \\ n &\rightarrow VnV^{-1} = {}^V\!n \end{aligned}$$

$$\Rightarrow \hat{A}({}^V\!A, {}^V\!n) = {}^V\!\hat{A}(A, n).$$

restricted and reducible gauge connection!

GENERALIZATION FOR $SU(N)$ (SIMPLY LACED):
n_i: r commuting orthogonal generators. Demand

$$\hat{A}({}^V\!A, {}^V\!n_i) = {}^V\!\hat{A}(A, n_i) \quad , \quad \hat{A}(dn_i{=}0, A) = P_n(A)$$

$$\hat{A} = P_n(A) + i \sum [n_i, dn_i]$$

reducible with holonomy group $\supset U^r(1)$.

Abelian Gauges/Projections

Extracting n from A. Decompose

$$A = \hat{A}(A, n) + X$$

$A, \hat{A}$ transform as gauge fields: X *in adjoint reps.*

MAG: given A, define

$$F[n] = \|A_\mu - \hat{A}_\mu(n, A)\|$$

$\hat{A}$ near A: minimizes F; $n = V^{-1} n^* V,\ n^*$ constant

$$\min_n F[n] = \min_V \left\| A - \hat{A}(A, V^{-1} n^* V) \right\|$$
$$= \min_V \left\| {}^V\!A - \hat{A}({}^V\!A, n^*) \right\| = \min_V \left\| \left({}^V\!A\right)^\perp \right\|.$$

PAG:

$\hat{A}_0 = (A_0, n)n + i[n, \partial_0 n], \quad \|A_0 - \hat{A}_0\|$ minimal

Singularities of n: position of magnetic monopoles

relation $n \longrightarrow$ instantons, monopoles

Abelian projections:

- Abelian gauge fixing: $\min F[n]$, residual $U(1)$
- Truncation: throw away X in 'measured' observable

Low Energy Approximations

- n encodes information about *topological defects*
- should be good variables for QCD in infrared (?)

Idea (lattice): $A = \hat{A} + X$

$$
\begin{aligned}
\int \mathcal{D}A e^{-S} O(A) &= \int \mathcal{D}\hat{A}\mathcal{D}X e^{-S[\hat{A},X]} O(\hat{A},X) \\
&\overset{\text{AG}}{\sim} \int \mathcal{D}C\mathcal{D}n\mathcal{D}X \text{ (J FP) } e^{-S[C,n,X]} O(C,n) \\
&= \int \mathcal{D}C\mathcal{D}n\, e^{-S_{\text{eff}}[C,n]} O(C,n)
\end{aligned}
$$

One step further

$$
e^{-S_{\text{eff}}[n]} \overset{\text{AG}}{=} \int \mathcal{D}C\mathcal{D}X \text{ (J FP) } e^{-S[C,n,X]}
$$

What is effective action $S_{\text{eff}}[n]$?

Skyrme–Faddeev–Niemi Action

$$\begin{aligned}\mathcal{L}_{\rm SFN} &= \frac{m^2}{2}(\partial_\mu \vec{n})^2 - \frac{1}{4e^2}H_{\mu\nu}^2 \\ &\equiv \text{'sigma–model'} + \text{‘Skyrme term’}\end{aligned}$$

- ‘field strength’ $H_{\mu\nu} \equiv (\vec{n}, \partial_\mu \vec{n} \times \partial_\nu \vec{n})$
- $\tilde{A} = \frac{1}{2i}[n, dn] \Rightarrow F_{\mu\nu}(\tilde{A}) = 8H_{\mu\nu}$ classical term
- dynamically generated mass scale m

FN: “*unique* local and Lorentz invariant action for the unit vector $\vec{n}$ which is at most quadratic (!) in time derivatives ... and involves *all* such terms that are either relevant or marginal in the infrared”

$S_{\rm SFN}$ supports stable knot–like (i.e. closed) *solitons*

FADDEEV–NIEMI CONJECTURE:

- $S_{\rm SFN}$ low–energy effective action for YM-theory
- knot solitons $\simeq$ glue-balls

- Bound on energy of static configurations: rescale x

$$\frac{2e}{m}E = \tilde{E} = \int d^3x \Big((\nabla \vec{n})^2 + \frac{1}{2} H_{ij} \Big)$$

configuration space consists of topological sectors

$$\vec{n}\big(|\vec{x}| \to \infty\big) = \vec{n}_0 \qquad \vec{n} : S^3 \to S^2$$

$$\pi_2(S^3) = \mathbb{Z} \qquad \text{classified by } Q$$

$\vec{n}(C_i) = \vec{n}_i \Rightarrow Q =$ linking number of loops C_1, C_2
Hopf invariant Q $\sim$ instanton number

