

# Tunnel Determinants

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## Abstract

Methods for computing the regularized determinants of fluctuation operators are being developed. The results follow from the fact that these determinants can be expressed by eigenmodes of the fluctuation operator. As an application the tunnel determinants of some one- and higher-dimensional models are computed. It is shown that every fluctuation operator defines a supersymmetric quantum mechanical system.

Keywords: vacuum decay; functional determinant; tunneling; instanton; bounce;  
semiclassical expansion

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## 1 Introduction

The theory of homogeneous nucleation has been a subject of research for at least fifty years. An important milestone was reached when Langer [1] developed the field theoretical formalism within the context of statistical mechanics. The link with quantum field theory was provided by Polyakov’s paper [2] in which he proposed the semiclassical approximation within the euclidean path integral formalism. Inspired by Polyakov’s work, Callan and Coleman [3] constructed the semiclassical decay amplitude by using the functional integral techniques pioneered by Langer. These authors made use of the characterization

$$E_0 = - \lim_{T \rightarrow \infty} \frac{1}{T} \log \langle x | e^{-TH} | x \rangle = - \lim_{T \rightarrow \infty} \frac{1}{T} \left( \frac{1}{N} \int_{w(0)=x}^{w(T)=x} e^{-S_E[w]} \mathcal{D}w \right) \quad (1)$$

of the ground state energy of  $H$ , in order to compute its semiclassical approximation  $E_{sc}$ . Then they changed the potential in  $H$ ; both a typical old potential  $V$  and new potential  $V^*$  are shown in Figure 1.

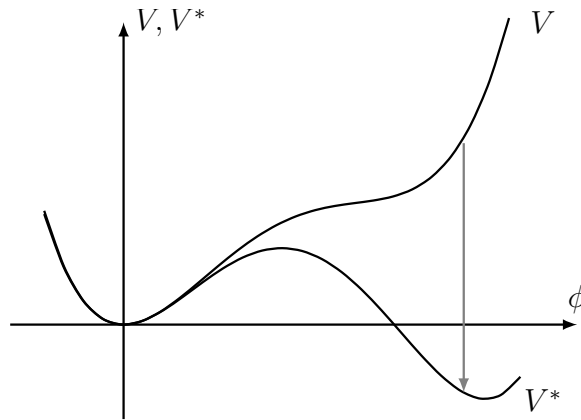


Figure 1: A change of the potential which causes a metastable state.

As a result  $E_0$  ceases to be the ground state energy; what is more interesting is that its analytical continuation develops an imaginary part due to the change of  $V$ .

In order to compute the analytical continuation of the semiclassical approximation  $E_{\text{sc}}$ , Callan and Coleman integrated over the quadratic fluctuations around multi-bounce configurations. By treating the collective coordinates in a suitable manner and assuming that the main contribution comes from multi-bounce configurations for which the bounces have negligible overlap (dilute gas approximation) they managed to exponentiate the analytic continuation of the sum

$$\langle x | e^{-TH} | x \rangle_{\text{sc}} = \sum_n \langle x | e^{-TH} | x \rangle_{n \text{ bounces}},$$

and found an explicit formula for  $\Gamma = 2\Im E_{\text{sc}}$ .

The analytic continuation of  $E_0$  is to be interpreted as the position of a second sheet resonance pole of the resolvent of the new Hamiltonian  $H^*$ . Whenever  $\Gamma/|E_0| < 1$  one expects the large time behaviour

$$p(T) = |(\psi_0, e^{itH^*T} \psi_0)|^2 \sim e^{-\Gamma T} + O\left(\frac{\Gamma}{E_0}\right) \left(\frac{\hbar}{E_0 T}\right)^\alpha \quad (2)$$

for the decay of the ground state  $\psi_0$  of  $H$  under the time evolution of  $H^*$  [4]. The positive constant  $\alpha$  is model dependent. Hence we interpret  $1/\Gamma$  as the decay time of the state  $\psi_0$ .

At this stage several remarks should be made:

- Firstly, the decay is exponentially for intermediate times only. For  $T \rightarrow \infty$  an exponential decay is not possible. This is a consequence of Khalfin's theorem [5] which states that for a Hamiltonian which is bounded below the function  $(1+x^2)^{-1} \cdot \log p(x)$  is integrable. However, in most situations this remark is of minor relevance since we expect that the decay already has happened when the second term in (2) dominates the first one.
- Secondly, the Callan-Coleman (CC) formula is valid only when the dilute gas condition is fulfilled, i.e. when  $\Gamma/V''(0) \ll 1$ . For the anharmonic oscillator model this inequality excludes the strong coupling regime [6].
- Thirdly, the derivation given by Callan and Coleman has been criticized. For instance, Patrascioiu [7] found a different answer for  $\Gamma$  for a simple quantum mechanical system by using the complex time method.
- Finally, it is far from being obvious that the analytic continuation of the approximated  $E_0$  is a good approximation to the analytic continuation of  $E_0$ .

Our interest in the vacuum decay process is due to the crucial role it plays in the inflationary cosmological models [8]. There one is forced to calculate, or at least estimate, the decay width of a metastable state. This symmetric state becomes metastable, due to the change of the effective potential with decreasing temperature. Since the CC approximation for  $\Gamma$  is the only result known to us which readily can be generalized to field theory, we adopt a pragmatical attitude and assume its validity in what follows. After all, even if the CC formula turns out to be not as good as we hope, then the following considerations may help to clarify this crucial problem.

The generalization of the CC formula to field theory yields

$$\frac{\Gamma}{V} = \left(\frac{S}{2\pi\hbar}\right)^{d/2} \left(-\det' \frac{-\Delta + V''(\phi)}{-\Delta + m^2}\right)^{-1/2} \exp(-S/\hbar) = \Theta \exp(-S/\hbar) \quad (3)$$

for the decay rate per volume and time of the metastable state. In this formula  $m^2 = V''(0)$ , where  $\phi = 0$  is the false vacuum. For the computation of the euclidean action

$$S = S[\phi] = \frac{1}{2} \|\nabla\phi\|^2 + \int d^2x V(\phi) \quad (4)$$

of the solution of the classical field equation (bounce)

$$\Delta\phi = V'(\phi), \quad \lim_{|x|\rightarrow\infty} \phi(x) = 0 \quad (5)$$

with least action, powerful variational methods are known [9]. In cosmological applications one typically uses dimensional arguments for a rough estimate of the radiative corrections  $\Theta$  to  $\exp(-S/\hbar)$ . It is the purpose of this paper to replace the rough estimate [10]

$$\Theta \approx \eta M^d, \quad (6)$$

where  $M$  is a characteristic mass of the theory and  $\eta$  is a dimensionless number of order unity, by a more accurate value.

Related results on functional determinants are found in the paper of de Vega [11] on the large- $N$  expansion. However, we have to deal with the additional difficulty that the fluctuation operator  $-\Delta + V''(\phi) = H + m^2$  has  $d$  zero modes [3] which must be dropped in evaluating the tunnel determinant (also called primed determinant). This truncation is indicated by the prime in the formula (3). As a by-product we show that every fluctuation operator defines a supersymmetric quantum mechanical system.

In section 2 we set up our problem in more detail, give the relevant properties of functional determinants and relate these objects to scattering data of the operator  $H$ . In two and more dimensions our tunnel determinants are UV-divergent. As a regularization we adopt the counterterm method [12]. This strategy seems to be more suitable for our task than the Pauli-Villars,  $\zeta$ -function or analytic regularizations. Finally we renormalize the divergent terms in the Feynman-graph expansion of the determinant.

