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Algebraic Solution of the Supersymmetric Hydrogen Atom

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Abstract

The $\mathcal{N} = 2$ supersymmetric extension of the SCHRÖDINGER-HAMILTONian with $1/r$-potential in d dimension is constructed. The system admits a supersymmetrized LAPLACE-RUNGE-LENZ vector which extends the rotational $SO(d)$ symmetry to a hidden $SO(d+1)$ symmetry. It is used to determine the discrete eigenvalues with their degeneracies and the corresponding bound state wave functions.

1 Classical motion in *Newton/Coulomb* potential

For a closed system of two non-relativistic point masses interacting via a central force the angular momentum $\boldsymbol{L}$ of the relative motion is conserved and the motion is always in the plane perpendicular to $\boldsymbol{L}$. If the force is derived from a $1/r$-potential, there is an additional conserved quantity: the LAPLACE-RUNGE-LENZ[1] vector,

$$\boldsymbol{C} = \frac{1}{m}\,\boldsymbol{p} \times \boldsymbol{L} - \frac{e^2}{r}\,\boldsymbol{r}.$$

This vector is perpendicular to $\boldsymbol{L}$ and points in the direction of the semi-major axis. For the hydrogen atom the corresponding Hermitian vector operator has the form

$$\boldsymbol{C} = \frac{1}{2m}\left(\boldsymbol{p} \times \boldsymbol{L} - \boldsymbol{L} \times \boldsymbol{p}\right) - \frac{e^2}{r}\boldsymbol{r} \tag{1}$$

with reduced mass m of the proton-electron system. By exploiting the existence of this conserved vector operator, PAULI calculated the spectrum of the hydrogen

[1]A more suitable name for this constant of motion would be HERMANN-BERNOULLI-LAPLACE vector, see [1].

atom by purely algebraic means [2, 3]. He noticed that the angular momentum $\boldsymbol{L}$ together with the vector operator

$$\boldsymbol{K}=\sqrt{\frac{-m}{2H}}\,\boldsymbol{C}, \tag{2}$$

which is well-defined and Hermitian on bound states with negative energies, generate a hidden $SO(4)$ symmetry algebra,

$$\begin{aligned}[][L_a,L_b] &= i\epsilon_{abc}L_c\,,\\ [L_a,K_b] &= i\epsilon_{abc}K_c\,,\\ [K_a,K_b] &= i\epsilon_{abc}L_c\,,\end{aligned} \tag{3}$$

and that the HAMILTON-Operator can be expressed in terms of $\mathcal{C}_{(2)}=\boldsymbol{K}^2+\boldsymbol{L}^2$, one of the two second-order CASIMIR operators of this algebra, as follows

$$H=-\frac{me^4}{2}\frac{1}{\mathcal{C}_{(2)}+\hbar^2}. \tag{4}$$

One also notices that the second CASIMIR operator $\tilde{\mathcal{C}}_{(2)}=\boldsymbol{L}\cdot\boldsymbol{K}$ vanishes and arrives at the bound state energies by purely group theoretical methods. The existence of the conserved vector $\boldsymbol{K}$ also explains the accidental degeneracy of the hydrogen spectrum.

We generalize the COULOMB-problem to d dimensions by keeping the $1/r$-potential. Distances are measured in units of the reduced COMPTON wavelength, such that the SCHRÖDINGER-operator takes the form

$$H=p^2-\frac{\eta}{r}\;,\quad p_a=\frac{1}{i}\,\partial_a\;,\quad a=1,\dots,d\;. \tag{5}$$

η is twice the fine structure constant. Energies are measured in units of $mc^2/2$.