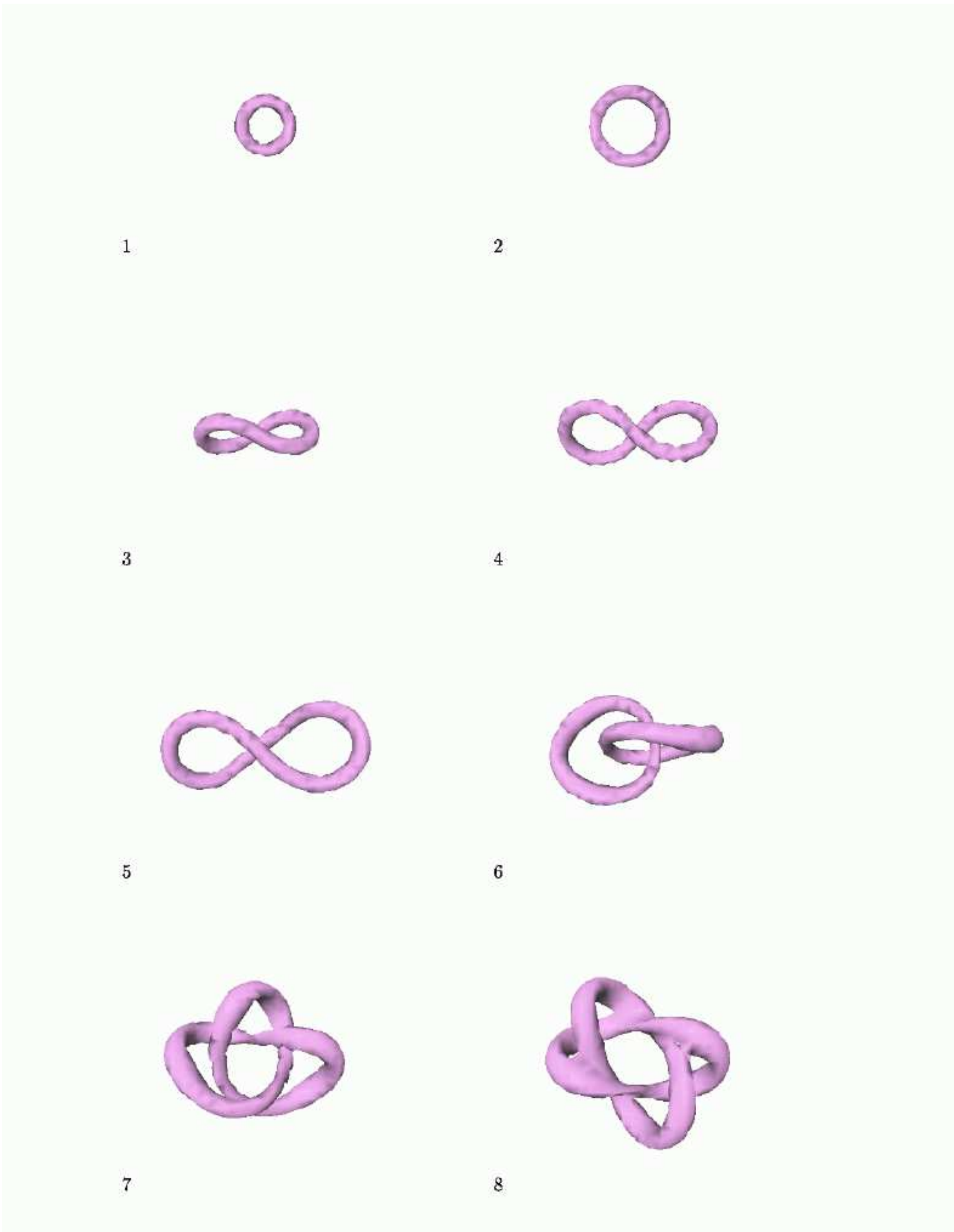
Valenko and Kapitansky (1979):

$$\tilde{E} \geq C|Q|^{3/4}, \qquad C = 16\pi^2 3^{3/8} \sim 238$$

Can be improved. Is this a BPS-bound?

