Faddeev-Niemi-Skyrme Models

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OUTLINE:

- CHO-connection and Abelian Projections
- Low Energy Approximations
- Toward calculating S_{eff} by Inverse MC

Cho connection and Abelian Projections

$SU(2)$: given $\boldsymbol{n}(x) \in S^2,\ \boldsymbol{n}\cdot\boldsymbol{n} = 1,\ n = n_a \sigma_a$

$$\hat{A}_\mu = (n, A_\mu)n + i[n, \partial_\mu n] = \hat{A}_\mu(A, n)$$

Properties:

- $\hat{D}_\mu n = 0$
- $(A_\mu, n) = (\hat{A}_\mu, n)$
- n^* constant: $\hat{A}_\mu = (A_\mu, n^*)n^*$

Conversely:

$$D_\mu n = 0 \Longrightarrow A_\mu = C_\mu n + i[n, \partial_\mu n] \quad \text{reducible}$$

Gauge transformations:

$$\begin{aligned} A_\mu &\longrightarrow {}^V\!A_\mu = V(A_\mu + i\partial_\mu)V^{-1} \\ n &\longrightarrow {}^V\!n = VnV^{-1} \end{aligned}$$

$$\Rightarrow \hat{A}_\mu({}^V\!A, {}^V n) = {}^V\hat{A}_\mu(A, n).$$

$\boxed{\text{restricted and reducible gauge connection!}}$

GENERALIZATION FOR $SU(N)$ (SIMPLY LACED):
n_i: r commuting orthogonal generators. Demand

$$\hat{A}_\mu({}^V\!A, {}^V n_i) = {}^V\hat{A}_\mu(A, n_i) \quad , \quad \hat{A}_\mu(n_i^*, A) = P_{n^*}(A)$$

$$\longrightarrow \hat{A}_\mu = P_n(A_\mu) + i \sum [n_i, \partial_\mu n_i]$$

reducible with holonomy group $\supset U^r(1)$.

Abelian Gauges/Projections for $SU(2)$

$\longrightarrow n(A)$. Decompose

$$A_\mu = \hat{A}_\mu(A, n) + X_\mu$$

$A_\mu, \hat{A}_\mu$ gauge potentials $\rightarrow$ X_μ *in adjoint reps.*

MAG: A_μ given

$$\text{minimize} \quad F_{\mathrm{MAG}}[n] = \|A_\mu - \hat{A}_\mu(A, n)\|$$

$$\min_n F[n] = \min_V \left\| A_\mu - \hat{A}_\mu(A, \overbrace{V^{-1} n^* V}^{n}) \right\|$$
$$= \min_V \left\| {}^V\!A_\mu - \hat{A}_\mu({}^V\!A, n^*) \right\| = \min_V \left\| \left({}^V\!A_\mu\right)^\perp \right\|.$$

PAG: A_0 given

$$\text{minimize} \quad F_{\mathrm{Pol}}[n] = \|A_0 - \hat{A}_0\|$$

Singularities of n: position of magnetic monopoles[a]

$$\boxed{\text{relation } n \longrightarrow \text{instantons, monopoles}}$$

Abelian projections:

- Abelian *gauge fixing*: $\min F[n]$, residual $U(1)$
- *Truncation*: throw away X_μ in 'measured' object

[a] H. Reinhardt et.al; F. Lenz et.al; A.W. et.al

Low Energy Approximations

- $n \to$ *topological defects*
- good variable for QCD in infrared?

