Phases of generalized Potts-Models and their Relevance for Gauge Theories

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O'Raifeartaigh Symposium
Budapest
22.–24. June 2006

Potts-Models

Polyakov-Loop Dynamics

Gluodynamics and Potts-Models

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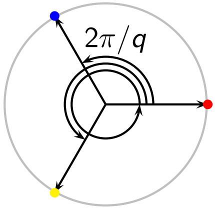
Conclusions

Potts-Models

- 

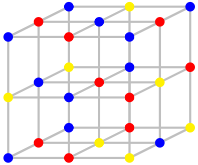
generalized Ising models:

$$\theta_x \in \{2\pi k/q\}\,,\quad 1 \le k \le q$$

$$H = -J\sum_{\langle xy\rangle} \cos\left(\theta_x - \theta_y\right)$$

$$\mathbb{Z}_q : \theta_x \to \theta_x + 2\pi n/q$$

- ferromagnetic phase: q ground states
  phase transition symmetric $\leftrightarrow$ ferromagnetic
  $d = 2$: second order $q \le 4$, first order $q > 4$
  $d = 3$: second order $q \le 2$, first order $q > 2$

- 

anti-ferromagnetic phase:
rich vacuum structures
symmetric $\leftrightarrow$ antiferrom:
$d = 3, q = 3$: second order
entropy of ground states?

- entropy $S_B(p) = -\sum p(w)\log p(w) \Rightarrow$ free energy

$$\beta F = \inf_p \left(\beta\langle H\rangle_\rho - S_B\right) \Rightarrow p_{\mathrm{Gibbs}} \sim e^{-\beta H}$$

- variational characterization of (convex) effective action:

$$\Gamma[m] = \inf_p \left(\beta\langle H\rangle_p - S(p)\,\middle|\,\langle e^{i\theta(x)}\rangle_p = m(x)\right)$$

- mean field approximation:

$$\rho(w) = \prod_x \rho_x(\theta_x) \Rightarrow \Gamma_{\mathrm{MF}}[m]$$

translational invariance: $p_x = p \Rightarrow m(x) = m$
effective potential: $\Gamma_{\mathrm{MF}}[m] = V\, u_{\mathrm{MF}}(m)$

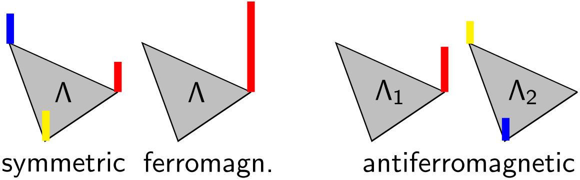
$$u_{\mathrm{MF}}(m) = \inf_p \left(-Kmm^* + \sum_\theta p(\theta)\log p(\theta)\right)$$

$$m = \sum_\theta p(\theta)\, e^{i\theta}, \qquad K = dJ.$$

- antiferromagnetic phase:
  translational invariance on sublattices $\Lambda = \Lambda_1 \cup \Lambda_2$
  two neighbours in different sublattices
  $p(x) = p_i \Rightarrow m(x) = m_i$ for $x \in \Lambda_i$

$$u_{\mathrm{MF}}(m_1, m_2) = \frac{1}{2}\left(K|m_1 - m_2|^2 + \sum_i u_{\mathrm{MF}}(m_i)\right),$$

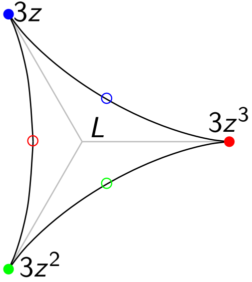
- $K > K_{f,c} > 0 \Rightarrow m_1 = m_2 \neq 0$, $\mathbb{Z}_q$-broken
  $K < K_{a,c} < 0 \Rightarrow m_1 \neq m_2 \neq 0$, $\mathbb{Z}_{2q}$-broken



Polyakov-Loop Dynamics

- finite temperature gluodynamics
  order parameter for confinement: Polyakov loop
  effective action:

$$e^{-S_{\text{eff}}[\mathcal{P}]} = \int \mathcal{D}U \delta\left(\mathcal{P}_{\boldsymbol{x}}, \prod_{t=0}^{N_t} U_{t,\boldsymbol{x};0}\right) e^{-S_{\text{w}}[U]}$$



gauge invariance:

$$S_{\text{eff}} = S_{\text{eff}}[L], \quad L_{\boldsymbol{x}} = \operatorname{Tr}\mathcal{P}_{\boldsymbol{x}}$$

global Z_3 center symmetry:

$$S_{\text{eff}}[L] = S_{\text{eff}}[z \cdot L]$$

- good ansatz for S_{eff}?

