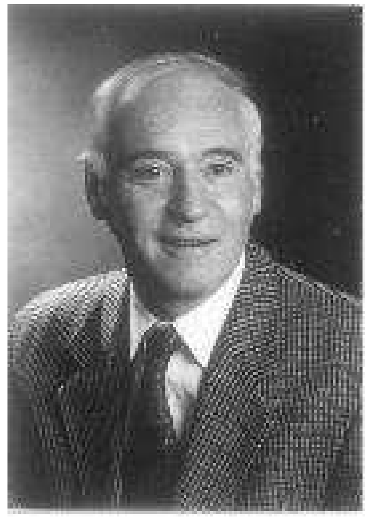
O'Raifeartaigh Memorial Conference



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The Supersymmetric Hydrogen Atom

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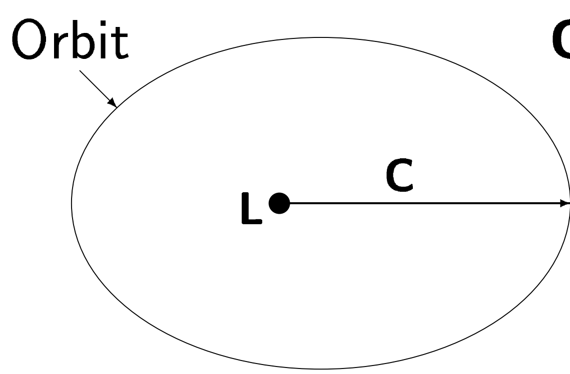
- Introduction
- $\mathcal{N} = 2$ Supersymmetric QM in d Dimensions
- The Supersymmetric H-Atom
  - ◇ Spectrum ◇ Eigenstates
- Example, Conclusions

Introduction

Classical motion in Newton/Coulomb potential
(*Hermann, Bernoulli, Laplace, Runge, Lenz*)

angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$

Runge-Lenz-vector $\mathbf{C} = \dfrac{1}{m}\mathbf{p} \times \mathbf{L} - \dfrac{e^2}{r}\mathbf{r}$



$\mathbf{C} \perp \mathbf{L}$

- Hydrogen atom in quantum mechanics:

(*Pauli, Hulthen, Bargmann, Fock, Zwanziger*)

$$\mathbf{C} = \frac{1}{2m}(\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{e^2}{r}\mathbf{r} \, .$$

On bound states

$$\mathbf{K} = \sqrt{\frac{-m}{2H}}\,\mathbf{C}$$

(hidden, dynamical) $SO(4)$ symmetry ($\hbar = 1$)

$$\begin{aligned} [L_a, L_b] &= i\epsilon_{abc} L_c \\ [L_a, K_b] &= i\epsilon_{abc} K_c \\ [K_a, K_b] &= i\epsilon_{abc} L_c \end{aligned}$$

Coulomb-Hamiltonian

$$H = -\frac{me^4}{2}\frac{1}{\mathbf{K}^2 + \mathbf{L}^2 + \hbar^2} \quad , \quad \mathbf{L} \cdot \mathbf{K} = 0$$

$\mathbf{L}^2 + \mathbf{K}^2$, $\mathbf{L} \cdot \mathbf{K}$ second order Casimirs

- bound state energies from *group theory*
- *accidental degeneracy* of the hydrogen spectrum
- analog: *scattering amplitudes* for hydrogen atom

Arbitrary dimensions

Schrödinger eq. in d dimensions (distances in $\hbar/mc$)

$$H\psi = E\psi, \quad H = p^2 - \frac{\eta}{r}, \quad p_a = \frac{1}{i}\partial_a$$

$a = 1, \ldots, d,\ \ \eta = 2\alpha,\ \ E$ in units of $mc^2/2$

- angular momenta $L_{ab} = x_a p_b - x_b p_a \longrightarrow SO(d)$

$$[L_{ab}, L_{cd}] = i(\delta_{ac}L_{bd} + \delta_{bd}L_{ac} - \delta_{ad}L_{bc} - \delta_{bc}L_{ad})\ ,$$