- Extensive numerical studies

Battye/Sutcliffe '98:



Problems

- interpretation of n ?
- relation to Yang Mills? (cf. Gies 2001)
- $H^2 = (\vec{n}\Box\vec{n})^2 - (\vec{n}\partial_\mu\partial_\nu\vec{n})^2$; why same coupling?
- $S_{\rm SFN} \to$ spontaneous SB: SO(3) $\to$ S0(2)

2 Goldstone bosons, no mass gap

Reformulations: (PLB 515 (01) 181)

- The CP_1 formulation: $n_a = z^\dagger \tau_a z,\; z \in \mathbb{C}^2,\; z^\dagger z = 1$

$$\begin{aligned} \mathcal{L}_{\rm SFN} &= \frac{m^2}{2}(D_\mu z)^\dagger D^\mu z - \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} \\ Q &= \frac{1}{4\pi^2}\int d^3x\, A \wedge F, \quad \text{CS-term} \end{aligned}$$

gauge potential = composite:

$$A = -iz^\dagger dz, \quad D_\mu = \partial_\mu - iA_\mu, \quad F = dA$$

- The $SU(2)/U(1)$ formulation

$$n_a(x) = \frac{1}{2}\text{tr}\Big(\tau_3 g^\dagger(x)\tau_a g(x)\Big).$$

Introduce currents $J = g^\dagger dg \sim$ *flat potential*

$$A \wedge F = -\tfrac{1}{2}\,\text{tr}\big(J \wedge dJ + \tfrac{2}{3}J \wedge J \wedge J\big)$$

$$E \sim \int d^3x \Big\{ \big(J_i^1 J_i^1 + J_i^2 J_i^2\big) + \tfrac{1}{2}\big(J_i^1 J_j^1 - J_j^2 J_i^1\big)^2 \Big\}$$

- $Q =$ gauge field winding number
- first term of E: defines MAG
- $E \sim$ gauge fixing functional for a non-linear MAG

minima of E (Q fixed) $\sim$ gauge fixed pure gauge potentials in a sector with winding number Q.

$\rightarrow$ relation of pure gauge theory vacua to knots

Toward calculating S_{eff}

non-local S_{eff} should vary slowly with $x \to$

- systematic derivative expansion

$$S_{\text{eff}}[n] = \sum_i \lambda_i \, S_i[n]$$

operators S_i, coupling λ_i

- list of symmetric operators:

$\partial\vec{n} \cdot \partial\vec{n}$	$\Box\vec{n} \cdot \Box\vec{n}$	$(\vec{n} \cdot \Box\vec{n})^2$	$(\vec{n} \cdot \partial_\mu\partial_\nu\vec{n})^2$

- list of symmetry breaking operators ('ext.field' h):

$\vec{n} \cdot \vec{h}$	$(\vec{n} \cdot \vec{h})^2$	$(\partial\vec{n} \cdot \partial\vec{n})(\vec{n} \cdot \vec{h})$

Explicit calculations for $SU(2)$:

- generate lattice configurations $\sim e^{-S_{\text{YM}}[A]}$

- fix to lattice Landau gauge by maximizing

$$F_{\mathrm{LLG}}[\,^{V}U] = \sum_{x,\mu} \mathrm{tr}\Big(^{V}U_{\mu}(x) + \mathrm{h.c.}\Big)$$

$\rightarrow$ residual global $SU(2)$

- gauge transform to MAG by maximizing

$$F_{\mathrm{MAG}} = \sum_{x,\mu} \mathrm{tr}\left(\tau_3 \,^{V}U_{\mu}(x) \tau_3 \,^{V}U^{\dagger}_{\mu}(x)\right)$$

residual local $U(1)$, 3–direction distinguished

- chose g_x, defined by

$$U_{\mathrm{LLG}} \xrightarrow{g} U_{\mathrm{MAG}}$$

to define (almost gauge invariant) $n_x \equiv g^{\dagger}_x \tau_3 g_x$

$\{n_x\}$ distributed $\sim \exp(-S_{\mathrm{YM}})$ in LLG

global $SU(2)$ broken $explicitly$ down to $U(1)$

Calculating the couplings

λ_i via *inverse MC* (Parisi et al. '86)

- SD-equations = overdetermined *linear system* for λ_i

$$\sum_j \left\langle F_i[n](\delta^{ab} - n^a n^b) S^j_{,b} \right\rangle \lambda_j = \left\langle I^a[n] \right\rangle$$