Section 3 is devoted to the treatment of the zero modes of the fluctuation operator. In section 4 we apply the obtained results to some typical models. We find explicit expressions for one-dimensional and higher-dimensional tunnel determinants. The latter correspond to the conformally invariant scalar theories.

In section 5 we study the sector with angular momentum  $j = 1$  more carefully and find that the fluctuation operator defines a supersymmetric quantum mechanical system on this subspace.

The two appendices are devoted to some more mathematical results. We first prove that the functional determinant is analytic in  $m^2$ . Then we show that the tunnel solution (the bounce)  $\phi$  is spherically symmetric, even in cases where it is a more-component field.

## 2 Tunnel determinants

In defining functional determinants like

$$w(z) = \det(H - z)(H_0 - z)^{-1}, \quad (7)$$

where  $H_0 = -\Delta$  and  $H = H_0 + q(x)$  are Schrödinger operators, one tries to generalize the matrix identity  $\log \det B = \text{tr} \log B$  to operators. With

$$(H - z)(H_0 - z)^{-1} = 1 + A(z), \quad \text{where} \\ A(z) = q(x)(H_0 - z)^{-1} = q(x)R_0(z), \quad (8)$$

one therefore defines

$$\log w(z) = \text{tr} \log (1 + A(z)). \quad (9)$$

For the trace of  $\log(1 + A(z))$  to exist it suffices that the trace of  $|A(z)|$  is finite [12]. Actually this condition is too strong. For example, the operator

$$B : \ell_2 \mapsto \ell_2 \\ \{x_n\}_1^\infty \mapsto \left\{ \frac{(-)^{n+1}}{2n-1} x_n \right\}_0^\infty$$

is not trace class because  $\text{tr}|B| = \sum_1^\infty 1/(2n-1)$  is infinite. Nevertheless  $\text{tr} \log(1+B)$  exists and is equal  $\frac{1}{2} \log 2$ . More generally, the finiteness of  $\text{tr} B$  implies the finiteness of  $\text{tr} \log(1+B)$  if and only if  $\text{tr}|B|^2 < \infty$ .

The resolvent  $R_0(z)$  in (8) is an integral operator with free two-point Schwinger function  $S_2(z; x-y)$  with mass  $m^2 = -z$  as integral kernel. Then  $A(z)$  is an integral operator as well with kernel  $A(z; x, y) = q(x)S_2(z; x-y)$ , such that

$$\text{tr} A(z) = \int A(z; x, x) d^d x = \int q(x) d^d x S_2(z; 0). \quad (10)$$

In two and more dimensions [13]

$$S_2(z; \xi) \sim \begin{cases} c_d |\xi|^{2-d} & \text{for } d > 2, \\ c \log(|\xi|) & \text{for } d = 2, \end{cases}$$

and  $\text{tr} A(z) = \infty$  due to the short-distance singularity of the free Schwinger function. Consequently we must regularize the determinant  $w(z)$ . There exist many regularization procedures in field theory which easily can be transformed into regularization schemes for determinants. We will use the counterterm strategy [12] which is shortly described as follows. In the power series expansion<sup>1</sup>

$$\log(1+A(z)) = - \sum_{n=1}^{\infty} \frac{(-)^n}{n} A^n(z), \quad (11)$$

we denote the sum of the first  $p = [\frac{d}{2}]$  terms by  $P(A(z))$ . These are the terms which have an infinite trace. Now one simply drops this polynomial in defining the *regularized determinant*, i.e.

$$w_R(z) = \det_R(1+A(z)) = \det\{(1+A(z))e^{-P(A(z))}\}. \quad (12)$$

plot [] function To define a renormalized determinant we should restore the terms that were deleted in  $\det_R$ . It is not legitimate to just leave them out since this would not correspond to local counterterms. For that purpose one should bear in mind that one normally encounters functional determinants as one-loop contributions to effective actions. As such they are always accompanied by local one-loop counterterms, i.e.

$$\Gamma_\Lambda[\phi] = S[\phi] + \frac{1}{2}\hbar \log \det_\Lambda(1+A(z)) + \hbar S_\Lambda^{\text{ct}}[\phi] \quad (13)$$

Here  $\Lambda$  indicates a regularization, say a momentum cutoff, of the corresponding quantities. Hence one defines the renormalized determinant as

$$\log \det_{\text{ren}}(1+A(z)) = \lim_{\Lambda \rightarrow \infty} \{\log \det_\Lambda(1+A(z)) + 2S_\Lambda^{\text{ct}}[\phi]\}. \quad (14)$$

To be more specific let us consider the model  $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$  in  $d \leq 4$  dimensions. In two and three dimensions

$$S_\Lambda^{\text{ct}}[\phi] = \frac{1}{2}\delta m^2(\Lambda) \int \phi^2 \quad (15)$$

and in four dimensions

$$S_\Lambda^{\text{ct}}[\phi] = \frac{1}{2}\delta m^2(\Lambda) \int \phi^2 + \frac{1}{4}\delta\lambda(\Lambda) \int \phi^4. \quad (16)$$

To proceed we have to choose normalization condition(s). It is convenient to use a zero-momentum or constant field renormalization. For a constant field  $\Phi$  the effective actions  $\Gamma$  yields the effective potential  $U$ , i.e.  $\Gamma_\Lambda[\Phi]/\text{Vol} = U_\Lambda(\Phi)$ , and  $U_\Lambda$  can be computed explicitly in one-loop approximation..

If we take the renormalization conditions

$$\left. \frac{d^2 U}{d\Phi^2} \right|_{\Phi=0} = m^2 \quad (17)$$

<sup>1</sup>since  $\text{tr}(1+uA)$  is analytic in  $u$  we may assume  $|A(z)| < 1$ .

in two and three dimensions, or

$$\left. \frac{d^2 U}{d\Phi^2} \right|_{\Phi=0} = m^2, \quad \left. \frac{d^4 U}{d\Phi^4} \right|_{\Phi=0} = 6\lambda \quad (18)$$

in four dimensions, then

$$\det_{\text{ren}}(1 + A(z)) = \det_{\text{R}}(1 + A(z)) \quad (19)$$

in two and three dimensions and

$$\det_{\text{ren}}(1 + A(z)) = \det_{\text{R}}(1 + A(z)) \exp \left\{ (3\lambda/4\pi)^2 \int \phi^2(x) G(x-y) \phi^2(y) d^4 x d^4 y \right\}, \quad (20)$$

in four dimensions, where

$$G(\xi) = \frac{1}{(2\pi)^4} \int e^{ip\xi} \hat{G}(p) d^4 p,$$

with

$$\hat{G}(p) = \left( 1 + \frac{4m^2}{p^2} \right)^{1/2} \text{Arcoth} \left( 1 + \frac{4m^2}{p^2} \right)^{1/2} - 1.$$

We see, that if one knows the regularized determinant one easily recovers the renormalized one. Therefore, in what follows we are mainly concerned with the problem of calculating the regularized determinants.