Would like: $A_\mu = \hat{A}_\mu + X_\mu$

$$
\begin{aligned}
\int \mathcal{D}A e^{-S} O(A) &= \int \mathcal{D}\hat{A}\mathcal{D}X e^{-S[\hat{A},X]} O(\hat{A},X) \\
&\overset{\text{AG}}{\sim} \int \mathcal{D}C\mathcal{D}n\mathcal{D}X \text{ (J FP)}\, e^{-S[C,n,X]} O(C,n) \\
&= \int \mathcal{D}C\mathcal{D}n\, e^{-S_{\text{eff}}[C,n]} O(C,n)
\end{aligned}
$$

One step further

$$
e^{-S_{\text{eff}}[n]} \overset{\text{AG}}{=} \int \mathcal{D}C\mathcal{D}X \text{ (J FP)}\, e^{-S[C,n,X]}
$$

$\boxed{\text{What is effective action } S_{\text{eff}}[n] = S_{\text{eff}}[\boldsymbol{n}]?}$

Faddeev–Niemi Proposal

$$\begin{aligned} \mathcal{L}_{\mathrm{SFN}} &= \frac{m^2}{2}(\partial_\mu \boldsymbol{n})^2 - \frac{1}{4e^2} H_{\mu\nu}^2 \\ &\equiv \text{'sigma–model'} + \text{'Skyrme term'} \end{aligned}$$

- 'field strength' $H_{\mu\nu} \equiv (\boldsymbol{n}, \partial_\mu \boldsymbol{n} \times \partial_\nu \boldsymbol{n})$
- $\tilde{A}_\mu = \frac{1}{2i}[n, \partial_\mu n] \Rightarrow F_{\mu\nu}(\tilde{A}) = 8 H_{\mu\nu}$ classical
- *dynamically* generated mass scale m!

FN: "*unique local and Lorentz invariant action for the unit vector* $\boldsymbol{n}$ *which is at most quadratic (!) in time derivatives ... and involves all such terms that are either relevant or marginal in the infrared*"

S_{SFN} supports stable knot–like *solitons*

FADDEEV–NIEMI CONJECTURE:

- S_{SFN} low–energy effective action for YM-theory
- knot solitons $\simeq$ glue-balls

- Bound on energy of static configurations: rescale x

$$\frac{2e}{m}E = \tilde{E} = \int d^3x\Big((\nabla \boldsymbol{n})^2 + \frac{1}{2}H_{ij}\Big)$$

$$\boxed{\text{configuration space consists of topological sectors}}$$

$$\boldsymbol{n}\big(|\vec{x}| \to \infty\big) = \boldsymbol{n}_0 \qquad \boldsymbol{n} : S^3 \to S^2$$

$$\pi_2(S^3) = \mathbb{Z} \qquad \text{classified by } Q$$

$\boldsymbol{n}(C_i) = \boldsymbol{n}_i \Rightarrow Q =$ linking number of loops C_1, C_2
Hopf invariant Q $\sim$ instanton number