F_i, I^a known functions of n

expectation values calculated with $\exp(-S_{\rm YM})$

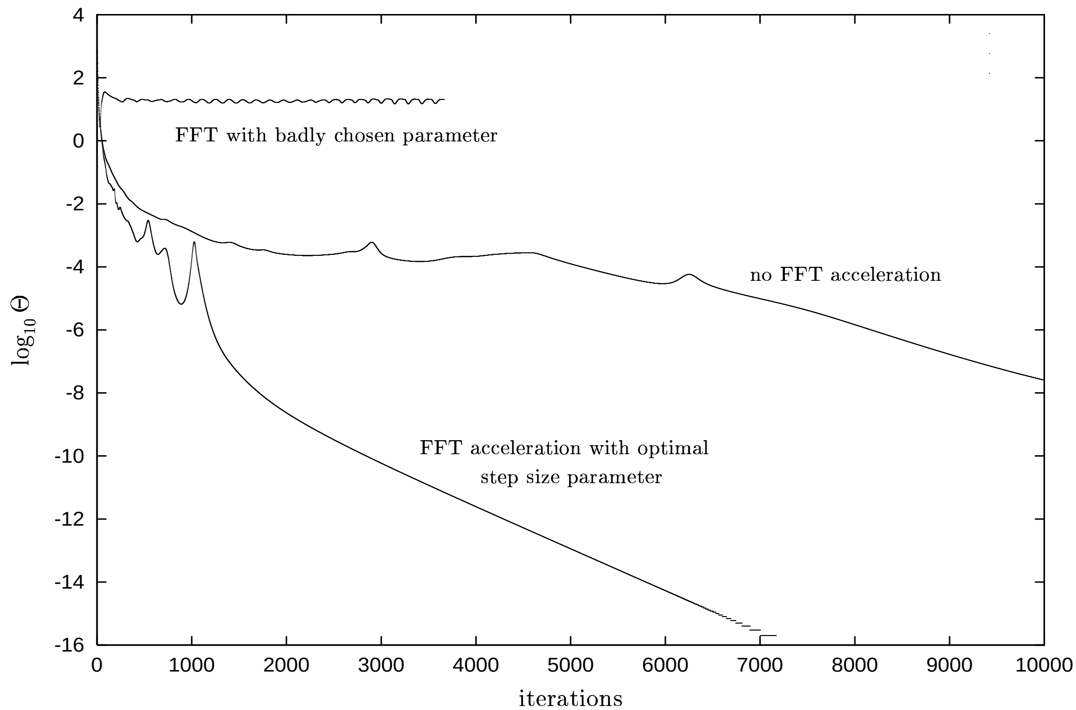
least square fit

Technicalities:

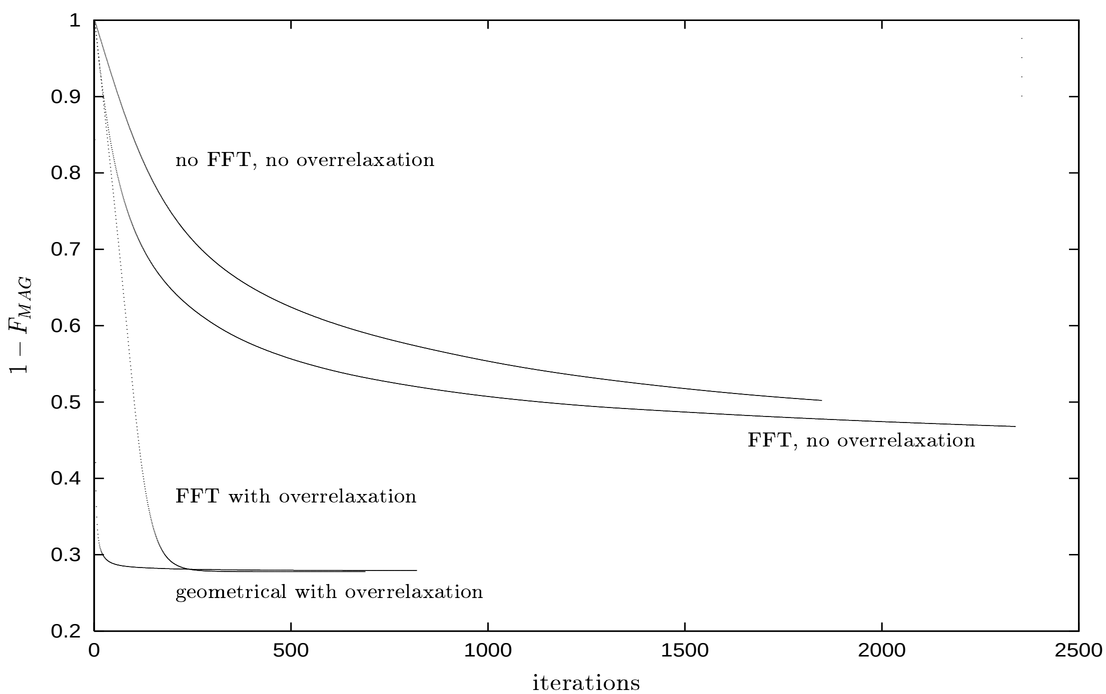
- lattice: $V = 16^4$, $\beta = 2.35$, $a \simeq 0.13$ fermi
- LLG: Fourier acc. steepest descent, 6000 iterations
- MAG: two independent algorithms