The Hermitian generators $L_{ab}=x_ap_b-x_bp_a$ of the rotation group satisfy the familiar $so(d)$ commutation relations

$$[L_{ab},L_{cd}]=i\left(\delta_{ac}L_{bd}+\delta_{bd}L_{ac}-\delta_{ad}L_{bc}-\delta_{bc}L_{ad}\right)\;. \tag{6}$$

It is not very difficult to guess the generalization of the LAPLACE-RUNGE-LENZ vector (1) in d dimensions [4],

$$C_a=L_{ab}p_b+p_bL_{ab}-\frac{\eta x_a}{r}\;. \tag{7}$$

These operators commute with H in (5) and form a $SO(d)$-vector,

$$[L_{ab},C_c]=i(\delta_{ac}C_b-\delta_{bc}C_a)\;. \tag{8}$$

The commutator of C_a and C_b is proportional to the angular momentum,

$$[C_a,C_b]=-4iL_{ab}H\;. \tag{9}$$

Now one proceeds as in three dimensions and defines on the negative energy subspace of $L_2(\mathbb{R}^d)$ the Hermitian operators

$$K_a = \frac{1}{2}\frac{C_a}{\sqrt{-H}} \quad \text{with} \quad [K_a, K_b] = iL_{ab}\ . \tag{10}$$

The operators $\{L_{ab}, K_a\}$ form a closed symmetry algebra and can be combined to form generators L_{AB} of the orthogonal group[1] $SO(d+1)$,

$$L_{AB} = \left(\begin{array}{c|c} L_{ab} & K_a \\ \hline -K_b & 0 \end{array}\right). \tag{11}$$

They obey the commutation relations (6) with indices running from 1 to $d+1$.

One finds a relation similar to (4) by solving

$$C_a C_a = -4K_a K_a H = \eta^2 + \left(2L_{ab}L_{ab} + (d-1)^2\right) H$$

for the Hamiltonian,

$$H = p^2 - \frac{\eta}{r} = -\frac{\eta^2}{(d-1)^2 + 4\mathcal{C}_{(2)}}. \tag{12}$$

$\mathcal{C}_{(2)}$ is the second-order CASIMIR operator of the dynamical symmetry group,

$$\mathcal{C}_{(2)} = \frac{1}{2}L_{AB}L_{AB} = \frac{1}{2}L_{ab}L_{ab} + K_a K_a\ . \tag{13}$$

It remains to find the admitted irreducible representations of $SO(d+1)$. In three dimensions they are fixed by the condition $\tilde{\mathcal{C}}_{(2)} = 0$ on the CASIMIR operator not entering the relation (4). In $d = 2n-1$ and $d = 2n$ dimensions there are n CASIMIR operators of the dynamical symmetry group and we expect $n-1$ conditions. The analysis in [5] lead to the following results:

- Only the completely symmetric representations of $SO(d+1)$ are realized.
- As in three dimensions the energies, degeneracies and eigenfunctions are determined by group-theoretic methods.

2 Susy Quantum Mechanics

The HILBERT-Space of a supersymmetric system is the sum of its bosonic and fermionic subspaces, $\mathcal{H} = \mathcal{H}_\mathrm{B} \oplus \mathcal{H}_\mathrm{F}$. Let A be a linear operator $\mathcal{H}_\mathrm{F} \to \mathcal{H}_\mathrm{B}$. We shall use a block notation such that the vectors in $\mathcal{H}_\mathrm{B}$ have upper and those in $\mathcal{H}_\mathrm{F}$ lower components,

$$|\psi\rangle = \begin{pmatrix} |\psi_\mathrm{B}\rangle \\ |\psi_\mathrm{F}\rangle \end{pmatrix}.$$

[1]For scattering states ($E > 0$) a similar redefinition leads to generators of the Lorentz group $SO(d,1)$. Here we are interested in bound states and will not further discuss this possibility.

