Determinants, Dirac Operators and One-Loop Physics

A. Wipf

Theoretisch-Physikalisches Institut, FSU Jena

with Abrikosov, Beneventano, Blau, Dettki, Duerr, Falomir, Kirsten
Pisani, Mukhanov, Sachs, Santangelo, Visser, Wiesendanger, Zelnikov,. . .

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Selected problems

Useful methods

Boundary conditions for spinor fields

Results for bag-BC

Selected problems

- Euclidean gauge theories on compact spaces

$$Z[\eta,\bar\eta] = \int \mathcal{D}(A,\psi)\, e^{-S_{YM}+\int \bar\psi i \not{D}\psi + \int \bar\eta\psi + \bar\psi\eta}$$

$$= \sum_{\text{sectors}} \int \mathcal{D}A\, e^{-\Gamma[A]+\int \bar\eta S'\eta} \prod_{k=1}^{n} (\bar\eta,\psi_k)(\bar\psi_k,\eta),$$

- needed: determinants, Greenfunction, zero-modes, ...

$$\Gamma[A] = S_{YM} - \log \det{}' i\not{D}, \quad S' = \frac{1}{i\not{D}'}, \quad i\not{D}\psi_k = 0$$

-  thermal boundary conditions $\Rightarrow$ finite temperature QFT

- solution of 2d gauge theories (T^2: with I. Sachs)

- Semiclassical approximation (one-loop)
  Generating functional: $\phi : \mathcal{M} \longrightarrow$ target space

$$Z[\mathcal{M}, g, j] = \int \mathcal{D}\phi e^{-S[\phi,g]+(j,\phi)} = e^{W[\mathcal{M},g,j]}$$

  Legendre transform: $W \rightarrow$ effective action $\Gamma[\mathcal{M}, g, \varphi]$
- scaling behaviour $\longrightarrow$ β-function, φ-renormalization ...

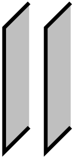
$$\Gamma[\lambda\mathcal{M}, \lambda^{d_g} g, \lambda^{d_\varphi}\varphi] = \Gamma[\mathcal{M}, g(\lambda), \sqrt{Z(\lambda)}\,\varphi]$$

- expand $\phi = \phi_{cl} + \delta\phi$, up to $(\delta\phi)^2 \Rightarrow$

$$\begin{aligned} \Gamma^{(1)} &= S[\varphi] + \frac{\hbar}{2}\log\det S''(\mathcal{M}, g, \varphi) \\ \mathcal{L} &= \frac{1}{2}(\partial_\mu\phi)^2 + V(\phi) \Longrightarrow S'' = -\triangle + V''(\varphi) \end{aligned}$$

- $\lambda_n(\lambda) = \lambda^{-2}\lambda_n(1)$, HK $\Rightarrow g^{(1)}(\lambda), Z^{(1)}(\lambda), \ldots$ (BC) (Duerr, Wiesendanger, W.)

- Casimir-effect



free field $H = \sum \omega_n (a_n^\dagger a_n + \frac{1}{2})$
ω_n geometry and field-dependent
Casimir energy $E_{\rm vac} = \frac{1}{2} \sum_n \omega_n$
better: $\zeta(s) = \sum \omega_n^{-s}$, $E_{\rm vac} = \frac{1}{2}\zeta(1)$.
$\to$ measured Casimir force

(with Blau, Visser)

- Fractional charges

massless fermions, $i\not{D}\psi_n = \lambda_n \psi_n$.
charge vs spectral asymmetry

$$\eta(s) = \sum \frac{\text{sign}(\lambda_n)}{|\lambda_n|^s}, \quad Q = \eta(0)$$

$\eta(0)$ related to scattering problem
depend on boundary conditions (APS, bag, ...)
(with Forgacs, O'Raifeartaigh, Kirsten, La Plata, Abrikosov)

- Hawking-radiation
  scalar field in curved space: effective action

$$
\begin{aligned}
e^{-\Gamma[g_{\mu\nu}]} &= \int \mathcal{D}\phi\, e^{-S[g_{\mu\nu},\phi]} \\
\langle T_{\mu\nu}\rangle &= \frac{2}{\sqrt{g}}\frac{\delta\Gamma[g_{\mu\nu}]}{\delta g^{\mu\nu}}
\end{aligned}
$$