- Generalized *Laplace-Runge-Lenz* vector

$$C_a = L_{ab}p_b + p_bL_{ab} - \frac{\eta x_a}{r}\ .$$

$$[L_{ab}, C_c] = i(\delta_{ac} C_b - \delta_{bc} C_a)$$

$$[C_a, C_b] = -4i L_{ab} H$$

(hidden) dynamical symmetry algebra $SO(d+1)$

$$L_{AB} = \left(\begin{array}{c|c} L_{ab} & K_a \\ \hline -K_b & 0 \end{array} \right), \quad K_a = \frac{1}{\sqrt{-4H}} C_a$$

$$\boxed{C_a C_a = -\underbrace{4K_a K_a}\, H = \eta^2 + \big(\underbrace{2L_{ab}L_{ab}} + (d-1)^2\big) H}$$

$$\Longrightarrow \qquad H = p^2 - \frac{\eta}{r} = -\frac{\eta^2}{(d-1)^2 + 4\,\mathcal{C}_{(2)}}$$

$$\mathcal{C}_{(2)} = \frac{1}{2} L_{AB} L_{AB} = \frac{1}{2} L_{ab} L_{ab} + K_a K_a$$

- *which representations* are realized in $L_2(\mathbb{R}^d)$ or what replaces $\mathbf{L} \cdot \mathbf{K} = 0$?

- explicit realization of Cartan- and step operators of $SO(d+1)$ as first/second order differential operators
- treat even- and odd-dimensional cases separately

Results for hydrogen atom in d dimensions:

1. only *symmetric representations*
2. *energies*

$$E_\ell\left(\boxed{1}\boxed{\cdot}\boxed{\cdot}\boxed{\ell}\right) = -\frac{mc^2}{2}\gamma_\ell^2, \quad \gamma_\ell = \frac{\alpha}{\ell + (d-1)/2}$$

3. *degeneracies*

$$\mathrm{dim}\left(\boxed{1}\boxed{\cdot}\boxed{\cdot}\boxed{\ell}\right) = \binom{\ell+d}{\ell} - \binom{\ell+d-2}{\ell-2}$$

4. highest weight *states*

$$\Psi\left(\boxed{1}\boxed{\cdot}\boxed{\cdot}\boxed{\cdot}\boxed{l}\right) = \exp(-\gamma_\ell r)\,(x_1 + ix_2)^\ell$$

5. *branching rules*

$$\left.\boxed{1}\boxed{\cdot}\boxed{\cdot}\boxed{\ell}\right|_{SO(d+1)} \longrightarrow \left\{ \mathbb{1} \oplus \square \oplus \ldots \oplus \boxed{1}\boxed{\cdot}\boxed{\cdot}\boxed{\ell} \right\}_{SO(d)}$$

$\mathcal{N} = 2$ Susy Quantum Mechanics

susy extension of d-dimensional Schrödinger operators

$$\boxed{H = \{Q, Q^\dagger\} = H^\dagger \quad \text{with} \quad Q^2 = Q^{\dagger\, 2} = 0}$$

◇ *supercharge* Q transforms 'bosons' into 'fermions'

$$[Q, H] = 0$$

- Q generates supersymmetry
- simplest models: 2×2-matrix differential operators in one dimension (*Nicolai, Witten*)
- higher dimensions (*Witten, Andrianov et.al, Wipf et.al*).

Here: $\mathcal{N}=2$-models

fermionic *creation* and *annihilation operators*

$$\{\psi_a,\psi_b^\dagger\}=\delta_{ab} \ \ , \ \ \{\psi_a,\psi_b\}=\{\psi_a^\dagger,\psi_b^\dagger\}=0, \ \ a,b\leq d$$