In the appendix we show that  $\log w_{\text{R}}(z)$  is an analytic function at points which are not in the spectra of both,  $H$  and  $H_0$ . Several authors [14, 15] proved that the bounce  $\phi$  in (5) is spherically symmetric for a large class of Higgs potentials. These proofs apply to a one-component field. In the appendix we generalize this result to the case when  $\phi$  has several components and the action is invariant under rotations of the coordinates and field. Therefore we may assume that the function  $q(x) = V''(\phi) - m^2$  in (8) is spherically symmetric. This allows for an angular momentum expansion of  $w_{\text{R}}(z)$ . Let  $\mathcal{H}_j$  denote the subspace with angular momentum  $j$ . We obtain

$$w_{\text{R}}(z) = \prod_{j=0}^{\infty} \left\{ \det(1 + A_j(z)) e^{-P(A_j(z))} \right\}^{\dim \mathcal{H}_j}, \quad (21)$$

where

$$1 + A_j(z) = (h_j - z)(h_j^0 - z)^{-1},$$

and

$$h_j = -\partial_r^2 - \frac{d-1}{r} \partial_r + \frac{j(j+d-2)}{r^2} + q(r) \quad (22)$$

is the radial Schrödinger operator on  $\mathcal{H}_j$ . Since  $h_j$  is a one-dimensional operator it follows that  $A_j(z)$  is of trace class. But for trace class operators we have [12]

$$\det \left\{ (1 + A_j(z)) e^{-P(A_j(z))} \right\} = \det (1 + A_j(z)) e^{-\text{tr} P(A_j(z))} \quad (23)$$

Using the abbreviations

$$w_j(z) = \det (1 + A_j(z)) \quad \text{and} \quad p_j(z) = \text{tr} P(A_j(z)), \quad (24)$$

we finally end up with the expansion

$$w_{\text{R}}(z) = \prod_{j=0}^{\infty} \left\{ w_j(z) e^{-p_j(z)} \right\}^{\dim \mathcal{H}_j}. \quad (25)$$

Still we are left with the problem of computing the one-dimensional determinants  $w_j(z)$ . Here we use the result [11, 14] that  $w_j(z)$  is given by the normalized Jost function of  $H$ ,

$$w_j(z) = F(\mu, k) \quad \text{with} \quad k^2 = z, \quad \mu = j - 1 + \frac{d}{2}. \quad (26)$$

The Jost function can be defined in the following way: take the solution of the radial Schrödinger equation  $h_j\psi = k^2\psi$  which fulfills

$$\lim_{r \rightarrow \infty} e^{ikr} r^{(d-1)/2} \psi(r) = 0.$$

Then the unnormalized Jost function is given by

$$f(\mu, k) = 2\mu \lim_{r \rightarrow \infty} r^j \psi(r). \quad (27)$$

To obtain the normalized Jost function one divides  $f$  by the Jost function  $f^0$  of  $H_0$ , i.e.  $F(\mu, k) = f(\mu, k)/f^0(\mu, k)$ .

At this stage we may use known properties of Jost functions [16]. For example,  $F(\mu, k)$  is analytic in  $\Im(k) < m$  and  $\Re(\mu) > 0$  and its zeros on the negative imaginary axis correspond to bound states of  $H$ . Because  $w_j(z)$  vanishes exactly if  $z$  is a bound state energy of  $H$ , this implies that we must choose the root  $k = \sqrt{z}$  in (26) on the lower half-plane.

### One dimension

In one dimension the product (25) contains only two factors, since  $\dim \mathcal{H}_j = 0$  for  $j > 1$ . No counterterms  $p_j$  are needed. With  $F(\frac{1}{2}, k)F(-\frac{1}{2}, k) = T^{-1}(k)$ , where  $T$  denotes the transmission coefficient, the expansion (25) reduces to

$$w(z) = \det \left( \frac{H - z}{H_0 - z} \right) = \frac{1}{T(k)}. \quad (28)$$

### Higher dimensions

In higher dimensions we must determine the counterterms in the sectors with fixed angular momentum. For that purpose we first expand  $F(\mu, k)$  in powers of  $q(r)$  [16],

$$F(\mu, k) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} K_n = \sum \int \frac{(-1)^n}{n!} K_n(\mu; r_1, \dots, r_n) q(r_1) \cdots q(r_n), \quad (29)$$

where

$$K_n(\mu, r_1, \dots, r_n) = \det \begin{pmatrix} C(\mu; r_1, r_1) & \cdots & C(\mu; r_1, r_n) \\ \vdots & & \vdots \\ C(\mu; r_n, r_1) & \cdots & C(\mu; r_n, r_n) \end{pmatrix}$$

with

$$C(\mu; r, r') = -\frac{i}{2} \sqrt{rr'} J_\mu(kr_{<}) H_\mu(kr_{>}) \quad \text{and} \quad r_{\leq} = \frac{1}{2}(r + r') \pm \frac{1}{2}|r - r'|.$$

According to the described counterterm strategy we next expand

$$\log w_j(z) = \log \left( 1 + \sum_1^{\infty} \frac{(-1)^n}{n!} K_n \right) \quad (30)$$

in powers of  $q$ . The counterterm  $p_j$  is equal to the sum of the terms up to order  $[\frac{d}{2}]$ .

Now it is straightforward to express the expansion in terms of the Jost function and the  $K_n$ 's. Especially in three and four dimensions

$$w_R(z) = \begin{cases} \prod \{F(j + \frac{1}{2}, k) \exp(K_1)\}^{2j+1} & \text{for } d = 3, \\ \prod \{F(j + 1, k) \exp(K_1 - \frac{1}{2}[K_2 - K_1^2])\}^{(j+1)^2} & \text{for } d = 4. \end{cases} \quad (31)$$

One must bear in mind that the  $K_n$  in the exponents depend via  $\mu$  on  $j$  and  $d$ .

### 3 Division by the zero modes

In differentiating the field equation (5) for a spherically symmetric solution with respect to the radial variable,

$$\begin{aligned} 0 &= \partial_r \left( -\partial_r^2 \phi - \frac{d-1}{r} \partial_r \phi + V'(\phi) \right) \\ &= -\partial_r^2 (\partial_r \phi) - \frac{d-1}{r} \partial_r (\partial_r \phi) + V''(\phi) \partial_r \phi = (h_1 + m^2) \partial_r \phi, \end{aligned} \quad (32)$$

we recognize  $\partial_r \phi$  as a zero mode of  $h_1 + m^2$ . So there exist  $\dim(H_1) = d$  zero modes of the fluctuation operator with angular momentum  $j = 1$ . These are the  $d$  Goldstone modes which are due to the translational invariance of  $S[\phi]$ . Therefore  $w_1(h)$  vanishes at the interesting point  $z = -m^2$ . We apply l'Hospital's rule

$$w'_1(-m^2) = - \lim_{k \rightarrow -im} \frac{F(\frac{d}{2}, k)}{k^2 + m^2} = \frac{\frac{df}{dk}(\frac{d}{2}, -im)}{2im f^0(\frac{d}{2}, -im)}$$

for dividing  $w_1$  by this zero mode. Here  $f^0$  denotes the free Jost function. Next we use the identity [16]

$$\frac{\frac{df}{dk}(\frac{d}{2}, -im)}{f(\frac{d}{2}, -im)} = i \int f(\frac{d}{2}, im, r)^2 r^{d-1} dr, \quad (33)$$

where  $f(\frac{d}{2}, im, r)$  is the radial eigenfunction with energy  $-m^2$  which is normalized by

$$\lim_{r \rightarrow \infty} r^{(d-1)/2} e^{mr} f(\frac{d}{2}, im, r) = 1. \quad (34)$$

Because of (32) this eigenstate is proportional to  $\partial_r \phi$ , i.e.