  MAGI: 'geometrical' iterations + over-relaxation

  MAGII: as LLG + over-relaxation



LLG: with and without FFT

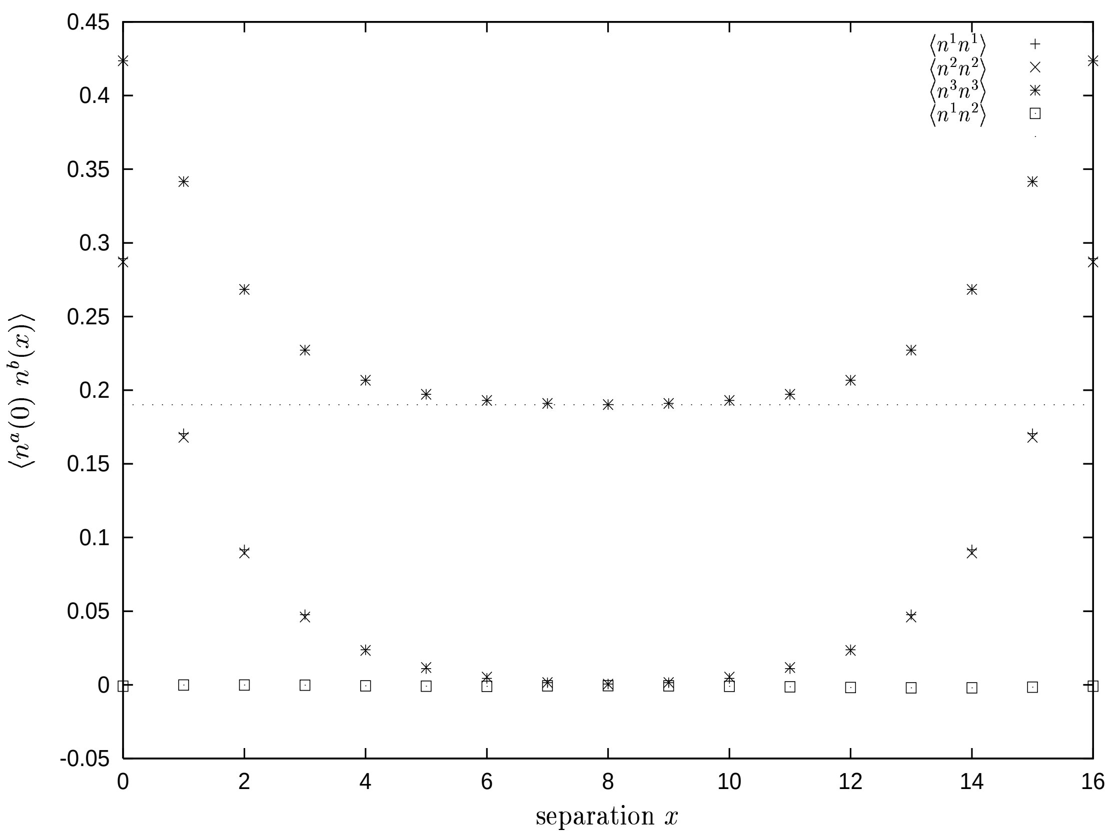


MAG: different algorithms

exhibits explicit SB $SO(3) \to SO(2)$

$$\langle n^1 \rangle = \langle n^2 \rangle = 0 \quad , \quad \langle n^3 \rangle = 0.44 \;\; (0.35)$$

$$\begin{aligned} G^{\|}(x) &= \Big\langle n^3(x) n^3(0) \Big\rangle \\ G^{\perp}(x) &= \frac{1}{2}\Big\langle n^a(x) n^a(0) \Big\rangle, \quad a = 1, 2 \end{aligned}$$