- *Fock space* and number operator

  $$\psi_a|0\rangle = 0\,, \quad |a_1\dots a_p\rangle = \psi^\dagger_{a_1}\cdots\psi^\dagger_{a_p}|0\rangle, \quad p\leq d$$

  $$N=\sum_{a=1}^d \psi_a^\dagger\psi_a, \quad N|a_1\dots a_p\rangle = p|a_1\cdot a_p\rangle$$

  number operator, $p=1,\dots,d$

- decomposition of Hilbert space

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_0\oplus\mathcal{H}_1\oplus\dots\oplus\mathcal{H}_d\\ \mathcal{H}_p\ni\Psi &= \sum_{a_1,\dots,a_p} f_{a_1\dots a_p}(x)\,|a_1\dots a_p\rangle\end{aligned}$$

$f_{a_1\dots a_p}$ antisymmetric:

$$\mathcal{H}_p\sim L_2(\mathbb{R}^d)\times\mathbb{C}^{n_p}, \qquad n_p=\binom{n}{p}.$$

- supercharge

$$Q = e^{-\chi} Q_0 e^{\chi} \quad , \quad Q_0 = i\psi_a \partial_a$$
$$Q^\dagger = \mathrm{e}^{\chi} Q_0^\dagger \mathrm{e}^{-\chi} \quad , \quad Q_0^\dagger = i\psi_a^\dagger \partial_a$$

$$\Longrightarrow Q_0^2 = 0 \quad \Longrightarrow \quad Q^2 = 0$$

$$[N, Q] = -Q \quad , \quad [N, Q^\dagger] = Q^\dagger$$

Hamiltonian $= 2^d \times 2^d$ matrix differential operator

$$H = \{ -\triangle + (\nabla\chi, \nabla\chi) + \triangle\chi \} \mathbb{1}_{2^d}$$
$$- 2 \sum_{a,b=1}^{d} \psi_a^\dagger \chi_{ab} \psi_b \ , \quad \chi_{ab} = \frac{\partial^2 \chi}{\partial x_a \partial x_b}$$

$$[N, H] = 0, \quad H\Big|_{\mathcal{H}_p} = -\triangle \mathbb{1} + V^{(p)} \ , \quad \mathrm{tr}\mathbb{1} = \binom{d}{p}$$

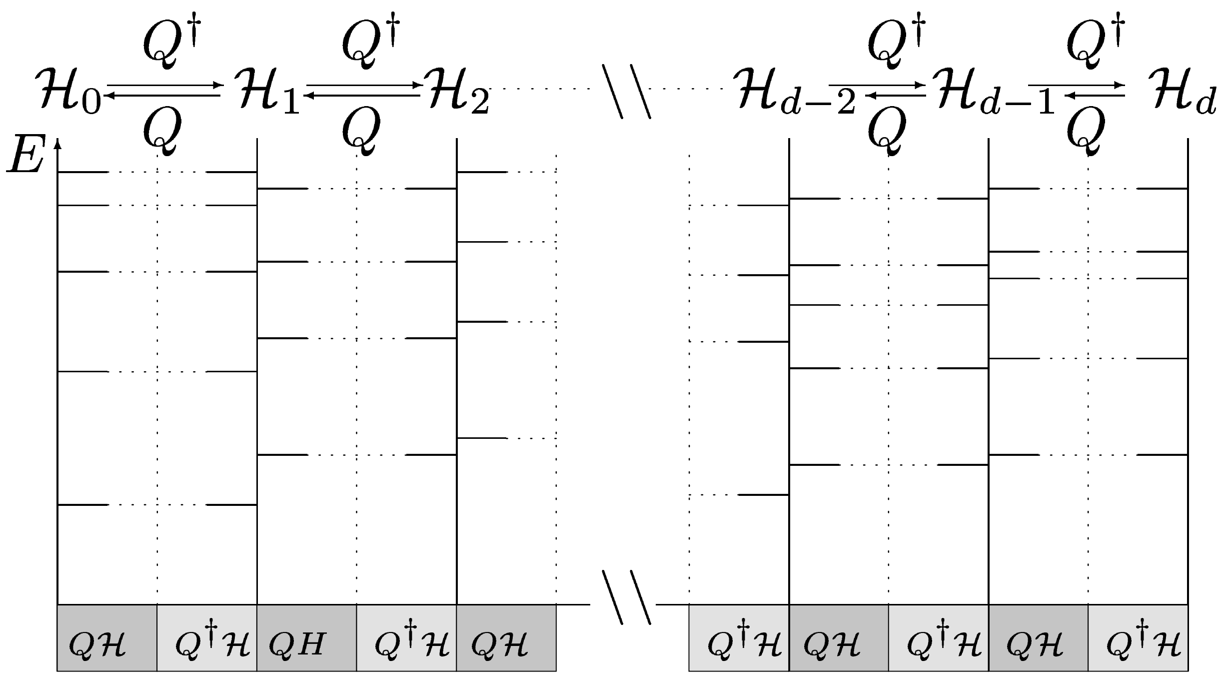
- $H|_{\mathcal{H}_0}$, $H|_{\mathcal{H}_d}$ ordinary Schrödinger operators