$$\sqrt{m} f(\frac{d}{2}, im, r) = \frac{1}{B} \frac{d\phi}{dr},$$

such that the right-hand side in (33) is equal to a constant times the kinetic energy of the bounce. Now we use the virial theorem  $S[\phi] = (2/d)T[\phi]$ , which relates the euclidean action and the kinetic energy of  $\phi$  [15], and obtain

$$w'_1(-m^2) = \frac{d}{2m^2 B^2 \omega_d} |F(\frac{d}{2}, im)| S[\phi], \quad \text{where } \omega_d = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \quad (35)$$

is the surface area of the  $n$ -sphere. So we end up with the angular momentum expansion

$$\begin{aligned} \det'_R \left( \frac{-\Delta + V''(\phi)}{-\Delta + m^2} \right) &= \{w'_1(-m^2) e^{-p_1(-m^2)}\}^d \\ &\times \prod_{j \neq 1} \{F(j-1 + \frac{d}{2}, -im) e^{-p_j(-m^2)}\}^{\dim \mathcal{H}_j} \end{aligned} \quad (36)$$

for the regularized tunnel determinant. Besides the Jost function we have to compute, or at least estimate, the action  $S$  of the bounce and the change of normalization between  $\partial_R \phi$  and  $f(\frac{d}{2}, im, r)$ , i.e.

$$\sqrt{m} B = \lim_{r \rightarrow \infty} \left\{ \frac{d\phi}{dr} r^{(d-1)/2} e^{mr} \right\}. \quad (37)$$

For one-dimensional models we will show how to calculate  $B$  explicitly. The action must be computed anyway in (3). Its value can be estimated with the help of powerful variational methods [9].



## 4 Examples

### 4.1 One-dimensional determinants

As explained in section 2 the product (36) contains only two factors in one dimension. Since  $F(\frac{1}{2}, -im)F(-\frac{1}{2}, -im) = -2$  we obtain

$$w'(-m^2) = -\frac{2}{(2mB)^2} S[\phi]. \quad (38)$$

The instanton solution is the inverse function of

$$x(\phi) = \int_0^\phi \{2V(\phi')\}^{-1/2} d\phi' \quad (39)$$

and the action of the bounce is

$$S[\phi] = 2 \int_0^\sigma \{2V(\phi')\}^{1/2} d\phi'. \quad (40)$$

There is a factor 2 since a bounce consists of an instanton and anti-instanton. The constant  $B$  in (37) becomes

$$B = \lim_{\phi \rightarrow 0} \sqrt{m} \phi(x) e^{mx(\phi)}. \quad (41)$$

The point  $\sigma$  up to which we integrate in (40) is the value to which the particle tunnels. A typical potential is shown in Figure 2. Let us illustrate these with three examples.

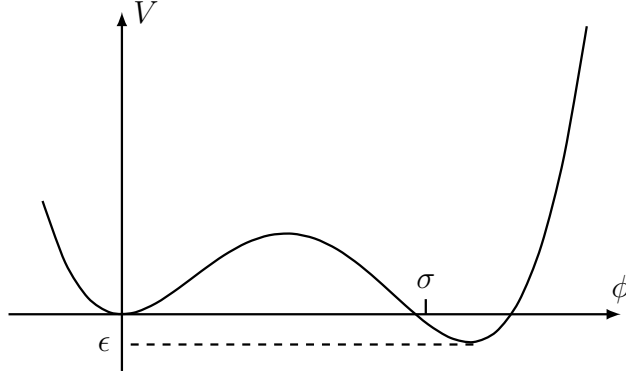


Figure 2: A typical Higgs potential which gives rise to a fluctuation operator with a negative and at least one zero eigenvalue.

#### Unstable potentials

As models for unstable potentials we use

$$V(\phi) = \frac{m^2}{2} \phi^2 \left(1 - \frac{\phi^p}{\sigma^p}\right), \quad p > 0. \quad (42)$$

It admits the tunnel solution  $\phi(x) = \sigma \cosh^{-2/p}(\frac{pm}{2}x)$  with action

$$S = \frac{2m\sigma^2}{p+4} B\left(\frac{1}{2}, \frac{2}{p}\right),$$

where  $B(a,b)$  denotes the beta-function. For the constant in (41) we find  $B = 2^{2/p} \sigma \sqrt{m}$  such that the

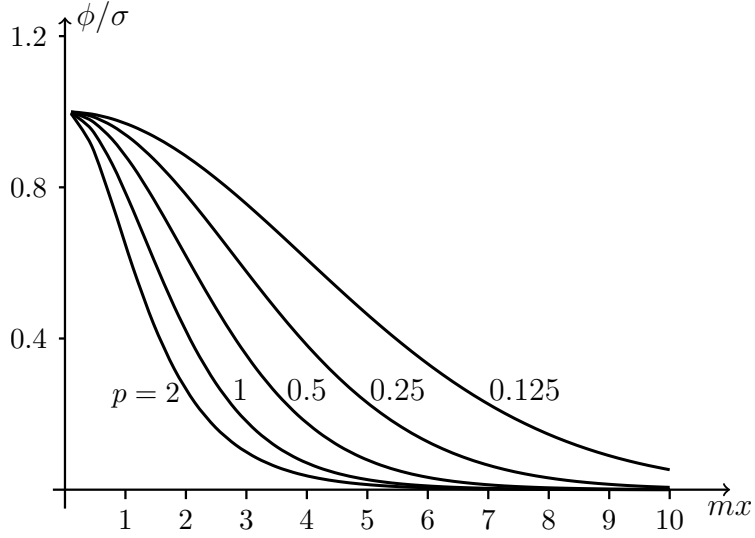


Figure 3: The tunnel solution for the model (42) for different  $p$ 's.

formula (38) gives the tunnel determinant

$$\det' \left( \frac{-\partial_x^2 + m^2 - \frac{1}{2}(p+1)(p+2)m^2 \cosh^{-2}(\frac{pm}{2}x)}{-\partial_x^2 + m^2} \right) = -\frac{1}{m^2} \frac{B(\frac{1}{2}, \frac{2}{p})}{(p+4)2^{4/p}}. \quad (43)$$

Inserting into (3) we obtain the quantum correction to the decay rate

$$\Theta = \left( \frac{S}{2\pi\hbar} \right)^{1/2} \det' \left( \frac{-\partial_x^2 + V''(\phi)}{-\partial_x^2 + m^2} \right)^{-1/2} = m\sigma \left( \frac{m}{\pi} \right)^{1/2} 2^{2/p}. \quad (44)$$

Now let  $p$  approach 0. Then the width of the bounce gets larger, for example  $\phi(x) = \sigma/e$  for  $x = (1 + \sqrt{4/p})/m$ . The limit  $p \rightarrow 0$  is somehow similar to the thin wall limit in higher dimensional theories [3]. There the width of the bounce diverges in the limit of a vanishing potential energy density difference  $\epsilon$  between the true and false vacuum. In using the asymptotic expansion for the beta-function  $B(\frac{1}{2}, \frac{2}{p})$  we find for the classical factor in the decay rate (3)

$$e^{-S} \sim \exp \left\{ - \left( \frac{p\pi}{2} \right)^{1/2} \frac{2m\sigma^2}{p+4} \right\}.$$

In Figures 3 and 4 we show the bounces for different values of  $p$  and the ratio  $\Theta/\exp(-S) = \Delta$  as function of  $p$ , respectively. We recognize that in the "thin-wall region"  $p \rightarrow 0$  the quantum corrections dominate the classical contribution. We conclude that the semiclassical expansion breaks down for small values of  $p$ .