Then the nilpotent *supercharge* and its adjoint take the forms

$$\mathcal{Q}=\begin{pmatrix}0 & A\\ 0 & 0\end{pmatrix},\quad \mathcal{Q}^\dagger=\begin{pmatrix}0 & 0\\ A^\dagger & 0\end{pmatrix}\Longrightarrow\{\mathcal{Q},\mathcal{Q}\}=0. \tag{14}$$

The block-diagonal super-HAMILTONian

$$H\equiv\{\mathcal{Q},\mathcal{Q}^\dagger\}=\begin{pmatrix}AA^\dagger & 0\\ 0 & A^\dagger A\end{pmatrix}=\begin{pmatrix}H_{\rm B} & 0\\ 0 & H_{\rm F}\end{pmatrix}, \tag{15}$$

commutes with the supercharge and the (fermion) number operator

$$\mathrm{N}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Bosonic states have $\mathrm{N}=0$ and fermionic states $\mathrm{N}=1$. The supercharge and its adjoint decrease and increase this conserved number by one.

In most applications in quantum mechanics A is a first order differential operator

$$A=i\partial_x+iW(x) \tag{16}$$

and yields the isospectral partner-HAMILTONians

$$H_{\rm B}=p^2+V_{\rm B}\ ,\quad H_{\rm F}=p^2+V_{\rm F},\quad \text{with}\quad V_{\rm B/F}=W^2\pm W'. \tag{17}$$

Such one-dimensional systems were introduced by NICOLAI and WITTEN some time ago [6, 7]. See the texts [8, 9] for a discussion of such models and in particular their relation to isospectral deformations and integrable systems.

3 SQM in Higher Dimensions

Supersymmetric quantum mechanical systems also exist in higher dimensions [7, 10]. The construction is motivated by the following rewriting of the supercharge

$$\mathcal{Q}=\psi\otimes A\quad\text{and}\quad \mathcal{Q}^\dagger=\psi^\dagger\otimes A^\dagger$$

containing the *fermionic* operators

$$\psi=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\quad\text{and}\quad \psi^\dagger=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

with anti-commutation relations

$$\{\psi,\psi\}=\{\psi^\dagger,\psi^\dagger\}=0\quad\text{and}\quad\{\psi,\psi^\dagger\}=\mathbb{1}.$$

In [10] this construction has been generalized to d dimensions. Then one has d fermionic annihilation operators ψ_k and d creation operators $\psi_k^\dagger$,

$$\{\psi_k,\psi_\ell\}=\{\psi_k^\dagger,\psi_\ell^\dagger\}=0\quad\text{and}\quad\{\psi_k,\psi_\ell^\dagger\}=\delta_{k\ell},\quad k,\ell=1,\dots,d. \tag{18}$$

For the supercharge one makes the ansatz

$$\mathcal{Q} = i\sum \psi_k \left(\partial_k + W_k(\boldsymbol{x})\right).$$

It is *nilpotent* (i.e. $\mathcal{Q}^2 = 0$) if and only if $\partial_k W_\ell - \partial_\ell W_k = 0$ holds true. Locally this integrability condition is equivalent to the existence of a potential $\chi(x)$ with $W_k = \partial_k\chi$. Thus we are lead to the following *nilpotent* supercharge

$$\mathcal{Q} = e^{-\chi}\mathcal{Q}_0\, e^{\chi} \quad \text{with} \quad \mathcal{Q}_0 = i\sum \psi_k\partial_k\,. \tag{19}$$

It acts on elements of the HILBERT-space

$$\mathcal{H} = L_2(\mathbb{R}^d)\otimes\mathbb{C}^{2^d},$$

which is graded by the 'fermion-number' operator $\mathrm{N} = \sum \psi_a^\dagger\psi_a$,

$$\mathcal{H} = \mathcal{H}_0\oplus\mathcal{H}_1\oplus\ldots\oplus\mathcal{H}_d, \qquad \mathrm{N}\big|_{\mathcal{H}_p} = p\mathbb{1}. \tag{20}$$