$$V^{(0)} = (\nabla\chi, \nabla\chi) + \triangle\chi \ , \ V^{(d)} = (\nabla\chi, \nabla\chi) - \triangle\chi$$

sector $\mathcal{H}_p$:

$$\langle a_1 \dots a_p | H \Psi \rangle = \left(- \triangle + V^{(0)} \right) f_{a_1 \dots a_p} + 2 \sum_{b,i=1}^{p} (-)^i \chi_{a_i b} \, f_{b a_1 \dots \check{a}_i \dots a_p}$$

$\mathcal{H} = Q\mathcal{H} \oplus Q^\dagger \mathcal{H} \oplus \mathrm{Ker}\, H$ 'Hodge'



pairing of states with $E > 0$.

The susy H-Atom and its Symmetries

- spherically symmetric systems

$$\chi(r) \Longrightarrow Q = i\psi_a\Big(\partial_a + x_a\frac{\chi'}{r}\Big)$$

ψ_a : $SO(d)$-scalar $\Rightarrow$ supplement L_{ab} by 'spin-part'

$$[S_{ab},\psi_c] = i(\delta_{ac}\psi_b - \delta_{bc}\psi_a), \quad S_{ab} = \frac{1}{i}(\psi_a^\dagger\psi_b - \psi_b^\dagger\psi_a)$$

total angular momenta

$$J_{ab} = L_{ab} + S_{ab} \Longrightarrow Q, Q^\dagger \text{ scalars}$$

Extension of Laplace-Runge-Lenz vector

$$C_a = J_{ab}p_b + p_bJ_{ab} + x_af(r)A$$

- C_a vector $\Longrightarrow A$ scalar, C_a known on $\mathcal{H}_0$
- $[J_{ab}, N] = 0 \Longrightarrow [C_a, N] = 0$

$$\Rightarrow A = \alpha \mathbb{1} - \beta N - \gamma S^\dagger S, \quad S = \hat{x}_a \psi_a$$

From $[C_a, Q] = 0 \Longrightarrow \alpha, \beta, \gamma, f(r) \Longrightarrow$

$$\boxed{\begin{aligned} C_a &= J_{ab} p_b + p_b J_{ab} - \lambda \hat{x}_a A \\ A &= (d-1)\mathbb{1} - 2N + 2S^\dagger S \end{aligned}}$$

$$\boxed{H = -\triangle + \lambda^2 - \tfrac{\lambda}{r} A}$$

- $\mathcal{H}_0$: hydrogen atom, $\mathcal{H}_d$: electron-antiproton

$$Q = Q_0 - i\lambda S \quad , \quad Q^\dagger = Q_0^\dagger + i\lambda S^\dagger$$