  $\langle T_{\mu\nu}\rangle \to$ vacuum polarization, Hawking radiation
- minimally coupled scalar field

$$
\Gamma[g_{\mu\nu}] = -\tfrac{1}{2}\log\det(-\triangle_g)
$$

- calculated in $2d$ with HK-method and $g^{\alpha}_{\mu\nu} = e^{2\alpha\sigma}\delta_{\mu\nu}$
- almost done for s-wave sector in $4d$
  heat kernel not sufficient (with Mukhanov, Zelnikov)
- $d = 4$ conformally coupled, $g_{\mu\nu}(x) \to e^{2\alpha\sigma(x)} g_{\mu\nu}(x)$

Useful methods

- Divergences:
  second order, elliptic differential operator A in $\mathcal{M}$

$$\text{Weyl:}\quad \lambda_n(A) \sim c\, n^{2/d}, \quad d = \dim(\mathcal{M})$$

  need regularization (definition) of divergent objects

$$\det(A), \quad \det_{d>1}\frac{A}{A_0}$$

- popular and very useful: heat kernel techniques

$$\log\det A = -\zeta_A'(0), \quad \zeta_A(s) = \mathrm{Tr}\, A^{-s} = \sum_n \lambda_n^{-s}$$

$$\zeta_A(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\, \mathrm{Tr}\, e^{-tA}$$

$$\mathrm{Tr}\, e^{-tA} = \int_{\mathcal{M}} dx\, K(t;x,x), \quad K(t;x,y) = \langle x|e^{-tA}|y\rangle$$

- deformations $\leftrightarrow$ integrating anomalies
  $A \equiv A_1 \in \{A_\alpha | \alpha \in [0,1]\}, \quad A_\alpha \psi_n^\alpha = \lambda_{n,\alpha} \psi_n^\alpha$, ortho

$$\begin{aligned} \frac{d\lambda_{n,\alpha}}{d\alpha} &= \lambda_{n,\alpha} \langle \psi_n^\alpha | F | \psi_n^\alpha \rangle \\ \frac{d}{d\alpha} \sum \lambda_{n,\alpha}^{-s} &= -s \sum \lambda_{n,\alpha}^{-s} \langle \psi_n^\alpha | F | \psi_n^\alpha \rangle = -s \,\mathrm{Tr}\left(A_\alpha^{-s} F \right) \\ \frac{d}{d\alpha} \mathrm{Tr}\, A_\alpha^{-s} &= -\frac{s}{\Gamma(s)} \int t^{s-1} \mathrm{Tr}\left(e^{-tA_\alpha} F \right) \end{aligned}$$

- α-variation of $\zeta'_{A_\alpha}(0)$ from heat kernel coefficients

$$\mathrm{Tr}\, e^{-tA_\alpha} F = \frac{1}{(4\pi t)^{\frac{d}{2}}} \Big(\sum t^n \int a_n F + \sum t^{n/2} \oint b_{n/2} F \Big)$$

  heat kernel coefficients $a^\alpha_{d/2}, b^\alpha_{d/2} \Rightarrow \zeta'_{A_1}(0) - \zeta'_{A_0}(0)$
  A_0 sufficiently simple $\Rightarrow$ det A
- $2d$ induced gravity, $2d$ gauge theories, β-functions, ...

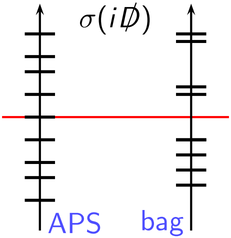
- explicit spectral analysis
  A sufficient simple (1-dimensional, radial, free field, highly symmetric background): calculate λ_n, $\det A$
  constant field strength (with Blau, Visser)
- scattering theory, dispersion relations
  $\Rightarrow$ spectral functions vs. scattering data
  e.g. $\eta(0) \sim$ phase shifts
  determinants $\sim$ Jost functions
  applied to
  sphalerons, monopoles, soliton quantization, ...

Boundary conditions (BC) for spinor fields

$\not{D}$ on compact $\mathcal{M} \subset \mathbb{R}^{d=2n}$ with boundary $\partial\mathcal{M}$

- nonlocal APS-BC
  $\{\not{D}, \gamma_*\} = 0 \Rightarrow$ spectrum symmetric, $\gamma_* \propto \gamma^0 \cdots \gamma^{d-1}$
  asymmetry only from zero-modes (index theorem)
  no breaking of CS in finite volume and $N_F > 1$
- attractive alternative: local bag-BC

$$\psi = B_\theta \psi \ \text{ on } \ \partial\mathcal{M}, \quad B_\theta = i\gamma_* e^{\theta\gamma_*}\gamma_n, \quad \gamma_n = n^\mu \gamma_\mu$$