### Double well potential

We consider the quartic double well potential

$$V(\phi) = \frac{m^2\phi^2}{2} \left( 1 - \frac{\phi}{\sigma} \right)^2, \quad (45)$$

which admits the bounce  $\phi(x) = \frac{1}{2}\sigma(1 + \tanh \frac{1}{2}mx)$ . Although the Goldstone mode has angular momentum  $j = 0$ , the formula (38) still holds (up to a sign). With  $S = \frac{1}{6}m\sigma^2$  and  $B = \sqrt{m}\sigma$  we find

$$\det' \left( \frac{-\partial_x^2 + m^2 - \frac{3}{2}m^2 \cosh^{-2}(\frac{1}{2}mx)}{-\partial_x^2 + m^2} \right) = \frac{1}{24m^2}. \quad (46)$$

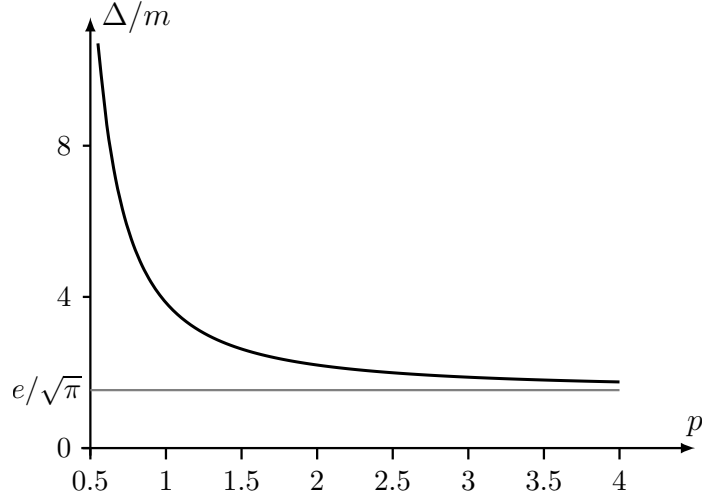


Figure 4: The ratio  $\Theta/e^{-S}$  for the model (42) as function of  $p$  for  $(m\sigma^2 = 1)$ .

### Sine-Gordon potential

The periodic potential

$$V(\phi) = \left(\frac{m\sigma}{2\pi}\right)^2 \left(1 - \cos \frac{2\pi\phi}{\sigma}\right) \quad (47)$$

has the instanton solution  $\phi(x) = (2\sigma/\pi) \tan^{-1}\{\exp(mx)\}$  with associated bounce action  $S = (2\sigma/\pi)\sqrt{m}$  and constant  $B = 2m\sigma^2/\pi^2$ . Hence we obtain for the tunnel determinant

$$\det' \left( \frac{-\partial_x^2 + m^2 - 2m^2 \cosh^{-2}(mx)}{-\partial_x^2 + m^2} \right) = \frac{1}{4m^2}. \quad (48)$$

### Fluctuation operators with Eckardt potentials

Actually, the tunnel determinants (43,46,48) are obtained more easily since for the Eckardt potential

$$q(x) = -\frac{\alpha(\alpha+1)}{\rho^2} \cosh^{-2}\left(\frac{x}{\rho}\right) \quad (49)$$

the transmission coefficient in (28) is explicitly known [17]

$$\begin{aligned} T^{-1}(ik, \alpha) &= \det \left( \frac{-\partial_x^2 - k^2 - \alpha(\alpha+1)/\rho^2 \cdot \cosh^{-2}(x/\rho)}{-\partial_x^2 - k^2} \right) \\ &= \frac{\Gamma(ik\rho + 1) \Gamma(ik\rho)}{\Gamma(ik\rho + 1 + \alpha) \Gamma(ik\rho - \alpha)}. \end{aligned} \quad (50)$$

It gives rise to a spectral flow with respect to the parameter  $\alpha$  as shown in Figure 5.

For  $\alpha = 1 + 2/p$  and  $m\rho = 2/p$  is the potential (49) just the potential in the fluctuation operator (43), for  $\alpha = 2$  and  $m\rho = 2$  the potential in the fluctuation operator (46) and for  $\alpha = 1$  and  $m\rho = 1$  the potential in the operator (48). Now one can directly divide (50) by the eigenvalue which vanishes for  $k = -im$  and perform the limit  $k \rightarrow -im$  for the quotient to recover the results (43,46,48) for the tunnel determinants.

## 4.2 Determinants in higher dimensions

Although the conformally invariant model in  $d$  dimensions

$$V(\phi) = -\frac{\lambda}{d_c} \phi^{d_c}, \quad \text{with} \quad d_c = \frac{2d}{d-2} \quad (51)$$

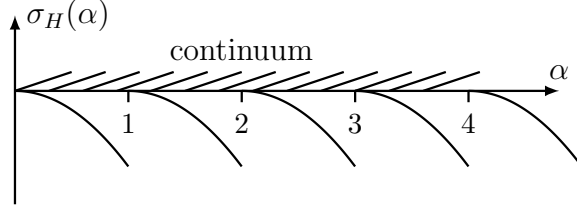


Figure 5: The spectral flow with  $\alpha$  for the Hamiltonian with Eckardt potential (49)

shows no tunnel effect in the usual sense, the corresponding fluctuation operator defines a tunnel determinant. Due to the scale invariance of this theory the instanton solution [18]

$$\phi(x) = \left\{ \frac{2d}{\lambda d_c} \right\}^{d/d_c} \left\{ \frac{\alpha}{1 + \alpha^2(x - x_0)^2} \right\}^{d/d_c} \quad (52)$$

depends on the additional parameter  $\alpha$ . Thus the fluctuation operator ( $\alpha = 1, x_0 = 0$ )

$$H = -\Delta - \frac{d(d+2)}{(1+r^2)^2} \quad (53)$$

has the zero modes

$$\begin{aligned} \partial_\alpha \phi &= c \frac{1-r^2}{1+r^2} \phi \approx r^{2-d} \quad \text{for } r \rightarrow \infty, \\ \partial_r \phi &= \tilde{c} \frac{r}{1+r^2} \phi \approx r^{1-d} \quad \text{for } r \rightarrow \infty, \end{aligned} \quad (54)$$

in the sectors with  $j = 0$  and  $j = 1$ , respectively. Because the Jost functions are only known at  $k = 0$  [18]

$$F(\mu, k = 0) = \frac{\Gamma(\mu)\Gamma(\mu+1)}{\Gamma(\mu - \frac{d}{2})\Gamma(\mu + \frac{d}{2} + 1)}, \quad (55)$$

we use a slightly different method for dividing by the zero modes.