Spectrum

$$[C_a, C_b] = -4iJ_{ab}\Big(- \triangle - \frac{\lambda}{r} A \Big) = -4iJ_{ab}\left(H - \lambda^2\right)$$

as before, but $H \to H - \lambda^2$, $L_{ab} \to J_{ab}$

$\Rightarrow$ Fock-Bargman $SO(d+1)$ generated by J_{ab} and

$$K_a = \frac{C_a}{\sqrt{4(\lambda^2 - H)}}$$

surprise $\qquad C_a C_a \neq f\big(\mathbb{1}, N, J_{ab}J_{ab}, H\big)$

$$\begin{aligned} C_a C_a &= 4(\lambda^2 - H) K_a K_a = -2\lambda^2 J_{ab} J_{ab} \\ &+ \left(2J_{ab}J_{ab} + (d-2N-1)^2\right) Q Q^\dagger \\ &+ \left(2J_{ab}J_{ab} + (d-2N+1)^2\right) Q^\dagger Q \end{aligned}$$

use $H|_{Q\mathcal{H}} = QQ^\dagger$ and $H|_{Q^\dagger\mathcal{H}} = Q^\dagger Q \;\Rightarrow$ spectrum

$$\begin{aligned} H\big|_{Q\mathcal{H}} &= QQ^\dagger = \lambda^2 - \frac{(d-2N-1)^2\lambda^2}{(d-2N-1)^2 + 4\mathcal{C}_{(2)}} \\ H\big|_{Q^\dagger\mathcal{H}} &= Q^\dagger Q = \lambda^2 - \frac{(d-2N+1)^2\lambda^2}{(d-2N+1)^2 + 4\mathcal{C}_{(2)}} \end{aligned}$$

- $\mathcal{C}_{(2)}$ second-order Casimir of $SO(d+1)$,

$$\mathcal{C}_{(2)} = \frac{1}{2} J_{AB} J_{AB} = \frac{1}{2} J_{ab} J_{ab} + K_a K_a \ .$$

Conclusions

- $\mathcal{C}_{(2)}\big|_{\text{Ker}\,H} = 0 \Longrightarrow$ zero-modes $SO(d+1)$ singlets.
- generalized $SO(d)$-spherical harmonics $\in \mathcal{H}_p$:

$$
\begin{aligned}
f_{a_1\ldots a_p}(x) &= \sum_{b_1,b_2,\ldots,b_\ell} f_{a_1\ldots a_p b_1\ldots b_\ell} x_{b_1} x_{b_2}\cdots x_{b_\ell} \\
\mathcal{D}^1_p \otimes \mathcal{D}^\ell_1 &= \mathcal{D}^\ell_{p-1} \oplus \mathcal{D}^{\ell-1}_p \oplus \mathcal{D}^{\ell+1}_p \oplus \mathcal{D}^\ell_{p+1}
\end{aligned}
$$

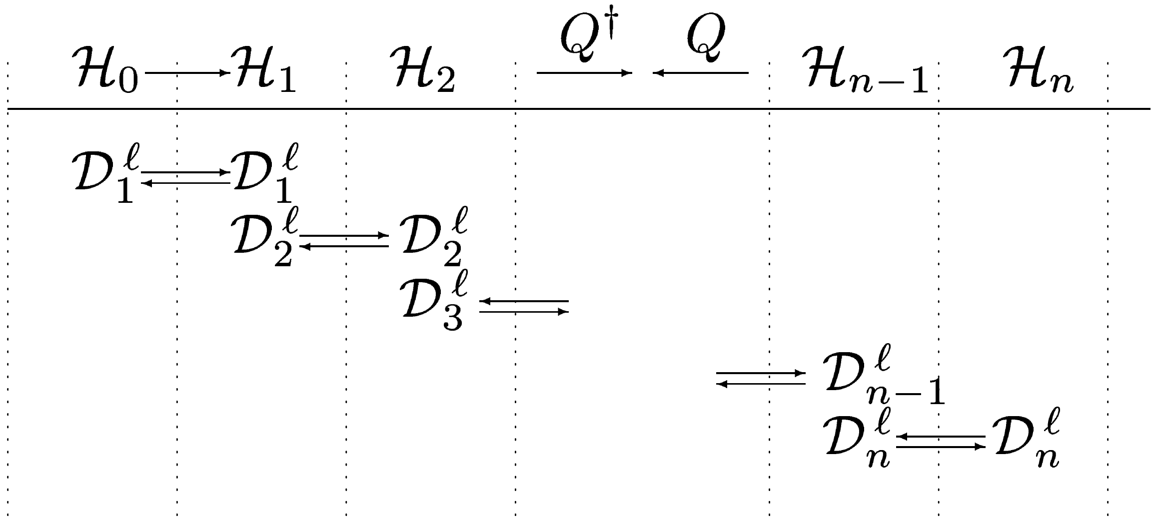
$$
\mathcal{D}^\ell_\wp \sim \begin{array}{|c|c|c|c|}\hline 1 & \cdot & \cdot & \ell \\ \hline \cdot \\ \cline{1-1} \cdot \\ \cline{1-1} \wp \\ \cline{1-1}\end{array}
\qquad
\begin{array}{c}(a_1,\ldots,a_p) \sim \mathcal{D}^1_p \\ (b_1,\ldots,b_\ell) \sim \mathcal{D}^\ell_1\end{array}
$$