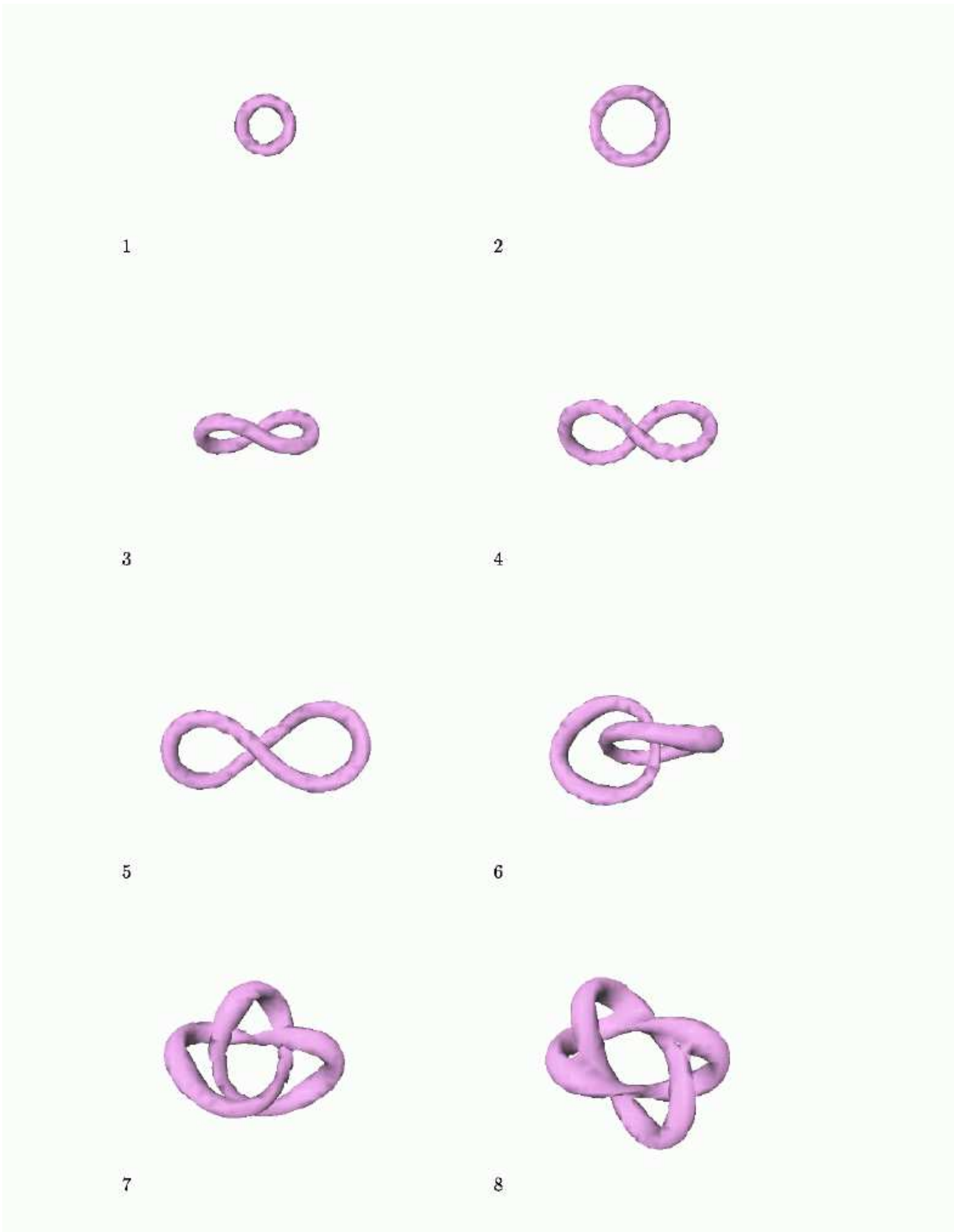
Valenko and Kapitansky (1979):

$$\tilde{E} \geq C|Q|^{3/4}, \qquad C = 16\pi^2 3^{3/8} \sim 238$$

Can be improved. Is this a BPS-bound?

- Extensive numerical studies

Battye/Sutcliffe '98:



Problems

- relation to Yang Mills? (cf. Gies 2001)
- $H^2 = (\boldsymbol{n}\Box\boldsymbol{n})^2 - (\boldsymbol{n}\partial_\mu\partial_\nu\boldsymbol{n})^2$; why same coupling?
- $S_{\mathrm{SFN}} \to$ SSB: SO(3) $\to$ S0(2)

$$\boxed{\text{2 Goldstone bosons, no mass gap???}}$$

where are these Goldstones?

Reformulations: (PLB 515 (01) 181)