Let  $E(\mu)$  be an eigenvalue of  $h_j$ . The constants  $\mu$  and  $j$  are related as in (26). Then the determinant  $w_j(z) = F(\mu, k)$  of the fluctuation operator on  $\mathcal{H}_j$  vanishes. It follows that  $F(\mu, k) = (E(\mu)^{1/2} - k)\tilde{F}(\mu, k)$ . As a consequence we find

$$-E(\mu)^{-1/2} \frac{\partial F}{\partial k} \Big|_{(\mu, E^{1/2}(\mu))} = \frac{1}{E'_\mu} \frac{\partial F}{\partial \mu} \Big|_{(\mu, E^{1/2}(\mu))}. \quad (56)$$

In our case  $E(\mu) = 0$  and since  $F(\mu, 0)$  is explicitly given in (55) it suffices to calculate  $E'_\mu$  for computing the primed determinant

$$E(\mu)^{-1/2} \frac{\partial F}{\partial k} \Big|_{(\mu, E^{1/2}(\mu))}.$$

It is enough to calculate  $E(\mu)$  in first order in the perturbation  $h_{j'} - h_j = \mu/r^2$ . The unperturbed wave functions are the zero modes (38). So we find for  $j = 0$  or equivalently  $\mu = \frac{d}{2}$  and for  $d > 4$

$$E'_0 = \left\langle \frac{1}{r} \right\rangle_{\partial_\alpha \phi} = \frac{d-4}{d+2}, \quad (57)$$

and for  $j = 1$  or equivalently  $\mu = \frac{d}{2}$  and  $d > 2$

$$E'_1 = \left\langle \frac{1}{r} \right\rangle_{\partial_r \phi} = \frac{d-2}{d}. \quad (58)$$

The IR-divergence of the theory shows itself in the non-normalizability of the Goldstone mode  $\partial_\alpha\phi$  or equivalently in the divergence of  $E_0'^{-1}$  in less than four dimensions. It should be canceled by the infrared divergent mass counterterm

$$\frac{1}{2}\delta m^2 \int \phi^2 = c\delta m^2 \frac{d-2}{d-4}.$$

With (55), (56) we end up with

$$w'_0 = -\frac{1}{E'_0} \frac{B(\frac{d}{2}, \frac{d}{2})}{(d-2)^2} \quad \text{and} \quad w'_1 = \frac{1}{E'_1} B(\frac{d}{2}, \frac{d}{2}).$$

The counterterms  $p_j(0)$  in (36) are easily found by expanding  $\log F(\mu)$  in powers of the potential  $q(r)$ . For the fluctuation operator (53) we therefore define  $\alpha = \alpha(\xi)$  through  $\xi d(d+2) = \alpha(\alpha+2)$  and expand  $\log F_\alpha(\mu) = C - \log B(\mu - \frac{1}{2}\alpha, \mu + 1 + \frac{1}{2}\alpha)$  in powers of  $\xi$ . Clearly the sum of the first  $[\frac{d}{2}]$  terms at  $\xi = 1$  is equal to  $p_j(0)$ .

We find in three dimensions

$$w'_R(0) = -\frac{1}{E'_0 E_1'^3} \frac{\pi}{4} e^{15} \left(\frac{\pi}{48}\right)^3 \prod_2^\infty \left\{ \frac{B(j + \frac{1}{2}, j + \frac{3}{2})}{B(j-1, j+3)} \exp\left(\frac{15}{4j+2}\right) \right\}^{2j+1}$$

and in four dimensions

$$w'_R(0) = -\frac{1}{E'_0 E_1'^4} \frac{e^{198}}{12} \left(\frac{1}{48}\right)^4 \prod_2^\infty \left\{ \frac{j(j-1)}{(j+2)(j+3)} \exp\left(\frac{6}{j+1} + 18\right) \right\}^{(j+1)^2}.$$

## 5 The supersymmetric sector

We have seen that

$$h_1 + m^2 = -\partial_r^2 - \frac{d-1}{r} \partial_r + \frac{d-1}{r^2} + V''(\phi(r)) \quad (59)$$

has always a zero mode - the Goldstone mode  $\partial_r\phi$ . The  $L_2$  norm of this mode is proportional to the action of the instanton and hence is finite. Furthermore, since the Goldstone mode has no zeros in  $(0, \infty)$  it is the normalizable ground state of  $h_1$ . For positive energies the spectrum of  $h_1$  is continuous. So the spectrum of  $h_1 + m^2$  looks as shown in Figure 6.

After these remarks it is not very surprising that  $h_1 + m^2$  is always a component of a supersymmetric Hamiltonian [19], i.e.  $h_1 + m^2 = S^\dagger S$ , where  $S = -r^{-D} \partial_r r^D + W$ ,  $D = \frac{1}{2}(d-1)$  and  $W = (d/dr) \log(r^D \partial_r \phi)$ .

Things become more transparent if we define the unitary map

$$U : L_2(R^+, r^{d-1} dr) \mapsto L_2(R, e^{2x} dx) \quad (60)$$

$$\psi(x) \mapsto \exp\left(\frac{d}{d_c} x\right) \psi(e^x),$$

so that the transformed Hamiltonians read

$$U h_j U^{-1} = e^{-2x} \left\{ -\partial_x^2 + \mu^2 + e^{2x} \left( V''(h(x)) - m^2 \right) \right\}, \quad (61)$$

with  $h(x) = \phi(e^x)$  and  $\mu = j - 1 + \frac{d}{2}$  was introduced in (26).

In using the field equation for the instanton solution it is easy to check the identity

$$e^{2x} V''(h(x)) = w^2(x) + w'(x) - \left(\frac{d}{2}\right)^2, \quad (62)$$

where

$$w(x) = \partial_x \log \partial_x h + \frac{d}{2} - 2. \quad (63)$$

Thus we end up with

$$U(h_j + m^2)U^{-1} = e^{-2x} \{S^\dagger S + \mu^2 - (\frac{d}{2})^2\} = e^{-2x} k_j, \quad (64)$$

where  $S = -\partial_x + w(x)$  and  $S^\dagger = \partial_x + w(x)$  is its adjoint in  $L_2(R, dx)$ .

Especially interesting is the case  $j = 1$  or equivalently  $\mu = \frac{d}{2}$  because  $k_1 = S^\dagger S$  is part of the supersymmetric Hamiltonian

$$K = \frac{1}{2}(Q_1^2 + Q_2^2) = (-\partial_x^2 + w^2)\mathbb{1} + w'\sigma_3 = \begin{pmatrix} S^\dagger S & 0 \\ 0 & SS^\dagger \end{pmatrix}, \quad (65)$$

where  $Q_1 = \sigma_1 p + \sigma_2 w$  and  $Q_2 = \sigma_2 p - \sigma_1 w$ .

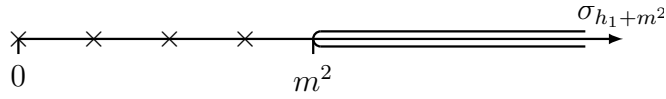


Figure 6: The spectrum of the supersymmetric Hamiltonian  $h_1 + m^2$

Because the spectra of  $S^\dagger S$  and  $SS^\dagger$  are, up to possible zero modes, exactly the same we naively expect

$$\det' \begin{pmatrix} S^\dagger S \\ SS^\dagger \end{pmatrix} = \det' \begin{pmatrix} -\partial_x^2 + w^2 + w' \\ -\partial_x^2 + w^2 - w' \end{pmatrix} \stackrel{?}{=} 1, \quad (66)$$

or, because

$$\begin{aligned} w_j(-m^2) &= \det(h_j + m^2)(h_j^0 + m^2)^{-1} \\ &= \det(U^{-1} e^{-2x} k_j U U^{-1} k_j^{0-1} e^{2x} U) = \det k_j k_j^{0-1} \end{aligned}$$

that

$$w_1(-m^2) = \det' \begin{pmatrix} S^\dagger S \\ S_0^\dagger S_0 \end{pmatrix} \stackrel{?}{=} \det \begin{pmatrix} SS^\dagger \\ S_0^\dagger S_0 \end{pmatrix}. \quad (67)$$

We will see that (66) generally is not true, due to the continuous part in the spectrum of  $h_1$ . Or in other words, the Hamiltonian (65) defines an unpairing supersymmetric quantum mechanics.