- strong coupling expansion for $S_{\text{eff}}[\mathcal{P}]$
  $\Rightarrow \mathbb{Z}_3$-invariant character expansion
  nearest neighbour interaction

$$S_{\text{eff}} = \lambda_{10}S_{10} + \lambda_{21}S_{21} + \lambda_{20}S_{20} + \lambda_{11}S_{11} + \dots$$
$$S_{10} = \sum \left(\chi_{10}(\mathcal{P}_{\boldsymbol{x}})\chi_{01}(\mathcal{P}_{\boldsymbol{y}}) + h.c\right), \; S_{21} = \dots$$

- center-transformation:

$$\chi_{pq}(z\mathcal{P}) = z^{p-q}\chi_{pq}(\mathcal{P}), \quad z^3 = z^* z = 1$$

With $L = \mathrm{Tr}\,\mathcal{P}$: leading terms

$$\begin{aligned} S_{\text{eff}} &= (\lambda_{10} - \lambda_{21}) \sum \left(L_{\boldsymbol{x}} L^*_{\boldsymbol{y}} + \text{h.c.}\right) \\ &+ \lambda_{21} \sum \left(L^2_{\boldsymbol{x}} L_{\boldsymbol{y}} + L^2_{\boldsymbol{y}} L_{\boldsymbol{x}} + \text{h.c.}\right) \end{aligned}$$

- complex field with compact target space, $\prod$(reduced Haar measures), close relation to 3-state Potts model

Gluodynamics and Pott-Models

- naive reduction to Potts: $\mathcal{P}_{\boldsymbol{x}} \to e^{i\theta_{\boldsymbol{x}}}\mathbb{1} \in$ centre

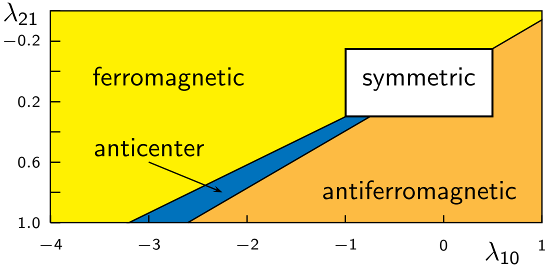
$$S_{\rm eff} \to H \quad \text{with} \quad J = 18(\lambda_{01} + 4\lambda_{21})$$

  true for all $S_{\rm eff} \Rightarrow S_{\rm eff}$ is extension of $\mathbb{Z}_3$ model.
- Conjecture (Svetitsky, Yaffe):

  effective finite-temperature $SU(N)$-gluodynamics in d dimensions $\cong \mathbb{Z}_N$ spin model in $d-1$ dimensions.
- same critical exponents $SU(2)$ and Ising (Engels et.al) same universality class (symmetric $\leftrightarrow$ ferrom.)

	β/ν	γ/ν	ν
$4d$ SU(2)	0.545	1.93	0.65
$3d$ Ising	0.516	1.965	0.63

- relevance for finite temperature $SU(N)$ with $N > 2$? transition first order! $\longrightarrow$ phase diagrams
- classical analysis: minimize S_{eff}



- quantum fluctuations $\Rightarrow$ include symmetric phase new ferromagnetic anti-center phase qualitatively correct phase diagram

Modified mean field approximation

- variational characterisation of Γ:
  fix $\langle \chi_j(\mathcal{P}_{\boldsymbol{x}}) \rangle$ for all χ_j in S_{eff}
- mean field approximation $\Rightarrow$ product measure

$$\mathcal{DP} \longrightarrow \prod_{\boldsymbol{x}} d\mu_{\text{red}}(\mathcal{P}_{\boldsymbol{x}})\, p_{\boldsymbol{x}}\,(\mathcal{P}_{\boldsymbol{x}})$$

- translational invariance on sublattices in $\Lambda = \Lambda_1 \cup \Lambda_2$
  $\Rightarrow$ nontrivial variational problem on two-sites
- most simple effective model (Polonyi)

$$S_{\text{eff}} = \lambda S_{10} = \lambda \sum \left(L_{\boldsymbol{x}} L_{\boldsymbol{y}}^* + \text{h.c} \right)$$