A state in $\mathcal{H}_p$ has the expansion

$$\Psi = \sum f_{a_1\ldots a_p}(x)\,|a_1\ldots a_p\rangle, \quad |a_1\ldots a_p\rangle = \psi^\dagger_{a_1}\cdots\psi^\dagger_{a_p}|0\rangle \tag{21}$$

with antisymmetric $f_{a_1\ldots a_p}$. $\mathcal{Q}$ decreases N by one and its adjoint increases it by one. It follows that the super-HAMILTONian

$$\begin{aligned} H = \{\mathcal{Q},\mathcal{Q}^\dagger\} &= H_0\otimes\mathbb{1}_{2^d} - 2\sum\psi_k^\dagger\psi_\ell\,\partial_k\partial_\ell\chi \\ &= H_d\otimes\mathbb{1}_{2^d} + 2\sum\psi_k\psi_\ell^\dagger\,\partial_k\partial_\ell\chi \end{aligned} \tag{22}$$

preserves the 'fermion-number'. The operators in the extreme sectors,

$$\begin{aligned} H_0 \equiv H\big|_{\mathcal{H}_0} &= -\triangle + (\nabla\chi,\nabla\chi) + \triangle\chi \\ H_d \equiv H\big|_{\mathcal{H}_d} &= -\triangle + (\nabla\chi,\nabla\chi) - \triangle\chi. \end{aligned} \tag{23}$$

are ordinary SCHRÖDINGER-operators, whereas the restriction of H to any other sector is a matrix-SCHRÖDINGER-operator,

$$H_p \equiv H\big|_{\mathcal{H}_p}: \quad 2^{\binom{d}{p}}\times 2^{\binom{d}{p}} \;-\; \text{matrix}.$$

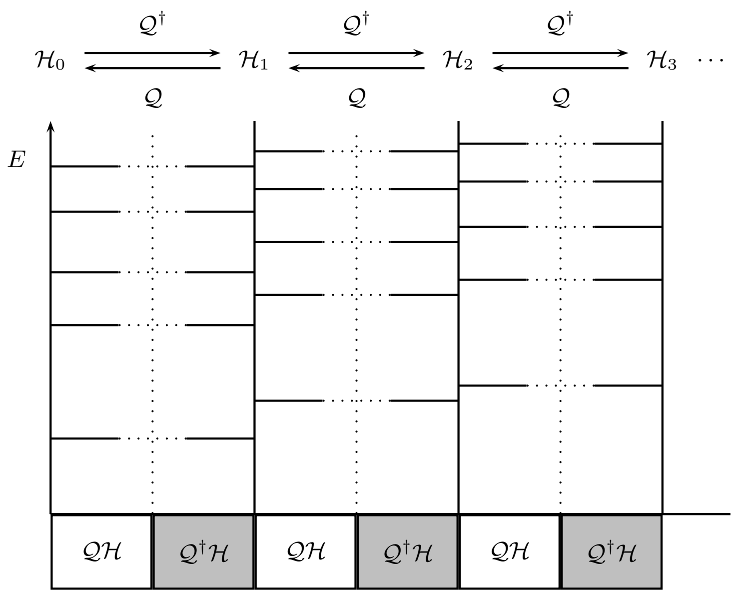
Due to the nilpotency of Q and $[Q,H]=0$ one has a HODGE-type decomposition of the HILBERT-space [5],

$$\mathcal{H} = \mathcal{Q}\mathcal{H}\oplus\mathcal{Q}^\dagger\mathcal{H}\oplus\operatorname{Ker}H\,. \tag{24}$$

Actually, the graded HILBERT-space is a $\mathcal{Q}$-complex of the following structure,

$$\mathcal{H}_0 \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \mathcal{H}_1 \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \mathcal{H}_2 \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \bullet\;\bullet\;\bullet\;\bullet \underset{\mathcal{Q}}{\overset{\mathcal{Q}^\dagger}{\rightleftarrows}} \mathcal{H}_d$$

Similarly as in the one-dimensional case one has a pairing of all H-eigenstates with non-zero energy. Every excited state is degenerate and the eigenfunctions are mapped into each other by $\mathcal{Q}$ and its adjoint. The situation is depicted in the following figure,



pairing of states with $E > 0$

4 The supersymmetric H-Atom

We supersymmetrized the H-atom along these lines and showed that it admits supersymmetric generalizations of the angular momentum and LAPLACE-RUNGE-LENZ vector [5]. As for the ordinary COULOMB problem the hidden $SO(d+1)$-symmetry allows for a purely algebraic solution. Here we discuss the construction for the 3-dimensional system and sketch the generalization to arbitrary dimensions.