Two-point functions with MAG I; clustering

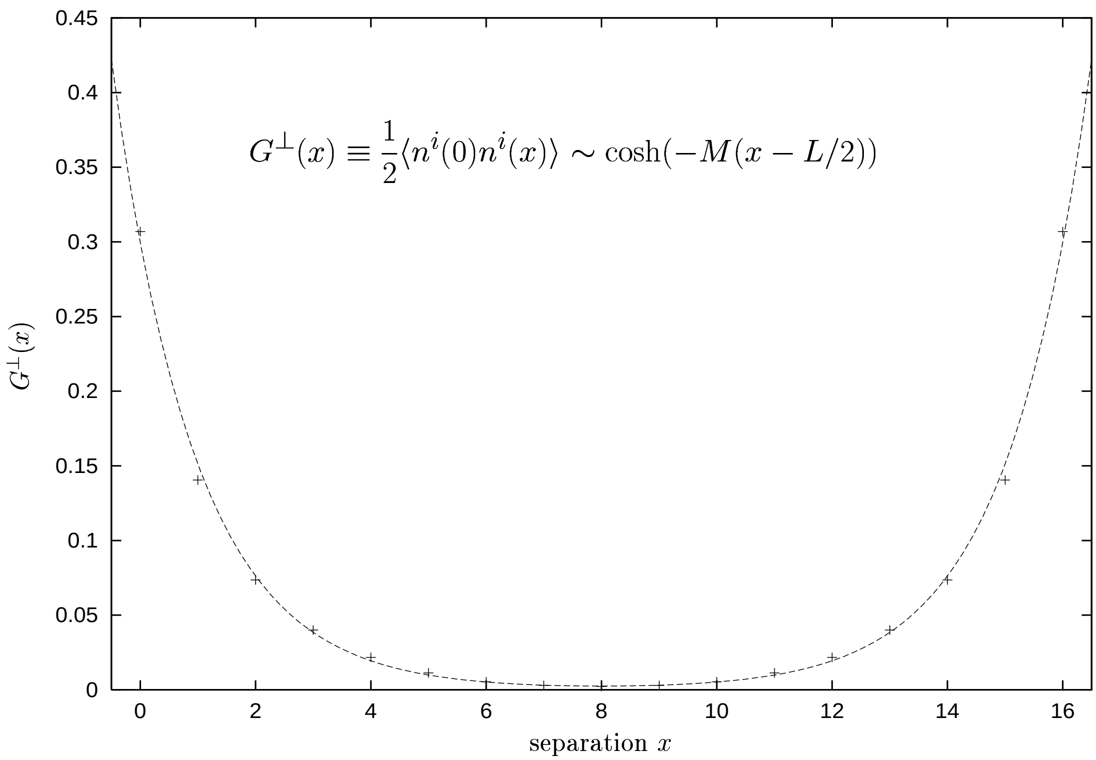
Magnetization $\langle n^a \rangle = \mathfrak{M}\,\delta^{a3}$

susceptibility $\chi^{\perp} \equiv \sum_x G^{\perp}(x)$

- Numerical values:

MAGI: $\mathfrak{M} = 0.436 \quad \chi^{\perp} = 0.636$

MAGII: $\mathfrak{M} = 0.352 \quad \chi^{\perp} = 0.596$



Cosh-fit to $G^{\perp} \rightarrow M = 1.0$ *GeV*

Comparison with minimal ansatz

$$S_{\text{eff}} = \frac{1}{2}\lambda_1(\partial_\mu \vec{n})^2 + \lambda_2\, \vec{n}\cdot\vec{h}$$

mass gap $M^2 = h + \mathcal{O}(p^4, h^2)$

Ward identity: $M \simeq (\mathfrak{M}/\chi^{\perp})^{1/2} \simeq 1.2$ GeV

From cosh-fit to $G^{\perp}(x)$: $M \sim 1.0$ GeV $\rightarrow$

gradient expansion of S_{eff} reasonable

CHECKED:

- explicit (not spontaneous) symmetry breaking
- $\langle \mathrm{tr}\mathcal{P}\rangle(\beta)$ ($N_s = 20, N_t = 4$), Binder cum: $\beta_c = 2.325$

FUTURE:

- λ_i from DS-equations via inverse MC
- inclusion of higher derivative terms