Lagrangean multiplier for $\bar{L}_i$ on Λ_i

- mean field effective potential for minimal model

$$2u_{\rm MF}(L_1,L_1^*,L_2,L_2^*) = -d\lambda|L_1-L_2|^2 + \sum v_{\rm MF}(L_i,L_i^*)$$

$$v_{\rm MF}(L,L^*) = d\lambda|L|^2 + \gamma_0(L,L^*)$$

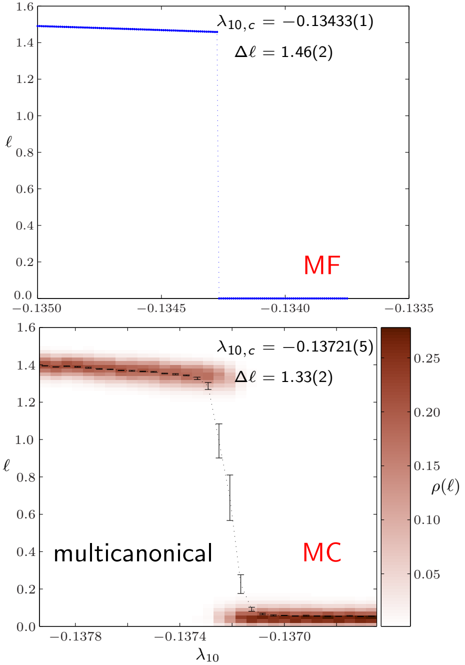
γ_0 Legendre-transform of

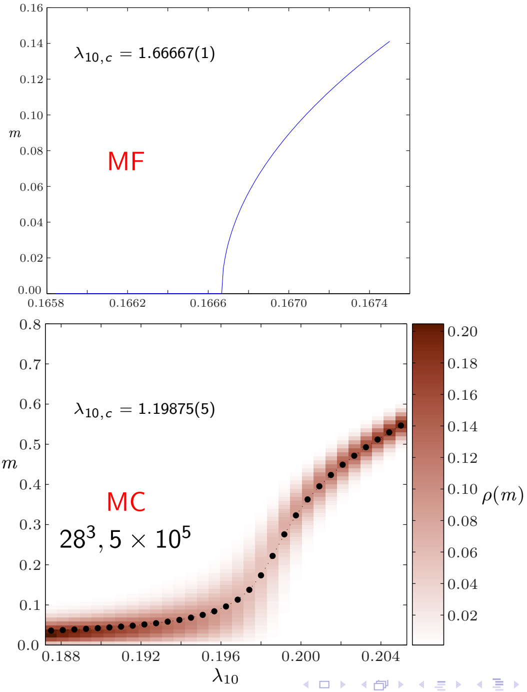
$$w_0(j,j^*) = \log \int d\mu_{\rm red} \exp\left(jL + j^*L^*\right)$$

- order parameters:

$$L = \frac{1}{2}(L_1+L_2),\quad M = \frac{1}{2}(L_1-L_2),\quad \ell = |L|,\quad m = |M|.$$

- group integral in closed form not known for $SU(3)$!!
  $\int \exp\left(j\,\mathrm{Tr}\,(U)\right) =$hypergeometric function


$\lambda_{10,c} = -0.13433(1)$
$\Delta\ell = 1.46(2)$
MF
ℓ
$\lambda_{10,c} = -0.13721(5)$
$\Delta\ell = 1.33(2)$
multicanonical
MC
$\rho(\ell)$
λ_{10}


$\lambda_{10,c} = 1.66667(1)$
MF
m
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0.1658
0.1662
0.1666
0.1670
0.1674
$\lambda_{10,c} = 1.19875(5)$
MC
$28^3, 5 \times 10^5$
m
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.188
0.192
0.196
0.200
0.204
λ_{10}
$\rho(m)$
0.20
0.18
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02

- Why is mean field so good?
  conjecture: $3 =$ upper crit. dimension for 3-state potts
- critical exponents of $S \leftrightarrow AF$:

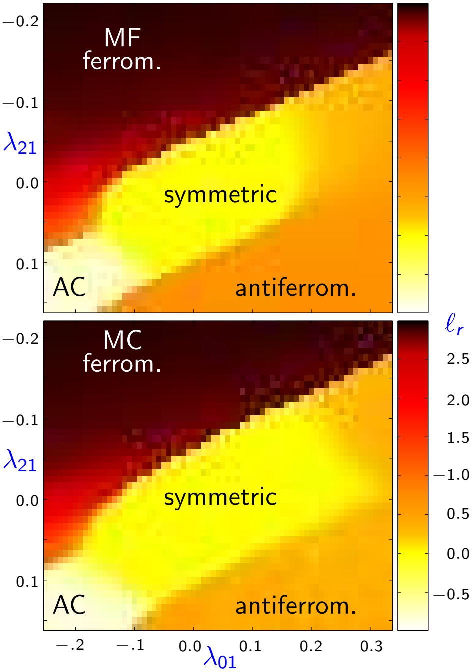
exponent	3-state Potts	minimal S_{eff}
ν	0.664(4)	0.68(2)
γ/ν	1.973(9)	1.96(2)

