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Temperature- and Curvature Dependence of the Chiral Symmetry Breaking in 2D Gauge Theories

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Abstract

The partition function and the order parameter for the chiral symmetry breaking are computed for a family of 2-dimensional interacting theories containing the gauged Thirring model. In particular we derive non-perturbative expressions for the dependence of the chiral condensate on the temperature and the curvature. Both, high temperature and high curvature supress the condensate exponentially and we can associate an effective temperature to the curvature.

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1 Introduction

Despite of the considerable amount of work devoted to the subject of chiral symmetry breaking in gauge theories and in particular QCD, the understanding of this non-perturbative phenomenon is still unsatisfactory [1]. Also the behaviour of quantum systems in a hot and dense enviroment (eg. in neutron stars or in the early universe) are still under active investigation [1]. On another front there has been much effort on the apparently different problem of quantizing self-interacting theories in a background gravitational field [2].

Rather than seeking new partial results for realistic 4-dimensional theories we analyse a family of interacting theories of charged fermions, scalars, pseudo-scalars and photons propagating in 2-dimensional curved spacetime in detail. These models are defined by the action

$$S = \int \sqrt{-g} \quad \Big[\bar{\psi} i\gamma^{\mu}(D_{\mu} - ig_1\partial_{\mu}\lambda + ig_2\eta_{\mu}{}^{\nu}\partial_{\nu}\phi)\psi \\ + g^{\mu\nu}(\partial_{\mu}\phi\partial_{\nu}\phi + \partial_{\mu}\lambda\partial_{\nu}\lambda) - g_3\mathcal{R}\lambda - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big], \qquad (1)$$

where $F_{\mu\nu}$ is the electromagnetic field strength and $D_{\mu} = \nabla_{\mu} - ieA_{\mu}$ the generally- and gauge covariant derivative. This family contains in particular the Schwinger model ($g_i = 0$, $i = 1, .., 3$)[3] and the gauged Thirring model ($g_1^2 = -g_2^2 = g^2$, $g_3 = 0$)[4, 5] in curved spacetime

$$S_{Th} = \int \sqrt{-g}\Big[\bar{\psi} i\gamma^{\mu} D_{\mu}\psi - \frac{g^2}{4} j^{\mu} j_{\mu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\Big].$$

The coupling constant g_3 has been introduced in order to test the effect of non-minimal coupling to the gravitational field. Finite temperature effects are then included by quantizing the system on an euclidean torus[1] $[0, \beta] \times [0, L]$ with arbitrary metric. We choose coordinates such that

$$g_{\mu\nu} = e^{2\sigma(x)} \begin{pmatrix} |\tau| & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{where} \quad \tau = i\frac{\beta}{L}.$$

β is the inverse temperature and L is the infrared cut-off which will be removed after the correlation have been calculated. Furthermore, finite temperature boundary conditions are imposed on the quantum fields [6].

[1] choosing a torus rather than a cylinder provides us with an infrared regularization [6].

On the torus a general gauge potential with non-vanishing flux can be decomposed as

$$A_\mu = A^k_\mu + t_\mu + \partial_\mu \alpha - \eta_{\mu\nu}\partial^\nu \varphi,$$

where the last 3 terms are recognized as Hodge decomposition of the single valued part of A and A^k is an instanton potential giving rise to a quantized flux $e\int F = 2\pi\, k$. As a consequence the corresponding Dirac operator has $|k|$ zero modes [6] of chirality $sign\{k\}$. These zero modes are responsible for a non vanishing chiral condensate $\langle\bar\psi\psi\rangle$ as can be seen by inspecting the fermionic generating functional [6] in the external fields[2] A, ϕ, λ, h and sources $\eta, \bar\eta$

$$Z_F[A^k, \phi, \lambda, h_\mu, \eta, \bar\eta] = \prod_{p=1}^{|k|}(\bar\eta, \psi_{0p})(\psi^\dagger_{0p}, \eta)\det{}'(i\not{D})\, e^{-\int \sqrt{g}\bar\eta(x)S_e(x,y)\eta(y)}. \quad (2)$$

Here $\psi_{0p}(x),\ p=1,...,|k|$, are the $|k|$ zero modes in the topological sector k and $\det'(i\not{D})$ denotes the zero mode truncated determinant. $S_e(x,y)$ is the excited fermionic Green's function. We shall restrict ourselves to the sectors $k=0$ and $k=1$, since these contribute to the partition function and the chiral condensate, respectively. Finally we introduce a chemical potential for the conserved electric charge. In the euclidean forumlation this is done by shifting the zero component of the gauge potential by an imaginary constant [7].