To construct the supersymmetrized H-atom in 3 dimensions we choose $\chi = -\lambda r$ in (19) and obtain the super-HAMILTONian [5]

$$H = (-\triangle + \lambda^2)\mathbb{1}_8 - \frac{2\lambda}{r}B, \quad B = \mathbb{1} - \mathrm{N} + S^\dagger S, \quad S = \hat{\boldsymbol{x}} \cdot \boldsymbol{\psi} \tag{25}$$

on the HILBERT-space

$$\mathcal{H} = L_2(\mathbb{R}^3) \times \mathbb{C}^8 = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3. \tag{26}$$

We defined the triplet $\boldsymbol{\psi}$ containing the 3 annihilation operators ψ_1, ψ_2, ψ_3. States in $\mathcal{H}_0$ are annihilated by S and states in $\mathcal{H}_3$ by $S^\dagger$. With $\{S^\dagger, S\} = \mathbb{1}$ we find the following HAMILTON-operators in these extreme subspaces,

$$H_0 = -\triangle + \lambda^2 - \frac{2\lambda}{r},$$
$$H_3 = -\triangle + \lambda^2 + \frac{2\lambda}{r}.$$

H_0 describes the proton-electron and H_3 the proton-positron system.

The conserved angular momentum contains a spin-type term,

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} = \boldsymbol{x} \wedge \boldsymbol{p} - i\boldsymbol{\psi}^\dagger \wedge \boldsymbol{\psi}. \tag{27}$$

The operators $\boldsymbol{x}$ and $\boldsymbol{\psi}$ are both vectors such that S and B in (25) commute with this total angular momentum. To find the susy extension of the RUNGE-LENZ vector is less simple. It reads [5]

$$\boldsymbol{C} = \boldsymbol{p} \wedge \boldsymbol{J} - \boldsymbol{J} \wedge \boldsymbol{p} - 2\lambda\, \hat{\boldsymbol{x}} B \tag{28}$$

with $\boldsymbol{J}$ from (27) and B from (25). The properly normalized vector

$$\boldsymbol{K} = \frac{1}{2} \frac{\boldsymbol{C}}{\sqrt{\lambda^2 - H}} \tag{29}$$

together with $\boldsymbol{J}$ form an $SO(4)$ symmetry algebra on the subspace of bound states for which $H < \lambda^2$.

To solve for the spectrum we would like to find a relation similar to (4). However, one soon realizes that there is no algebraic relation between the conserved operators $\mathbb{1}, \mathrm{N}, \boldsymbol{J}^2, \boldsymbol{K}^2$ and H. However, we can prove the equation

$$\begin{aligned} \lambda^2 \mathcal{C}_{(2)} = \boldsymbol{K}^2 H \quad &+ \quad \left(\boldsymbol{J}^2 + (1-\mathrm{N})^2\right) \mathcal{Q}\mathcal{Q}^\dagger \\ &+ \quad \left(\boldsymbol{J}^2 + (2-\mathrm{N})^2\right) \mathcal{Q}^\dagger \mathcal{Q}\,, \end{aligned} \tag{30}$$

where $\mathcal{C}_{(2)}$ is the second-order CASIMIR (4). This relation is sufficient to obtain the energies since each of the three subspaces in the HODGE-decomposition (24) is left invariant by H and thus we may diagonalize it on each subspace separately. Since $H|_{\mathcal{Q}\mathcal{H}} = \mathcal{Q}\mathcal{Q}^\dagger$ and $H|_{\mathcal{Q}^\dagger\mathcal{H}} = \mathcal{Q}^\dagger\mathcal{Q}$ we can solve (30) for H in both subspaces,