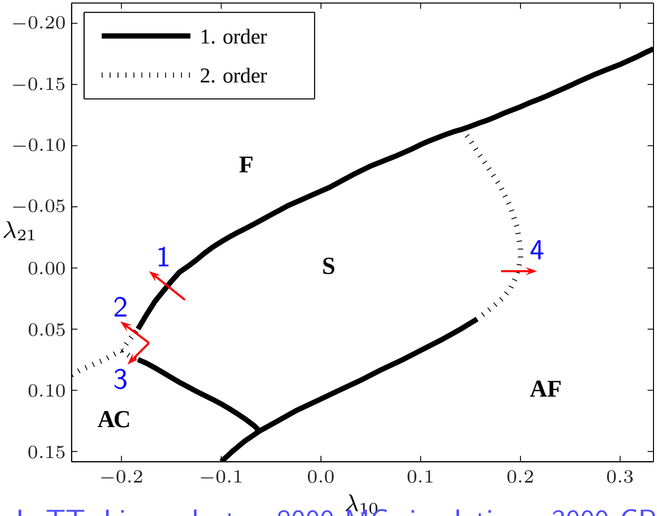
  critical exponents in mean field?
- finite temperature gluodynamics
  $\longrightarrow$ effective $\mathbb{Z}_3$ models with compact target spaces
  $\longrightarrow$ 3-state Potts-model

universality test in 'unphysical region'
(for gluodynamics)

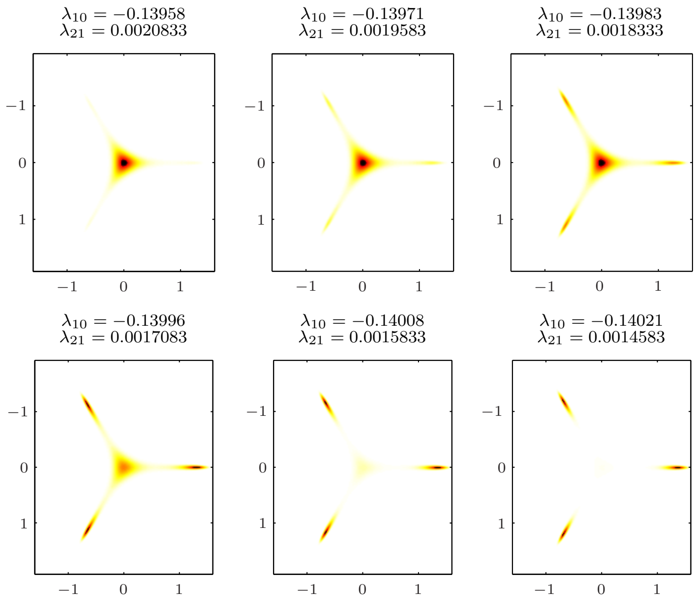
Results of MC-Simulations

- phase diagram and transitions → histograms
  large statistics, expensive → fast algorithms!
  standard Metropolis: 5% to 10% accuracy
- multicanonical algorithm: up to 20^3 lattices near first order transitions
- new cluster algorithm near second order transitions: auto-correlation times down by two orders of magnitude on larger lattices
- comparison with mean field results for two-coupling (costy).
- rich phase structure: 4 different phases, second und first order transitions, tricritical points(?), mean field very good.

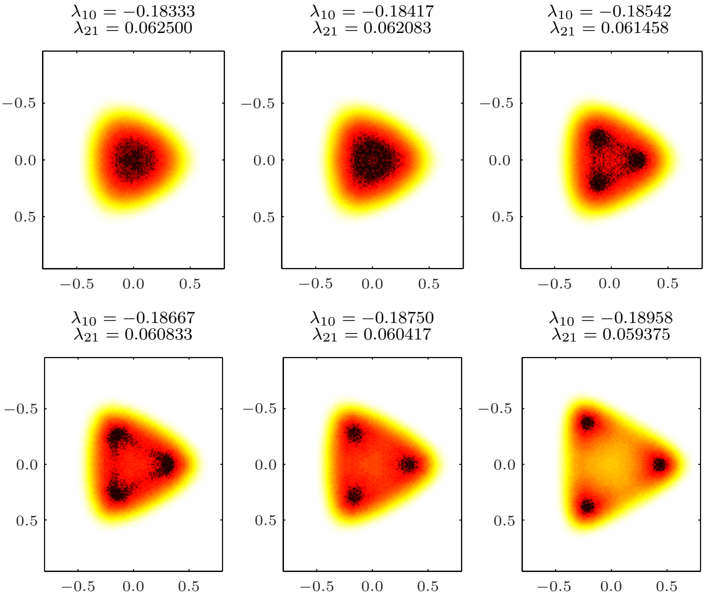

MF
ferrom.
symmetric
AC
antiferrom.
MC
ferrom.
symmetric
AC
antiferrom.
−0.2
−0.1
0.0
0.1
λ_{21}
λ_{01}
−.2
−.1
0.0
0.1
0.2
0.3
ℓ_r
2.5
2.0
1.5
−1.0
0.5
0.0
−0.5