- CP_1 formulation: $n_a = z^\dagger \tau_a z,\ z \in \mathbb{C}^2,\ z^\dagger z = 1$

$$\begin{aligned} \mathcal{L}_{\mathrm{SFN}} &= \frac{m^2}{2}(D_\mu z)^\dagger D^\mu z - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} \\ Q &= \frac{1}{4\pi^2}\int d^3x\ A \wedge F, \quad \text{CS-term} \end{aligned}$$

gauge potential = composite:

$$A = -iz^\dagger dz, \quad D_\mu = \partial_\mu - iA_\mu, \quad F = dA$$

- The $SU(2)/U(1)$ formulation

$$n_a(x) = \frac{1}{2}\mathrm{tr}\Big(\tau_3 g^\dagger(x)\tau_a g(x)\Big).$$

current $J = g^\dagger dg \sim$ *flat* potential

$$A \wedge F = -\tfrac{1}{2}\,\mathrm{tr}\big(J \wedge dJ + \tfrac{2}{3} J \wedge J \wedge J\big)$$
$$E \sim \int d^3x \Big\{ \big(J_i^1 J_i^1 + J_i^2 J_i^2\big) + \tfrac{1}{2}\big(J_i^1 J_j^1 - J_j^2 J_i^2\big)^2 \Big\}$$

- $Q =$ gauge field winding number
- first term of E: defines MAG
- $E \sim$ gauge fixing functional for a non-linear MAG

minima of E (Q fixed) $\sim$ gauge fixed pure gauge potentials in a sector with winding number Q.

pure gauge theory vacua $\longleftrightarrow$ knots

Toward calculating S_{eff}

non-local S_{eff} should vary slowly with $x \to$

- systematic derivative expansion

$$S_{\text{eff}}[n] = \sum_i \lambda_i \, S_i[\boldsymbol{n}] + \sum_i \lambda'_i \, S'_i[\boldsymbol{n}] = S + S'$$

- list of symmetric operators S_i:

$\partial \boldsymbol{n} \cdot \partial \boldsymbol{n}$	$\Box \boldsymbol{n} \cdot \Box \boldsymbol{n}$	$(\boldsymbol{n} \cdot \Box \boldsymbol{n})^2$	$(\boldsymbol{n} \cdot \partial_\mu \partial_\nu \boldsymbol{n})^2$

- list of non-symmetric operators S'_i:

$\boldsymbol{n} \cdot \boldsymbol{h}$	$(\boldsymbol{n} \cdot \boldsymbol{h})^2$	$(\partial \boldsymbol{n} \cdot \partial \boldsymbol{n})(\boldsymbol{n} \cdot \boldsymbol{h})$

Explicit calculations for $SU(2)$:

- generate lattice configurations $\sim e^{-S_{\text{YM}}[A]}$
- gauge fixings + projections: $A_\mu \longrightarrow \boldsymbol{n}$ (??)

- $\boldsymbol{n}$ distributed according

$$e^{-S_{\mathrm{YM}}[A]} \longrightarrow e^{-S_{\mathrm{eff}}[\boldsymbol{n}]}.$$

- gradient expansion for S_{eff}

$$\boxed{\text{calculate couplings via inverse MC}}$$

Schwinger Dyson and Ward identities

SO(3)-invariant $\quad \mathcal{D}\boldsymbol{n} = \prod_x d\boldsymbol{n}_x \delta\big(\boldsymbol{n}^2 - 1\big)$

- Isometries of $S^2 \rightarrow \; i\boldsymbol{L}_x = \boldsymbol{n} \wedge \frac{\partial}{\partial \boldsymbol{n}_x}$

$$\begin{aligned} \int \mathcal{D}\boldsymbol{n}\, \boldsymbol{L}_x \Big\{ G[\boldsymbol{n}] \exp\big(- S_{\mathrm{eff}}[\boldsymbol{n}, \boldsymbol{h}]\big) \Big\} &= 0 \\ \Longrightarrow \big\langle \boldsymbol{L}_x G[\boldsymbol{n}] - G[\boldsymbol{n}] \boldsymbol{L}_x S_{\mathrm{eff}} \big\rangle &= 0. \end{aligned}$$

- SD-equations: field monomials $G_i \to$

$$\sum_j \left\langle G_i \boldsymbol{L}_x S_j \right\rangle \lambda_j + \sum_j \left\langle G_i \, \boldsymbol{L}_x S'_j \right\rangle \lambda'_j = \langle \boldsymbol{L}_x G_i \rangle .$$

exact $\langle \ldots \rangle$, linear system $\longrightarrow \lambda_i, \lambda'_i$ (least squares)