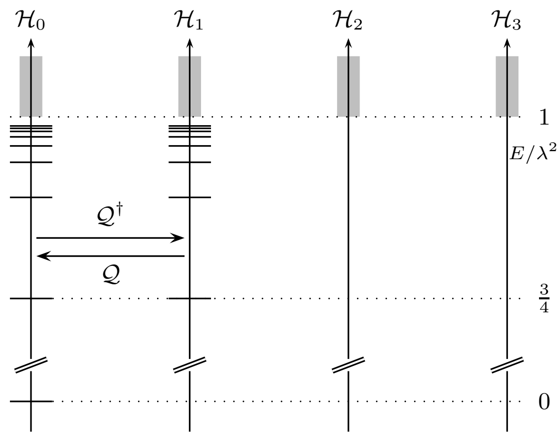
$$\begin{aligned} H\big|_{\mathcal{Q}\mathcal{H}} &= \lambda^2 \, \frac{\mathcal{C}_{(2)}}{(1-\mathrm{N})^2 + \mathcal{C}_{(2)}}\,, \\ H\big|_{\mathcal{Q}^\dagger\mathcal{H}} &= \lambda^2 \, \frac{\mathcal{C}_{(2)}}{(2-\mathrm{N})^2 + \mathcal{C}_{(2)}}\,. \end{aligned} \tag{31}$$

States with zero energy are annihilated by both $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, and according to (30) the second-order CASIMIR must vanish on these modes, such that

$$\mathcal{C}_{(2)}\big|_{\mathrm{Ker}\, H} = 0\,.$$

We conclude that every supersymmetric ground state of H is an $SO(4)$ singlet.

In the figure below we have plotted the spectrum of the supersymmetric H-atom in 3 dimensions. The bound states reside in the sectors with fermion numbers 0 and 1. In the sectors with fermion numbers 2 and 3 there are only scattering states. All bound states transform according to the symmetric representations of $SO(4)$. This is particular to 3 dimensions. The energies with degeneracies and the wave functions for all bound states can be found in [5].



realization of $so(4) \Rightarrow$ all bound states

5 Higher dimensions

The super-HAMILTONian (22) with $\chi = -\lambda r$ describes a supersymmetrized COULOMB-problem in d dimensions. As in 3 dimensions it can be solved with the help of a supersymmetrized angular momentum and RUNGE-LENZ vector generating a dynamical symmetry $SO(d+1)$. The supersymmetric extension of the angular momenta reads

$$J_{ab} = L_{ab} + S_{ab} \quad \text{with} \quad S_{ab} = \frac{1}{i}\left(\psi_a^\dagger \psi_b - \psi_b^\dagger \psi_a\right). \tag{32}$$

The supercharge, HAMILTONian and $\mathcal{S} = \hat{\boldsymbol{x}} \cdot \boldsymbol{\psi}$ are scalars with respect to the rotations generated by the J_{ab}. The supersymmetric extension of LAPLACE-RUNGE-LENZ vector

$$C_a = J_{ab} p_b + p_b J_{ab} - 2\lambda \hat{x}_a B \tag{33}$$

and the super-HAMILTONian

$$H = -\triangle + \lambda^2 - \frac{2\lambda}{r} B \tag{34}$$

both contain the scalar operator

$$B = \frac{1}{2}(d-1)\mathbb{1} - \mathrm{N} + \mathcal{S}^\dagger \mathcal{S}. \tag{35}$$

Again the FOCK-BARGMANN symmetry group $SO(d+1)$ is generated by

$$L_{AB} = \begin{pmatrix} L_{ab} & K_a \\ -K_b & 0 \end{pmatrix}, \quad K_a = \frac{C_a}{\sqrt{4(\lambda^2 - H)}},$$

and the second-order CASIMIR

$$\mathcal{C}_{(2)} = \frac{1}{2} J_{AB} J_{AB}, \tag{36}$$

together with λ, d, N enter the formulas for

$$H\big|_{\mathcal{Q}\mathcal{H}} \quad \text{and} \quad H\big|_{\mathcal{Q}^\dagger\mathcal{H}}.$$