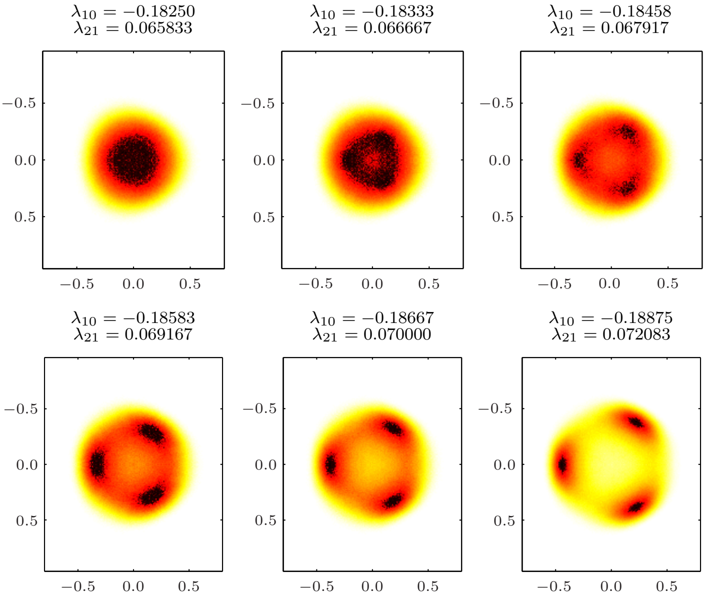
jenLaTT, Linux cluster, 8000 MC simulations, 3000 CPUh



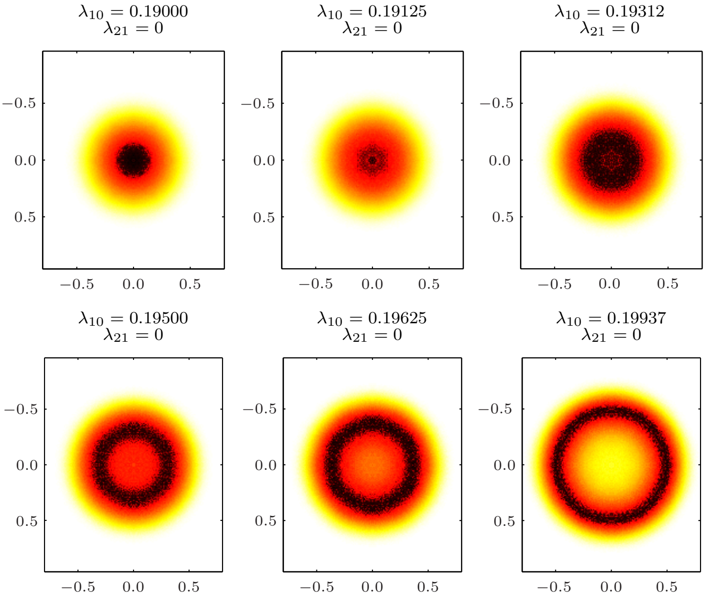
- 1: Histogramm of L, S $\leftrightarrow$ FM, 1st



- 2: Histogramm of L, S $\leftrightarrow$ FM, 2nd



- 3: Histogramm of L, S ↔ AC, 2nd



- 4: Histogramm of M, S $\leftrightarrow$ AF, 2nd

Conclusions

- strong coupling for Polykaov-loops effective action
- modified MF for non-translationally invariant states
- new efficient cluster algorithm!

- rich phase structure for simple $\mathbb{Z}_3$-models
- mean field unexpectedly accurate ($d_c = 3$?)

- calculate $\lambda_j(\beta)$ via IMC (cp. $SU(2)$)
- efficient 'group-theoretic' Schwinger-Dyson equations!
- vacuum-sector of AF phase? $SU(N)$ group integrals?
- is AC-phase relevant for gluodynamics?
- include fermions in effective Polyakov-loop dynamics.

JHEP 06 (2004) 005, PRD 72 (2005) 065005, hep-lat/0605012