- Ward-Identities (consistency checks)

$$\boldsymbol{L} = \sum_x \boldsymbol{L}_x \Longrightarrow \boldsymbol{L} S_i = 0, \quad \boldsymbol{L} S_{\text{eff}} = \boldsymbol{L} S'$$

$$\langle \boldsymbol{L} G[\boldsymbol{n}] - G[\boldsymbol{n}] \boldsymbol{L} S'[\boldsymbol{n}, \boldsymbol{h}] \rangle = 0.$$

Minimal ansatz

$$S_{\text{eff}} = \lambda_1 \sum_x (\partial_\mu \boldsymbol{n})^2 + \sum_x \boldsymbol{n}_x \cdot \boldsymbol{h}$$

Ward identity

$$G = 1 \longrightarrow \mathfrak{M} \wedge \boldsymbol{h} = 0, \qquad \mathfrak{M} = \langle \boldsymbol{n}_x \rangle$$

$$G = \boldsymbol{n}_y \cdot \boldsymbol{m}^* \longrightarrow \mathfrak{M} = -\chi^{\perp}\boldsymbol{h}$$

Technicalities:

- fix to lattice Landau gauge by maximizing

$$F_{\mathrm{LLG}}[\,{}^V U] = \sum_{x,\mu} \mathsf{tr}\Big({}^V U_\mu(x) + \mathrm{h.c.}\Big)$$

$\rightarrow$ residual global $SU(2)$

- gauge transform to MAG by maximizing

$$F_{\mathrm{MAG}} = \sum_{x,\mu} \mathsf{tr}\left(\tau_3 {}^V U_\mu(x) \tau_3 {}^V U_\mu^\dagger(x)\right)$$

residual local $U(1)$, 3–direction distinguished

- chose g_x, defined by

$$U_{\mathrm{LLG}} \xrightarrow{g} U_{\mathrm{MAG}}$$

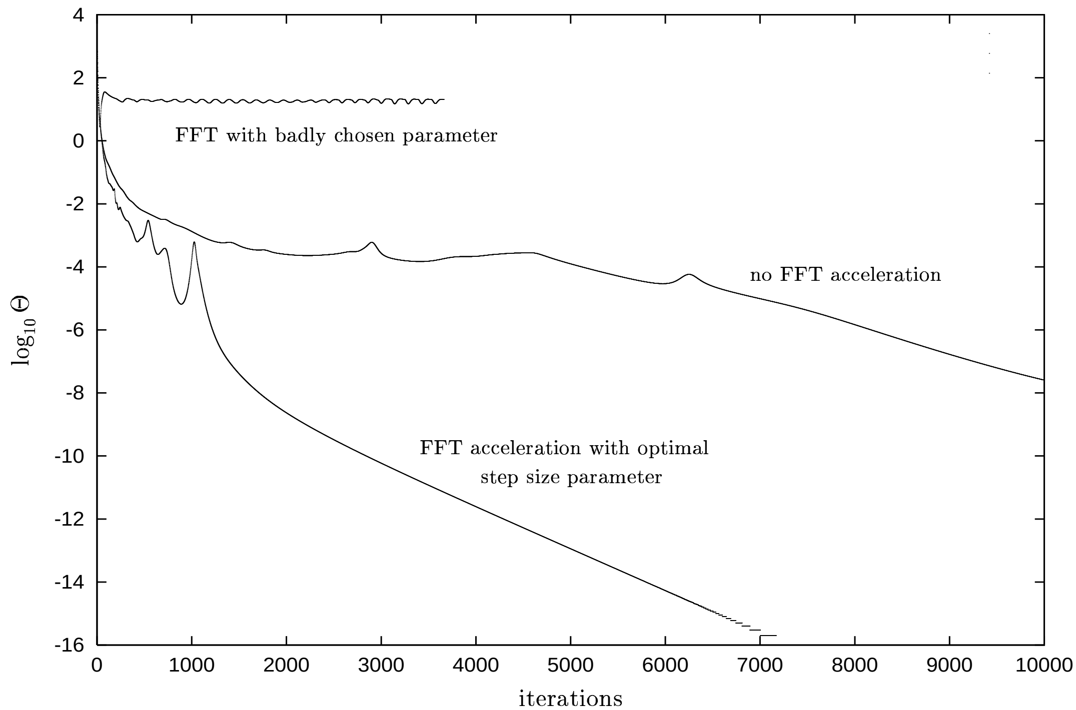
to define (almost gauge invariant) $n_x \equiv g_x^\dagger \tau_3 g_x$
$\{n_x\}$ distributed $\sim \exp(-S_{\rm YM})$ in LLG