The analysis parallels the one in 3 dimensions. To find the allowed representations one uses the branching-rules from the dynamical symmetry $SO(d+1)$ to the rotational symmetry $SO(d)$ generated by the J_{ab}. Only those representation for which the YOUNG-diagram has exactly one row and exactly one column give rise to normalizable states. The construction of the bound state wave function uses the realization of the CARTAN- and step operators H_α, E_α as differential operators. This way one finds the highest weight state in each representation [5].

6 Conclusions

We have succeeded in supersymmetrizing the celebrated construction of PAULI, FOCK and BARGMANN. For the COULOMB-problem with extended $\mathcal{N} = 2$ supersymmetry we have found the conserved angular momentum and conserved RUNGE-LENZ vector. Together they generate the FOCK-BARGMANN symmetry group $SO(d+1)$. A general relation of the type

$$\mathcal{Q}\mathcal{Q}^\dagger = f_1\left(\lambda, d, \mathbf{N}, \mathcal{C}_{(2)}\right) \quad \text{and} \quad \mathcal{Q}^\dagger\mathcal{Q} = f_2\left(\lambda, d, \mathbf{N}, \mathcal{C}_{(2)}\right) \tag{37}$$

has been derived which allows for an algebraic treatment of the supersymmetrized hydrogen atom in d dimensions. The energies depend on the fine structure constant, the dimension of space, the fermion number and the second order CASIMIR-operator. The bound states transform according to particular irreducible $SO(d+1)$-representations. The allowed representations, the explicit form of the bound states and their energies have been determined.

We have not discussed the scattering problem. It is well-known how to extend supersymmetric methods from bound to scattering states in supersymmetric quantum mechanical systems [12]. Thus one may expect that the construction generalizes to the scattering problem, for which the non-compact dynamical symmetry group will be $SO(d,1)$.

ITZYKSON and BANDER [13] distinguished between the infinitesimal and the global method to solve the COULOMB problem. The former is based on the LAPLACE-RUNGE-LENZ vector and is the method used here. In the second method one performs a stereographic projection of the d-dimensional momentum space to the unit sphere in $d+1$ dimensions which in turn implies a $SO(d+1)$

symmetry group. It would be interesting to perform a similar global construction for the supersymmetrized systems.

Every multiplet of the dynamical symmetry group appears several times [5] and there is a new 'accidental' degeneracy: in higher dimensions some eigenvalues of the Hamiltonian appear in many different particle-number sectors. It may very well be, that the algebraic structures discussed in the present work have a more natural setting in the language of superalgebras or the $SO(d,2)$-setting in [4]. We have not investigated this questions.

There exist earlier results on the supersymmetry of both the non-relativistic and relativistic hydrogen atom. In [14] the RUNGE-LENZ vector or its relativistic generalization, the JOHNSON-LIPPMANN operator, enter the expressions for the supercharges belonging to the *ordinary* SCHRÖDINGER- or DIRAC-operators with $1/r$ potential. This should be contrasted with the present work, where the COULOMB-problem is only a particular channel of a manifestly supersymmetric matrix-SCHRÖDINGER operator. Our HAMILTONians incorporate both the proton-electron and the proton-positron systems as particular subsectors.

The supercharge (19) and super-HAMILTONian (22) describe a wide class of supersymmetric systems, ranging from the supersymmetric oscillator in d dimensions to lattice WESS-ZUMINO-models with $\mathcal{N} = 1$ or $\mathcal{N} = 2$ supersymmetries in 2 dimensions [11]. In passing we mention, that the supercharge in d dimension is actually a dimensionally reduced DIRAC operator in $2d$ dimensions. During the reduction process the ABELian gauge potential A_μ in $2d$ dimensions transforms into the potential χ in (19), see [11].