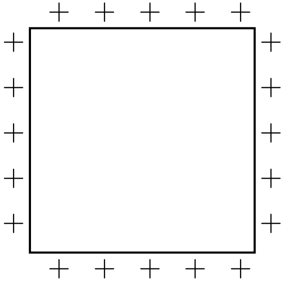
bag boundary conditions: local,
selfadjoint, strongly elliptic
spectrum not symmetric $\Rightarrow$
breaking of γ_*-symmetry
no zero-modes
no flux quantisation

- text book formula for generating functional

$$Z[\eta,\bar\eta]=\int\mathcal{D}A\,e^{-S_{YM}[A]+\int\bar\eta S_\theta\eta}\det\nolimits_\theta(i\not{D})$$

determinant and Greenfunction θ-dependent
classical symmetry $\psi\to e^{\alpha\gamma_*}\psi$, broken by bag-BC

- Boundary conditions and symmetry breaking



Ising model:
$\mathbb{Z}_2$ broken below T_c
for symmetric (periodic)
boundary cond. $\langle\sigma_x\rangle=0$
need trigger:
a) constant magnet field
b) $\mathbb{Z}_2$-breaking bc

- how to probe broken phase

	spin models	gauge theories
symmetry	$\mathbb{Z}_2 : \sigma_x \to -\sigma_x$	$U_A(1) : \psi \to e^{\alpha\gamma_*}\psi$
order par.	$\langle\sigma_x\rangle$	$\langle\bar{\psi}(x)\psi(x)\rangle$
trigger 1	magnetic field	quark mass
trigger 2	+ bc	bag bc

- parity transformation

$$\lambda_n(\theta, A) = -\lambda_n(-\theta, \tilde{A})$$

constraints on S_θ, $\det_\theta i\not{D}$ and asymmetry

$$\begin{aligned} \zeta(\theta, A) &= \zeta(-\theta, \tilde{A}) \\ \eta(\theta, A) &= -\eta(-\theta, \tilde{A}) \end{aligned}$$

Results for bag-BC

- green function for $A = 0$ in spherical bag

$$S_\theta^{(0)}(x,y) = S^{(0)}(x,y) + \frac{\Gamma(n)}{2R\pi^n}\gamma_* e^{\theta\gamma_*}\frac{R^2 - (\gamma,x)(\gamma,y)}{\left(R^2 - 2xy + \frac{x^2y^2}{R^2}\right)^n}$$

  green function for $A \neq 0$ in $2d$ spherical bag

$$S_\theta(x,y) = G^\dagger(A,x) S_\theta^{(0)}(x,y) G(A,y)$$

  similar results on cylinders

- heat kernel on cyliner $\mathcal{M} = [0,\beta] \times [0,L]$
  finite temperature: antiperiodic bc in x^0
  bag bc in spatial direction
  $K(t,x,y)$ explicitly constructed (with Duerr, Santangelo, Beneventano)

- η-function of $i\partial\!\!\!/$ in ball

$$\eta(s,i\partial\!\!\!/) = \sum_{\ell=0}^{\infty} \frac{a_\ell(d)}{2\pi i} \int_\Gamma \frac{d}{dz} \log\left(H(\theta,zR) - H(-\theta,zR)\right)$$

$$H(\theta,x) = \frac{J_{\nu-1/2}(x) - e^\theta J_{\nu+1/2}(x)}{J_{\nu-1/2}(x) + e^\theta J_{\nu+1/2}(x)}, \quad \nu = \ell + d - \tfrac{1}{2}$$

- asymmetry in 2 and 4 dimensions

$$\eta(0,i\partial\!\!\!/) = -\tfrac{1}{2}\tanh\tfrac{\theta}{2}$$

$$\eta(0,i\partial\!\!\!/) = \frac{\tanh\frac{\theta}{2}}{6144\cosh^6\frac{\theta}{2}}\Big(259 + 344\cosh\theta + 161\cosh 2\theta + 16\cosh 3\theta\Big)$$

odd functions (parity transformation)
(with Kirsten, Kirchberg, Santangelo)

- determinants: deformation technique,

$$\frac{d}{d\theta}\lambda_n = -\lambda_n(\psi_n, \gamma_*\psi_n) \Rightarrow$$
$$\frac{d}{d\theta}\log\det{}_\theta(i\not{D}) = -\int \text{Chern}(F) + \dots$$

induced θ-term

- chiral condensates

for $2d$ Abelian gauge theories on disk and cylinder:
single-flavour QED_2: condensate, CSB
multi-flavour QED_2: no condensate, no CSB
QCD_2 with Dirac-fermions: no CSB

bag-BC as good as small quark masses
toy-model calculations much simpler
useful: heat-kernel + deformation methods