$\boxed{\text{global } SU(2) \text{ broken } \textit{explicitly} \text{ down to } U(1)}$

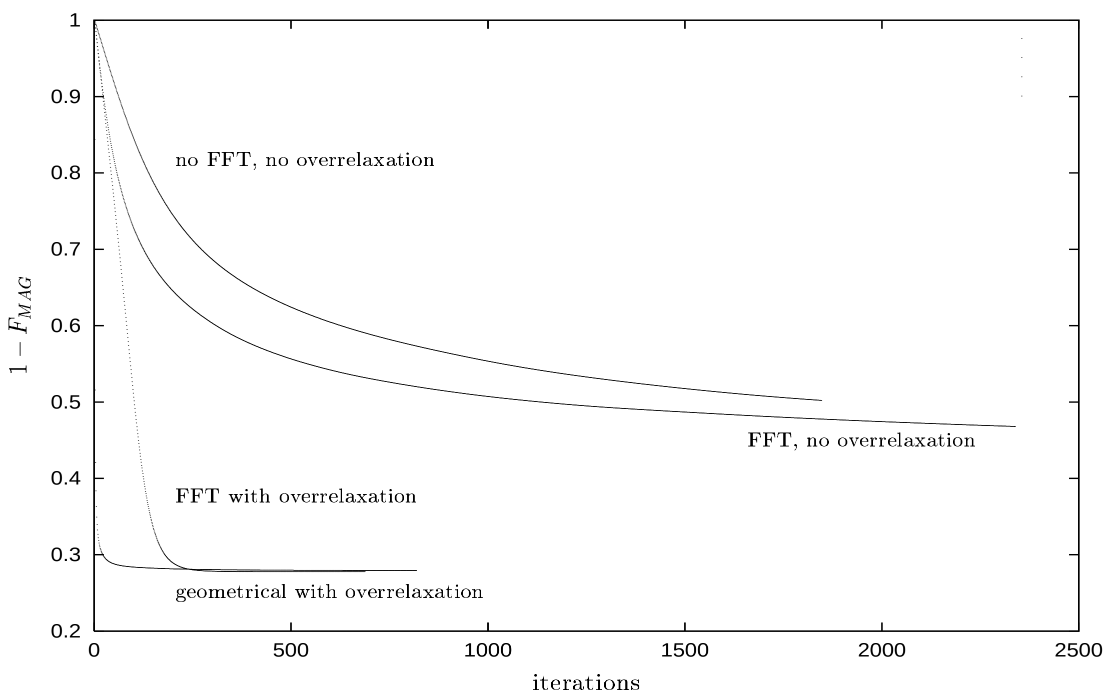
- lattice: $V = 16^4$, $\beta = 2.35$, $a \simeq 0.13$ fermi
- LLG: Fourier acc. steepest descent, 6000 iterations
- MAG: two independent algorithms