More generally, one may ask for which gauge- and gravitational background field the DIRAC-operator admits an extended supersymmetry. This question has been answered in full generality in [15]. For example, on a 4-dimensional hyper-KÄHLER space with self-dual gauge field the DIRAC-operator admits an $\mathcal{N} = 4$ supersymmetry. The extended supersymmetry may be used to construct possible zero-modes of the DIRAC-operator. Earlier results on the supersymmetries of DIRAC-type operators can be found in [16], for example. COMTET and HORVATHY investigated the solutions of the DIRAC-equation in the hyper-KÄHLER TAUB-NUT gravitational instanton [17]. The spin 0 case can be solved with the help of a KEPLER-type dynamical symmetry [18] and the fermion case by relating it to the spin 0 problem with the help of supersymmetry.

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References

[1] H. Goldstein, *Am. J. Phys.* **43**, 737 (1975); *Am. J. Phys.* **44**, 1123 (1976).

[2] W. Pauli, *Z. Phys.* **36** (1926), 336 (1926).

[3] V. Fock, *Z. Phys.* **98**, 145 (1935); V. Bargmann, *Z. Phys.* **99**, 76 (1936).

[4] E.C.G. Sudarshan, N. Mukunda and L. O'Raifeartaigh, *Phys. Lett.* **19**, 322 (1965).

[5] A. Kirchberg, J.D. Länge, P.A.G. Pisani and A. Wipf, *Annals Phys.* **303**, 359 (2003).

[6] H. Nicolai, *J. Phys.* **A9**, 1497 (1976).

[7] E. Witten, *Nucl. Phys.* **B188**, 513 (1981).

[8] F. Cooper, A. Khare and U. Sukhatme, *Phys. Rep.* **251**, 267 (1995); G. Junker, *Supersymmetric Methods in Quantum Mechanics*, Springer, Berlin 1996.

[9] A. Wipf, *Non-perturbative Methods in Supersymmetric Theories*, arXiv: hep-th/0504180.

[10] A.A. Andrianov, N.V. Borisov and M.V. Ioffe, *Phys. Lett.* **A105**, 19 (1984); A.A. Andrianov, N.V. Borisov, M.V. Ioffe and M.I. Eides, *Phys. Lett.* **A109**, 1078 (1985); F. Cooper, A. Khare, R. Musto and A. Wipf, *Annals Phys.* **187**, 1 (1988).

[11] A. Kirchberg, J.D. Länge and A. Wipf, *Annals Phys.* **316**, 357 (2005).

[12] F. Cooper, J.N. Ginocchio and A. Wipf, *Phys. Lett.* **A29**, 145 (1988).

[13] M. Bander and C. Itzykson, *Rev. Mod. Phys.* **38**, 330 (1966).

[14] R.D. Tangerman and J.A. Tjon, *Phys. Rev.* **A48**, 1089 (1993); J.P. Dahl and T. Jorgenson, *Int. J. Quantum Chem.* **53**, 161 (1995); F. Bloore and P. Horvathy, *Journ. Math. Phys.* **33**, 1869 (1992); H. Katsura and H. Aoki, *Exact supersymmetry in the relativistic hydrogen atom in general dimensions - supercharge and the generalized Johnson-Lippmann operator*, arXiv:quant-ph/0410174.

[15] A. Kirchberg, J.D. Länge and A. Wipf, *Annals Phys.* **315**, 467 (2005).

[16] M. de Crombrugghe and V. Rittenberg, *Annals Phys.* **151**, 99 (1983); W. van Holten, *Phys. Lett.* **B342**, 47 (1995).

[17] A. Comtet and P.A. Horvathy, *Phys. Lett.* **B349**, 49 (1995).

[18] G.W. Gibbons and N. Manton, *Nucl. Phys.* **B274**, 183 (1986); G.W. Gibbons and P. Ruback, *Phys. Lett.* **B188**, 226 (1987); B. Cordani, L. Feher and P.A. Horvathy, *Phys. Lett.* **B201**, 481 (1988).