branching-rules $SO(d+1) \longrightarrow SO(d)$

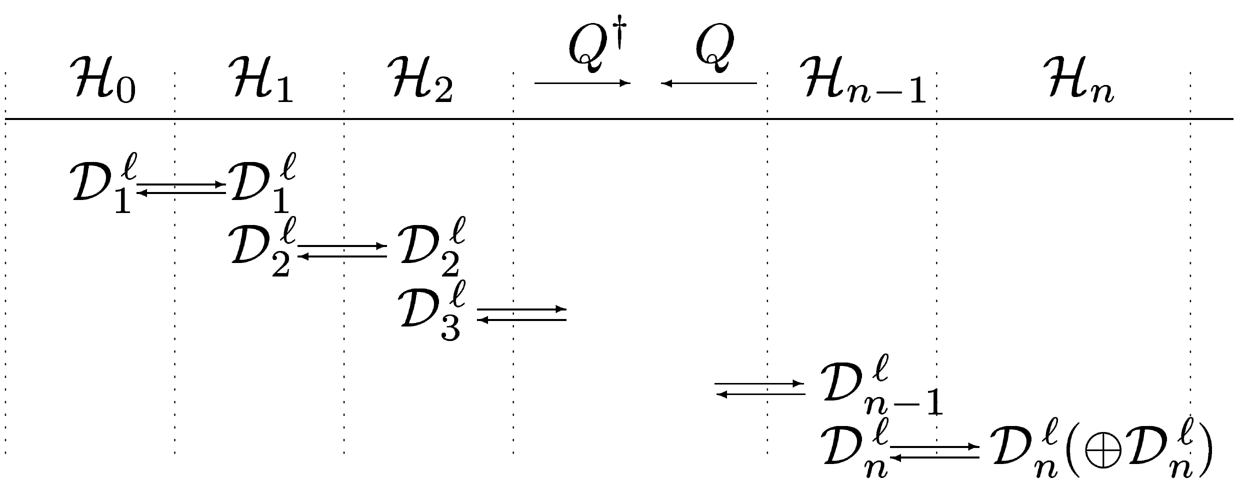
$$
\mathcal{D}^\ell_\wp\big|_{SO(d+1)} \longrightarrow \Big\{\mathcal{D}^\ell_\wp \oplus \mathcal{D}^{\ell-1}_\wp \oplus \ldots \oplus \mathcal{D}^1_\wp
\oplus \mathcal{D}^\ell_{\wp-1} \oplus \mathcal{D}^{\ell-1}_{\wp-1} \oplus \ldots \oplus \mathcal{D}^1_{\wp-1}\Big\}\Big|_{SO(d)}
$$

$\Longrightarrow$ identification of $SO(d+1)$ representations.

odd dimensions $d = 2n + 1$:



even dimensions $d = 2n$:



- second order Casimir

$$\mathcal{C}_{(2)}\left(\mathcal{D}^{\ell}_{\wp}\right) = d(\ell+\wp-1)+\ell(\ell-1)-\wp(\wp-1)$$