To be more specific we apply these results to the conformally invariant model (51). With (63,64) and (52) we obtain for the transformed radial Hamiltonians

$$\begin{aligned} k_j &= -\partial_x^2 + \mu^2 - \frac{1}{4}d(d+2) \cosh^{-2}(x), \\ k_j^0 &= -\partial_x^2 + \mu^2 \end{aligned}$$

and hence with (28)

$$\det \begin{pmatrix} S^\dagger S - z \\ SS^\dagger - z \end{pmatrix} = \det \begin{pmatrix} S^\dagger S - z \\ S_0^\dagger S_0 - z \end{pmatrix} \det \begin{pmatrix} S_0^\dagger S_0 - z \\ SS^\dagger - z \end{pmatrix} = \det \begin{pmatrix} T_{S^\dagger S}(k) \\ T_{SS^\dagger}(k) \end{pmatrix}. \quad (68)$$

For computing the ratio of the two transmission coefficients we assume that  $\psi$  is a solution of  $S^\dagger S\psi = k^2\psi$  with boundary conditions

$$\psi \rightarrow \begin{cases} e^{ikx} + R_{S^\dagger S}(k) e^{-ikx} & \text{for } x \rightarrow +\infty, \\ T_{S^\dagger S}(k) e^{ikx} & \text{for } x \rightarrow -\infty. \end{cases}$$

Then  $S(S^\dagger S)\psi = (SS^\dagger)S\psi = k^2S\psi$ , what means that  $S\psi$  is an eigenfunction of  $SS^\dagger$  with the same energy  $k^2$ . Since

$$S\psi \rightarrow \begin{cases} (-ik + w(\infty)) e^{ikx} + (ik + w(\infty)) R_{S^\dagger S}(k) e^{-ikx} & \text{for } x \rightarrow +\infty \\ (-ik + w(-\infty)) T_{S^\dagger S}(k) e^{ikx} & \text{for } x \rightarrow -\infty, \end{cases}$$

we obtain

$$\det \left( \frac{S^\dagger S - z}{SS^\dagger - z} \right) = \frac{ik - w(-\infty)}{ik - w(+\infty)}. \quad (69)$$

Again we use l'Hospital's rule to compute the primed determinant (66). With  $w = w(\infty = -w(-\infty))$  we end up with

$$\det' \left( \frac{S^\dagger S}{SS^\dagger} \right) = \frac{1}{4w^2} \quad (70)$$

instead of (66), or with

$$w'_1(-m^2) = \frac{1}{4w^2} \det' \left( \frac{SS^\dagger}{S_0 S_0^\dagger} \right) \quad (71)$$

instead of (67).

Phrased in other words, the Witten index  $\dim \ker(S) - \dim \ker(S^\dagger)$  is not given by the trace  $\text{tr} \{ \sigma_3 \exp(-tK) \}$ . This can be seen from

$$\begin{aligned} - \int_0^\infty \frac{dt}{t} \left\{ \text{tr} \sigma_3 e^{-tK} - \text{index}(S) \right\} &= - \int_0^\infty \frac{dt}{t} \left\{ \text{tr} (e^{-tS^\dagger S} - e^{-tSS^\dagger}) - \text{index}(S) \right\} \\ &= \log \det' \frac{S^\dagger S}{SS^\dagger} = -\log(4w^2) \neq 0, \end{aligned}$$

and is due to the continuous part of the spectrum of  $S^\dagger S$  or the fact that the theory is defined on an open space. For the model (53) we have  $w(x) = \frac{d}{2} \tanh x$  and the corresponding identity

$$\det' \left( \frac{-\partial_x^2 + \frac{1}{4}d^2 - \frac{1}{4}d(d+2) \cosh^{-2} x}{-\partial_x^2 + \frac{1}{4}d^2 - \frac{1}{4}d(d-2) \cosh^{-2} x} \right) = \frac{1}{d^2}$$

can be checked directly by using (50).

## 6 Summary

In this paper I have presented a method for calculating the tunnel determinant of fluctuation operators. The result for one-dimensional systems, eq. (38), has been applied to a one-parametric class of unstable potentials, the anharmonic double well potential and finally the sine-Gordon potential. We found that, in cases where the width of the tunnel solution becomes large, the one-loop quantum corrections dominate the classical result.

Next we applied the result (36) for the regularized determinant to the fluctuation operator of the conformally invariant model (51). We computed  $w'_R$  in any dimension and gave the explicit expressions in three and four dimensions.

Our methods are applicable to any model for which we know, or at least can estimate, the Jost function. In [14] the Jost function for the "thin-wall" model

$$q(r) = \begin{cases} \text{const} & \text{for } r < R, \\ 0 & \text{for } r > R, \end{cases} \quad (72)$$

was used for computing the tunnel determinant of the corresponding fluctuation operator. Again the quantum corrections compete with the classical part for large values of  $R$ .

For both models (43) and (72) we found an exponential dependence of the quantum corrections on the width of the tunnel-solution, i.e.

$$\Theta \approx \text{const} \exp \left\{ (m \text{ width}^{-d_c/d})^\alpha \right\},$$

where  $\alpha$  depends on the dimension of spacetime. Thus the rough dimensional estimate (6) gives a wrong result in this interesting limit.

With (71) we gave an alternative method for computing the primed determinant in the  $j = 1$  sector. The primed determinant of  $h_1$  is, up to a surface term, given by the unprimed determinant of its "supersymmetric partner". As a by-product we showed that every fluctuation operator defines an unpairing supersymmetric quantum mechanical system.

The author wishes to thank J. Albery, L. O'Raiheartaigh, N. Straumann and H. Yoneyama for useful discussions.

## A Tunnel determinants are analytic

In this appendix we show that

$$\log w_{\mathbb{R}}(z) = \text{tr} \{ \log (1 + A(z)) - P(A(z)) \},$$

where  $A(z) = q(x)R_0(z)$  and

$$P(A(z)) = A(z) - \frac{1}{2}A^2(z) + \frac{1}{3}A^3(z) - \dots - \frac{(-)^p}{p}A^p(z), \quad p = \lfloor \frac{d}{2} \rfloor,$$

is an analytic function of  $z$  at points which are not in the spectra of both,  $H$  and  $H_0$ . In the following we shall often skip the argument  $z$  in order to simplify the notation. We first observe that because of the identities

$$\begin{aligned} \frac{dA}{dz} &= AR_0, \\ R_0(1 + qR_0)^{-1}qR_0 &= R_0\{(H - z)R_0\}^{-1}A = R_0R_0^{-1}RA = RA, \quad R = \frac{1}{H - z}, \end{aligned}$$

we have

$$\frac{d}{dz} \log w_{\mathbb{R}} = (-)^{p+2} \text{tr} R_0(1 + A)^{-1}A^{p+1} = (-1)^p \text{tr} (RA^{p+1}).$$

Now we use the analytic nature of the resolvent operator, i.e.