2 Partition function

As a first step in analyzing structure of the quantum theory we evaluate the partition function formally defined by

$$Z_0 = \int \mathcal{D}(A,\phi,\lambda,h)\, Z_F[A,\phi,\lambda,h] e^{-S_B(A,\phi,\lambda,h)}, \quad (3)$$

where

$$S_B(A,\phi,\lambda,h) = -\int \sqrt{g}\phi\triangle\phi + \lambda\triangle\lambda - h_\mu h^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

[2] the harmonic field h is needed for a consistent quantization on the torus, analogous to the harmonic part t_μ of the gauge field. On the torus the action (1) is changed to $S \to S + \int\sqrt{g}[g_2 h_\mu j^\mu + h_\mu h^\mu]$

After a covariant gauge fixing (3) is promoted to a well defined quantity. From (2) it is clear that only the trivial topological sector contributes to the partition function. Then $Z_F[A,\phi,\lambda,h]$ equals the determinant of the Dirac operator $\not{D}$ which is related by conformal- and chiral transformations to

$$i\hat{\not{D}} \equiv \hat{\gamma}^\mu(\partial_\mu - \frac{2\pi i}{L}a_\mu), \qquad \text{where} \qquad \frac{2\pi}{L}a_\mu = et_\mu + g_2 h_\mu - \tau\mu\delta_{0,\mu}.$$

Hatted quantities refer to flat metric and constant gauge potentials. The chemical potential is contained in the last term in a_μ. Integrating the chiral- and conformal anomalies [8] we find

$$\det(i\not{D}) = \det(i\hat{\not{D}})\, \exp\,[\frac{1}{24\pi}S_L + \frac{1}{2\pi}\int\sqrt{g}G\frac{1}{\triangle}G], \tag{4}$$

where

$$S_L = \frac{1}{4}\int \mathcal{R}\frac{1}{\triangle}\mathcal{R}$$

is the Liouville action and $G = g_2\varphi + e\phi$. One must be careful in computing the hatted determinant since the gauge potential is complex. This has been done in [9] with the result

$$\det(i\hat{\not{D}}) = \frac{1}{|\eta(\tau)|^2}\Theta\Big[\begin{smallmatrix} a_1 \\ -a_0 \end{smallmatrix}\Big](0,\tau)\bar{\Theta}\Big[\begin{smallmatrix} \bar{a}_1 \\ -\bar{a}_0 \end{smallmatrix}\Big](0,\tau). \tag{5}$$

The remaining functional integrals in (3) turn out to be of iterative Gaussian type and yield after substitution of (5)

$$Z_0 = \sqrt{2\pi V}\,\frac{e}{m_\gamma}\frac{L}{\beta|\eta(\tau)|^4}\,\frac{1}{\det'^{\frac{1}{2}}(-\triangle + m_\gamma^2)}\,\exp\Big((\frac{1}{12\pi} + g_3^2)\,S_L\Big) \tag{6}$$

where $V = \int\sqrt{g}$ and

$$m_\gamma^2 = \frac{e^2}{\pi}\frac{2\pi}{2\pi + g_2^2}$$

is the dynamically generated "photon" mass. This result already indicates that in the trivial topological sector the theory (1) should be equivalent to a free, massive, neutral, boson even in curved space-time. Note that the mass depends on g_2. In particular (6) shows that only the transversal part of the current-current interaction contributes to the mass renormalization in the