  MAGI: 'geometrical' iterations + over-relaxation

  MAGII: as LLG + over-relaxation



LLG: with and without FFT

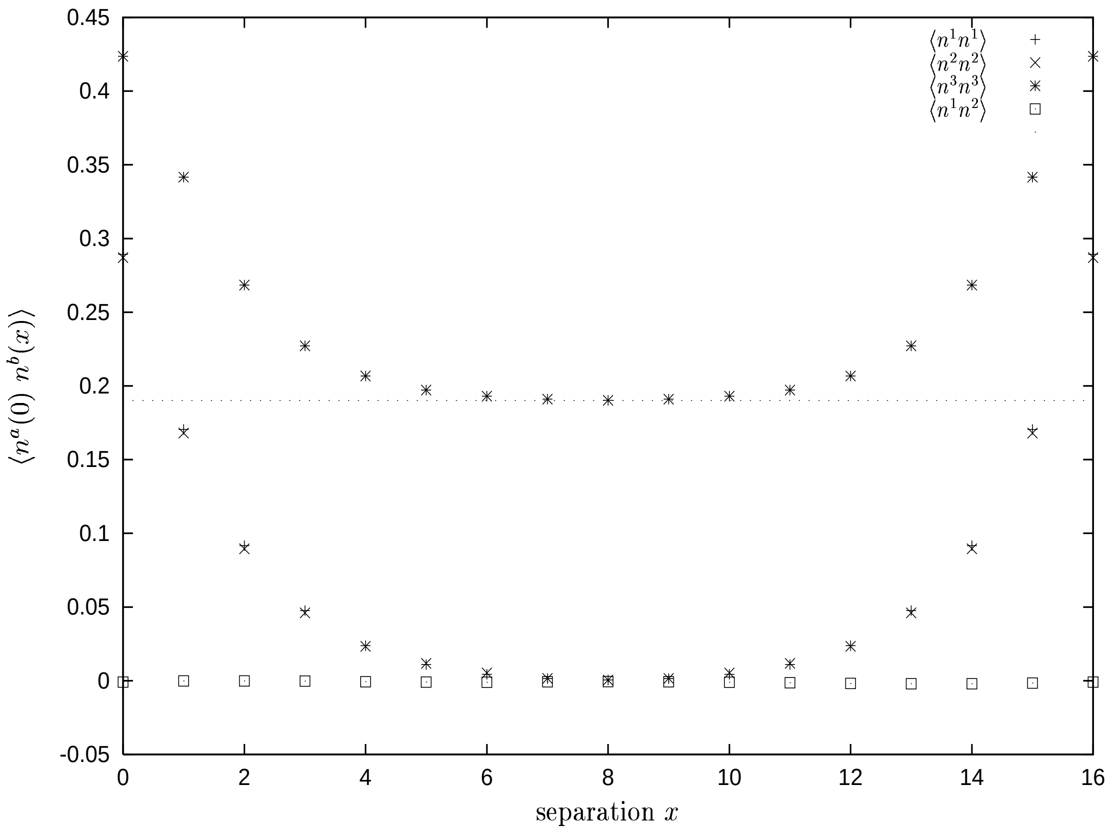


MAG: different algorithms

exhibits explicit SB $SO(3) \to SO(2)$

$$\langle n^1 \rangle = \langle n^2 \rangle = 0 \quad , \quad \langle n^3 \rangle = 0.44 \ \ (0.35)$$

$$\begin{aligned} G^{\parallel}(x) &= \big\langle n^3(x) n^3(0) \big\rangle \\ G^{\perp}(x) &= \frac{1}{2} \big\langle n^a(x) n^a(0) \big\rangle, \quad a = 1, 2 \end{aligned}$$