- bound state energies

$$E_p(\mathcal{D}^{\ell}_{p}) = Q^{\dagger}Q\big|_{\mathcal{H}_p}\left(\mathcal{D}^{\ell}_{p}\right) = \lambda^2 - \left(\frac{d+1-2p}{d-1+2\ell}\right)^2\lambda^2$$

$$E_p(\mathcal{D}^{\ell}_{p+1}) = QQ^{\dagger}\big|_{\mathcal{H}_p}\left(\mathcal{D}^{\ell}_{p+1}\right) = \lambda^2 - \left(\frac{d-1-2p}{d-1+2\ell}\right)^2\lambda^2$$

Eigenstates

susy $\Rightarrow$ need only h.w.states $\Psi_p(\mathcal{D}^{\ell}_{p+1})$

$$\Psi_{p+1}(\mathcal{D}^{\ell}_{p+1}) = Q^{\dagger}\Psi_p(\mathcal{D}^{\ell}_{p+1})\ .$$

Ψ h.w.state of $SO(d+1) \Longrightarrow \Psi$ h.w.state of $SO(d)$

$$\Longrightarrow \Psi_p(\mathcal{D}^{\ell}_{p+1}) = f(r)\mathcal{Y}_a(\ell, p+1)$$

$$E_n \Psi_p = 0 \quad \Longrightarrow \quad f$$

$$\Psi_p(\mathcal{D}^{\ell}_{p+1}) = e^{-\gamma_{\ell p} r}\, \mathcal{Y}_a(\ell, p+1), \;\; \gamma_{\ell p} = \frac{d-1-2p}{d-1+2\ell}\, \lambda$$

bound states for $p < n$

Examples, Conclusions

$$d = 3: \quad \mathcal{H} \;=\; \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$$

$$\Psi \;=\; f_0|0\rangle + \big(f_1|1\rangle + f_2|2\rangle + f_3|3\rangle\big) + \ldots$$

bound states in $\mathcal{H}_0$ and $\mathcal{H}_1$

$$\mathcal{H}_0: \qquad H^{(0)} = -\triangle + \lambda^2 - \frac{2\lambda}{r}$$

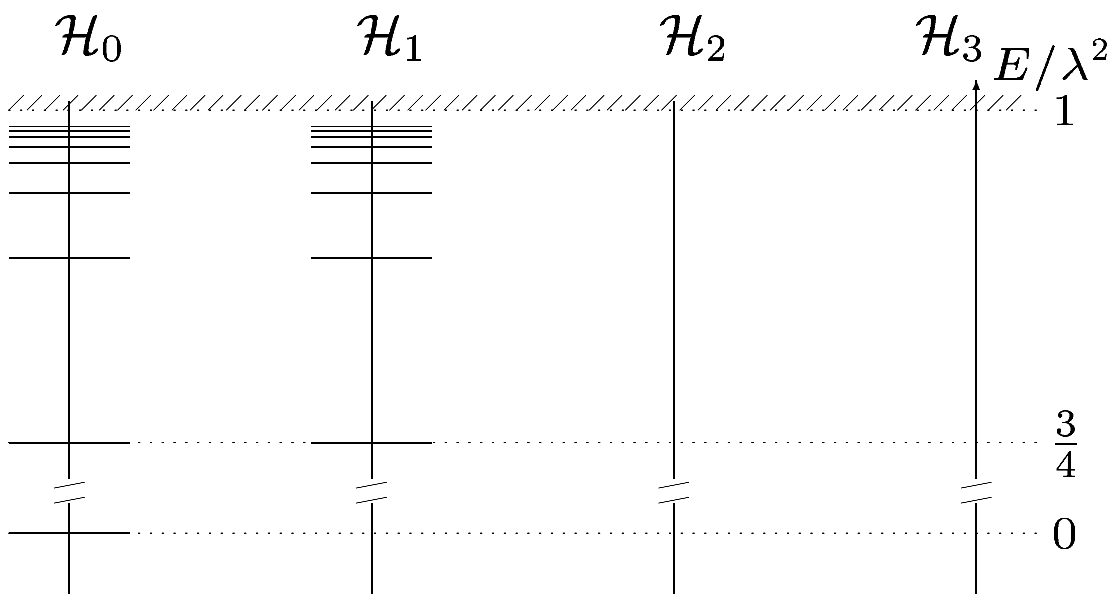
$$\mathcal{H}_1: \qquad \langle a|H\Psi\rangle = \big(-\triangle + \lambda^2\big) f_a - \frac{2\lambda}{r} \hat{x}_a \hat{x}_b f_b$$