Thirring model. Note also that the chemical potential does not appear in the final result for the partition function. This may not come as a surprise, because of the equivalence to a uncharged boson. Also $\partial_\mu Z[\mu] = 0$ is the only result consistent with Gauss's law. We consider this consistency as a confirmation of our definition of the fermionic determinant which differs from previous ones in the literature [10]. The non-minimal coupling to gravity (for $g_3 \neq 0$) contributes to the gravitational anomaly and therefore affects the intensity of the Hawking radiation

3 Chiral Condensate

The chiral condensate $\langle\bar{\psi}\psi\rangle$ is the order parameter for the chiral symmetry breaking, responsible for the mass term in (6). Here we evaluate the dependence of the order parameter on temperature and curvature. Recalling (2) we see that only configurations within the topological sectors $k = \pm 1$ can contribute to this expectation value. More precicely

$$\langle\bar{\psi}(x)P_+\psi(x)\rangle = \frac{1}{Z_0}\int \mathcal{D}(\ldots)\psi^\dagger_{01}(x)\psi_{01}(x)\det{}'(i\not{D})\, e^{-S_B[A^1,\phi,\lambda,h]}|_{k=1}, \quad (7)$$

where $P_+ = \frac{1}{2}(1+\gamma_5)$ is the projector on states with positive chirality. Z_0 has been computed in the previous section (6). The generalization of (4) to non-zero k reads

$$\begin{aligned}\det{}'(i\not{D}) &= \det\frac{\mathcal{N}_\psi}{\hat{\mathcal{N}}_\psi}\det{}'(i\hat{\not{D}})\exp\left(\frac{1}{24\pi}S_L\right)\\ &\quad\cdot\exp\left(\frac{1}{2\pi}\int\sqrt{g}G\frac{1}{\triangle}G + \frac{2k}{V}\int\sqrt{g}G + \frac{2\pi k^2}{\hat{V}}\int\sqrt{\hat{g}}\chi\right), \quad (8)\end{aligned}$$

where the hatted determinant now also contains the instanton potential. $\mathcal{N}$ is the normmatrix of the zero modes

$$\psi^p_{0+}(x) = e^{iF-\gamma_5(G+2k\pi\chi)-\frac{1}{2}\sigma}\hat{\psi}^p_{0+}(x)$$

and $\chi(x)$ satisfies the differential equation

$$\sqrt{g}\triangle\chi = \sqrt{g}\frac{2\pi}{V} - \sqrt{\hat{g}}\frac{2\pi}{\hat{V}}.$$

All information about the harmonics and the chemical potential is contained in the zero modes. However,

$$\int d^2t \, \det{}'(i\hat{\not{D}})\psi_{01}^\dagger\psi_{01} = \frac{1}{\sqrt{2\beta L}}$$

and hence

$$\langle\psi^\dagger P_+\psi\rangle = \sqrt{\frac{-i\tau}{\hat{V}}}\,|\eta(\tau)|^2 e^{-2\pi^2/e^2V+2\pi/\hat{V}\int\sqrt{\hat{g}}\chi}\Big\langle e^{-2(g\phi+e\varphi)(x)-\sigma(x)}\Big\rangle_{\phi\varphi}, \qquad (9)$$

where the expectation value is evaluated with

$$S_{eff} = \int\sqrt{g}\Big[\frac{1}{2}\varphi(\triangle^2 - \frac{e^2}{\pi}\triangle)\varphi - \frac{e^2}{\pi m_\gamma^2}\phi\triangle\phi - \frac{e\,g_2}{\pi}\phi\triangle\varphi\Big].$$

A formal calculation of the resulting Gaussian integrals yields

$$\begin{aligned}\langle\psi^\dagger P_+\psi\rangle \;&=\; \sqrt{\frac{-i\tau}{\hat{V}}}|\eta(\tau)|^2 e^{-2\pi^2/e^2V+2\pi/\hat{V}\int\sqrt{\hat{g}}\chi}\, e^{-\sigma(x)-4\pi\chi(x)} \\ &\quad\cdot\exp\Big[\frac{2\pi^2m_\gamma^4}{e^2}\,K(x,x)\Big]\exp\Big[\frac{2\pi g_2^2}{2\pi+g_2^2}\,G_0(x,x)\Big], \qquad (10)\end{aligned}$$

where

$$K(x,y) = \langle x|\frac{1}{\triangle^2 - m_\gamma^2\triangle}|y\rangle = \frac{1}{m_\gamma^2}(G_0(x,y) - G_{m_\gamma}(x,y)) \qquad (11)$$

and G_m, G_0 are the massive and massless Green's functions respectively.