Two-point functions with MAG I; clustering

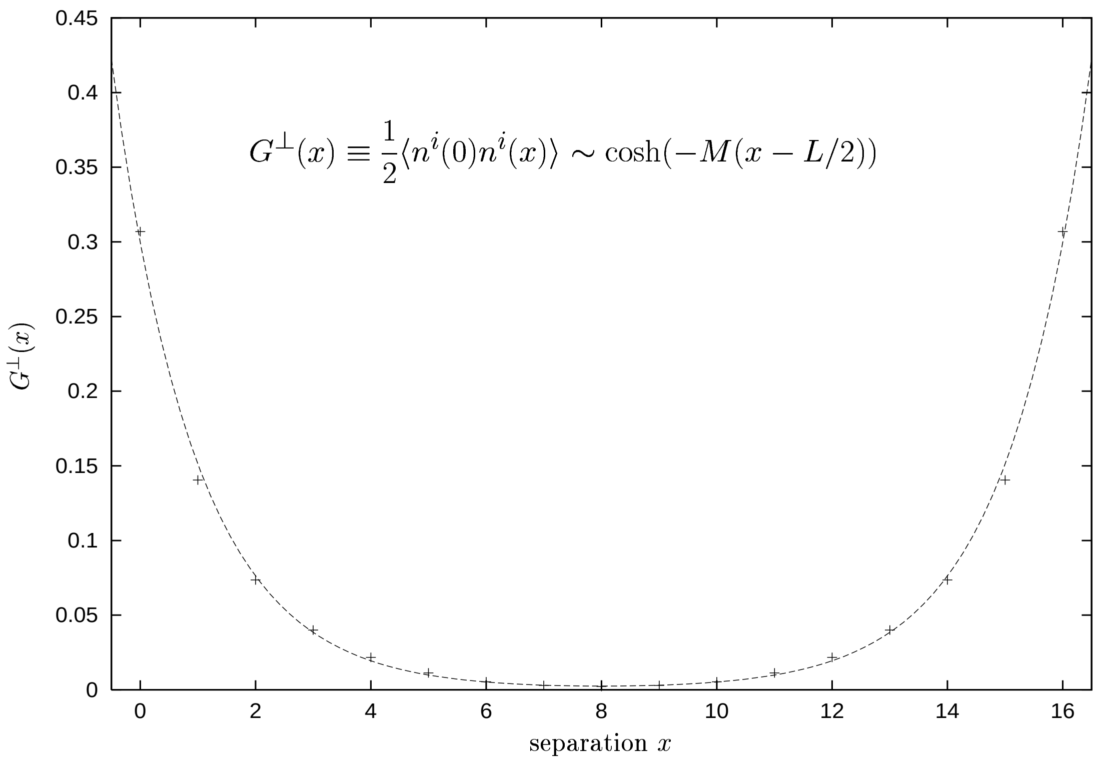
Magnetization $\quad \langle n^a \rangle = \mathfrak{M}\, \delta^{a3}$

susceptibility $\quad \chi^\perp \equiv \sum_x G^\perp(x)$

- Numerical values:

MAGI: $\quad \mathfrak{M} = 0.436 \quad \chi^\perp = 0.636 \quad M = 0.95$

MAGII: $\quad \mathfrak{M} = 0.352 \quad \chi^\perp = 0.596 \quad M = 1.01$



Cosh-fit to $G^\perp \to M = 1.0$*GeV*

Comparison with minimal ansatz:

- Ward identity: $M \simeq (\mathfrak{M}/\chi^{\perp})^{1/2} \simeq 1.2$ GeV
- cosh-fit to $G^{\perp}(x)$: $M \sim 1.0$ GeV $\rightarrow$

$$\boxed{\text{gradient expansion of } S_{\text{eff}} \text{ reasonable}}$$

CHECKED:

- explicit (not spontaneous) symmetry breaking
- $\langle \mathrm{tr}\mathcal{P} \rangle(\beta)$ ($N_s = 20, N_t = 4$), Binder cum: $\beta_c = 2.325$

FUTURE:

- λ_i from DS-equations via inverse MC
- inclusion of higher derivative terms
- how sensitive to choice of $\boldsymbol{n}$