- all h.w.states

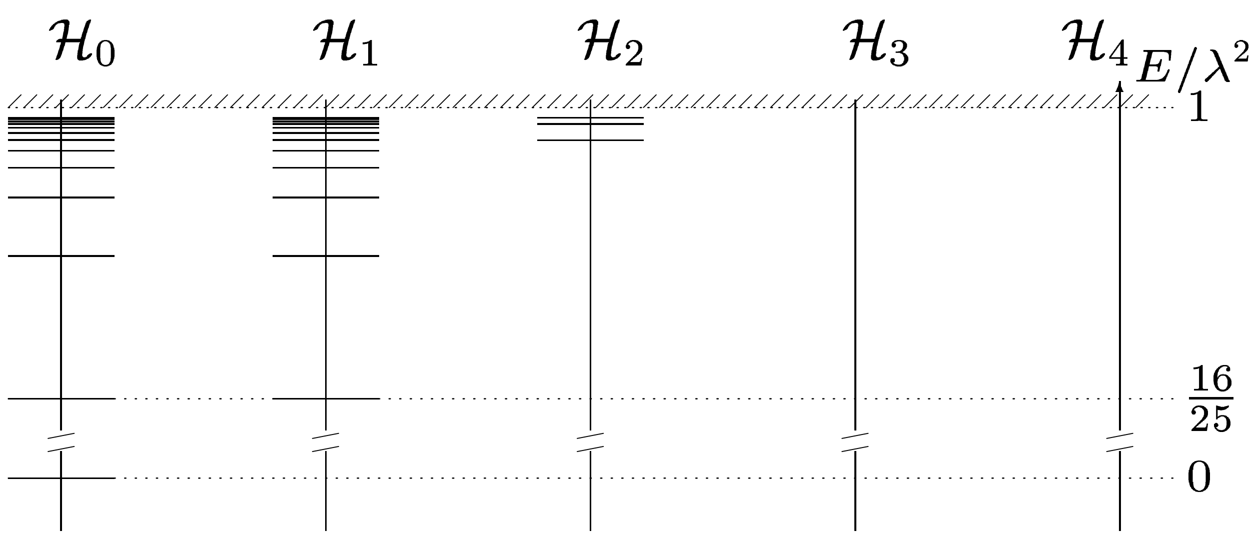
$$\Psi_0(\mathcal{D}^{\ell}_1) = e^{-\gamma_{\ell 0} r} \mathcal{Y}_a(\ell, 1) \in \mathcal{H}_0 \quad , \quad Q^{\dagger}\Psi_0(\mathcal{D}^{\ell}_1) \in \mathcal{H}_1$$

$$E_\ell = \lambda^2 - \gamma^2_{\ell 0} \qquad , \qquad \gamma_{\ell 0} = \frac{\lambda}{1+\ell}$$

- both multiplets contain $(\ell+1)^2$ states
- one normalizable zero-mode with $\ell = 0$ in $\mathcal{H}_0$
- remaining states are paired



Eigenvalues of H in $d = 3$ dimensions



Eigenvalues of H in $d = 4$ dimensions

- have extended results of Pauli, Fock, Bargmann and others to arbitrary d and $\mathcal{N} = 2$ susy $H-$atom.
- found generalized angular momentum
- constructed extended Laplace-Runge-Lenz vector
- obtained relation $QQ^\dagger, Q^\dagger Q \longrightarrow \mathcal{C}_{(2)}$
- got bound state spectrum: E, Ψ, degeneracies

- global construction for the susy systems?
- language of superalgebras?
- relation to Killing-Yano superchages?
- $SO(d,2)$ content? (see E. Sudarshan, N. Mukunda, L. O'Raifeartaigh, Physics Letters 19 (1965) 322