$$R(z) = \sum_0^{\infty} (z - z_0)^n R(z_0)^{n+1}$$

to show that  $(d/dz) \log w_{\mathbb{R}}$  has an absolute convergent Taylor expansion. In

$$\begin{aligned} \text{tr} R(z)A^{\alpha} &= \sum_{m, n_1, \dots, n_{\alpha}} (z - z_0)^{m+n_1+\dots+n_{\alpha}} \text{tr} R(z_0)^{m+1} qR_0(z_0)^{n_1+1} \dots qR_0(z_0)^{n_{\alpha}+1} \\ &= \sum_k a_k (z - z_0)^k, \end{aligned}$$

where  $\alpha = p + 1$ , and the coefficients  $a_k$  are bounded by

$$\begin{aligned} |a_k| &\leq \sum_{m+n_1+\dots+n_{\alpha}=k} \|R(z_0)^{m+1} qR_0(z_0)^{n_1+1} \dots qR_0(z_0)^{n_{\alpha}+1}\|_1 \\ &\leq \sum_{m+n_1+\dots+n_{\alpha}=k} \|R(z_0)\|^{m+1} \|qR_0(z_0)^{n_1+1} \dots qR_0(z_0)^{n_{\alpha}+1}\|_1. \end{aligned}$$

We used the inequality  $\|A_1 A_2\|_1 \leq \|A_1\| \|A_2\|_1$  for a bounded  $A_1$  and a trace class operator  $A_2$ . Here  $\|\cdot\|$  denotes the usual operator norm and  $\|A\|_p^p = \text{tr} (AA^{\dagger})^{p/2}$ . Next we iterate the inequality (see, for example [12])

$$\|A_1 A_2\|_q \leq \|A_1\|_{q_1} \|A_2\|_{q_2}, \quad \text{where} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}, \quad A_k \in P_{q_k},$$



and obtain

$$\|A_1 \cdots A_\alpha\|_q \leq \|A_1\|_{q_1} \cdots \|A_\alpha\|_{q_\alpha}, \quad \text{with} \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_1} + \cdots + \frac{1}{q_\alpha},$$

Now we choose  $q = 1$  and  $q_i = \alpha$  which yields

$$\begin{aligned} |a_k| &\leq \sum \|R\|^{m+1} \|qR_0^{n_1+1}\|_\alpha \cdots \|qR_0^{n_\alpha+1}\|_\alpha \\ &\leq \sum \|R\|^{m+1} \|R_0\|^{n_1+\cdots+n_\alpha} \cdot \|qR_0\|_\alpha. \end{aligned}$$

With the definition  $C := \max\{\|R(z_0)\|, \|R_0(z_0)\|\}$ ,

$$|a_k| \leq C^{k+1} \|A(z_0)\|_\alpha^\alpha \sum_{m+n_1+\cdots+n_\alpha=k} 1 = C^{k+1} \|A(z_0)\|_\alpha^\alpha \frac{(\alpha+1) \cdots (\alpha+k)}{k!}.$$

On the other side

$$\|A(z_0)\|_\alpha^\alpha \leq (2\pi)^{-d/\alpha} \|q\|_\alpha^\alpha \|\tilde{S}_2\|_\alpha^\alpha,$$

where  $\tilde{S}_2(p) = (p^2 - z_0)^{-1}$  is the free propagator in momentum space. The right-hand side is finite for  $\alpha > \frac{d}{2}$ ,  $z_0 \in \rho(H_0)$  and sufficiently fast decaying  $q$ . The radius of convergence  $R = 1/C$  is finite for  $z_0 \in \rho(H) \cap \rho(H_0)$ . This finishes the proof of our statement

## B Multifield bounces are spherically symmetric

The results in this paper heavily rely on the spherical symmetry of the bounce. Several authors [14, 15] showed that a one-component tunnel solution is indeed spherically symmetric. In this appendix we generalize this result to the case when  $\phi(x)$  transforms non-trivially under the action of the rotation group  $SO(d)$  and the rotations act transitively on the unit sphere  $S^{n-1}$  in the representation space  $V^n$  of the scalar field. Hence for every  $x \in R^d$  we can find a rotation  $g(x)$  such that the transformed field

$$U(g(x))\phi(x) = \|\phi(x)\|e_1 = \chi(x) \quad (73)$$

points always into the direction of  $e_1$ .

We assume that the potential energy is invariant under rotations such that

$$V[\phi] = V[\chi]. \quad (74)$$

Next we show that the kinetic energy of  $\chi$  is less or equal than the kinetic energy of  $\phi$ . For that purpose we use

$$\nabla\chi = \nabla(\phi, \phi)^{1/2}e_1 = (\phi, \phi)^{-1/2} \Re(\phi, \nabla\phi) e_1$$

and find

$$\begin{aligned} (\nabla\chi, \nabla\chi) &= (\phi, \phi)^{-1} \{\Re(\phi, \nabla\phi)\}^2 \leq (\phi, \phi)^{-1} |(\phi, \nabla\phi)|^2 \\ &\leq (\phi, \phi)^{-1} (\phi, \phi) (\nabla\phi, \nabla\phi) = (\nabla\phi, \nabla\phi), \end{aligned}$$

where we only used the Schwartz inequality. After integrating over  $R^d$  we end up with

$$T[\chi] \leq T[\phi]. \quad (75)$$

We use the symbol  $\mathcal{H}$  for the space of all Higgs fields (typically a Sobolev space) and let  $\mathcal{H}_1$  denote the subspace of fields which point into the  $e_1$  direction.

We will use the fact that an extremum of  $S$ , restricted to  $\mathcal{H}_1$ , is also an extremum of  $S$  on the whole space  $\mathcal{H}$ . This is an immediate consequence of the principle of symmetric criticality [20]. This theorem states that if  $S$  is invariant under the action of a compact group  $G$ , i.e.

$$S[g\phi] = S[\phi]$$

then every critical point of  $S$ , restricted to the subset of those  $\phi$  which do not change under the group action, is automatically a critical point of  $S$  on the space of all fields. Now we choose for  $G$  the little group of  $e_1$  and the above statement is proven.

Now we use the (in)equalities (74) and (75) to show that to every critical point on  $\mathcal{H}$  there exists a critical point on  $\mathcal{H}_1$  which has a smaller action.

So let us assume that  $\phi \in \mathcal{H}$  is a critical point of  $S$ . The virial theorem

$$S[\phi] = \frac{2T[\phi]}{d} \left\{ \frac{2T[\phi]}{-d_c V[\phi]} \right\}^{d/d_c}, \quad d_c = \frac{2d}{d-1}, \quad (76)$$

together with (74) and (75) implies

$$S[\phi] \geq \frac{2T[\chi]}{d} \left\{ \frac{2T[\chi]}{-d_c V[\chi]} \right\}^{d/d_c}. \quad (77)$$

Next we use the result (see [14]) that  $S$ , restricted to  $\mathcal{H}_1$ , has a spherically symmetric and monotonically decreasing tunnel solution  $\eta$  and  $S[\eta]$  is given by the variational principle

$$S[\chi] = \inf_{\substack{\zeta \in \mathcal{H}_1 \\ V[\zeta] < 0}} \frac{2T[\zeta]}{d} \left\{ \frac{2T[\zeta]}{-d_c V[\zeta]} \right\}^{d/d_c}. \quad (78)$$

Because of the virial theorem the potential energy of  $\phi$  is negative. With (74) the function  $\chi$  is admissible on the right-hand side in (78). Together with (77) we conclude

$$S[\phi] \geq S[\eta].$$

In going through the inequalities one sees that the equality sign holds exactly if  $\phi$  is a spherically symmetric and monotonically decreasing one-component field.

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