As it stands (10) is still a formal expression since $G_0(x,y)$ is logarithmically divergent when x tends to y. To extract a finite answer we need to renormalize the operator $\exp(\alpha\phi)$. This wave function renormalization is equivalent to the renormalization of the fermion field in the Thirring model and thus is very much expected already in flat space time [11]. Its generalization to curved space-time is found to be

$$G_0^{reg}(x,x) = -\frac{1}{2\pi}\log\Big[\frac{2\pi|\eta(\tau)|^2}{Lm_\gamma}\Big].$$

To determine the chiral condensate we also need to determine $K(x,y)$ on the diagonal. In a first step we shall obtain it for the flat torus. Its curvature dependence is then determined in a second step.

For $\sigma = 0$ the Green's function K has been computed in [6]. Substitution of this Green's function leads, after removing the infrared cut-off, to the following exact formula for the *chiral condensate on flat space*

$$\langle \psi^\dagger P_+ \psi \rangle_\beta = -T\Big(\frac{m_\gamma}{2\pi T}\Big)^{\frac{g_2^2}{2\pi+g_2^2}} \exp\Big[-\frac{\pi^2 m_\gamma}{e^2} T + \frac{2\pi}{2\pi+g_2^2} F\Big], \qquad (12)$$

where

$$F(\beta) = \sum_{n>0} \Big[\frac{1}{n} - \frac{1}{\sqrt{n^2 + (\beta m_\gamma/2\pi)^2}}\Big].$$

For arbitrary values of the temperature and g_2 the infinite sum F is evaluated on a computer (Fig.1). It is however interesting to discuss some limiting cases.

For low temperatures, compared to m_γ we have

$$F(\beta) \to \gamma + \log\frac{\beta m_\gamma}{4\pi} + \frac{\pi}{\beta m_\gamma}, \qquad (13)$$

where $\gamma = 0.57721\ldots$ is the Euler constant. Substitution of (13) yields the *zero temperature* result

$$\langle \psi^\dagger P_+ \psi \rangle = -\frac{m_\gamma}{4\pi}\, 2^{g_2^2/(2\pi+g_2^2)} \exp\Big(\frac{2\pi}{2\pi+g_2^2}\gamma\Big) \qquad \text{for} \quad T \to 0. \qquad (14)$$

On the other hand for temperatures large compared to the induced photon mass F vanishes. Thus we obtain the *high temperature behaviour*

$$\langle \psi^\dagger P_+ \psi \rangle_T = -T\Big(\frac{m_\gamma}{2\pi T}\Big)^{\frac{g_2^2}{2\pi+g_2^2}} \exp\Big(-\frac{\pi^2 m_\gamma}{e^2} T\Big) \quad \text{for} \quad T \to \infty. \qquad (15)$$

Hence the chiral condensate decays exponentially for high temperatures approaching zero assymptotically. The coupling to the pseudoscalars ϕ weakens the effect of the temperature while the scalar field λ has no effect. For the gauged Thirring model this result implies that only the transversal part of the current-current coupling affects the chiral condensate. Finally note that, as the partition function, the chiral condensate does not depend on the chemical potential.

How does the gravitational field affect the chiral condensate? To answer this question we need to know the massive Green's function, entering in (11),

for non-trivial gravitational fields (for simplicity we assume $T=0$). Let us first consider a space with constant positive curvature. Then G_{m_γ} has been computed explicitely [13]. Here we only need the short distance expansion, given by

$$G_{m_\gamma}(x,y) = -\frac{1}{4\pi}\{2\gamma + \log(\frac{s^2\mathcal{R}}{8}) + \psi(\frac{1}{2}+\alpha) + \psi(\frac{1}{2}-\alpha) + O(s^2)\}, \quad (16)$$

where $\alpha^2 = \frac{1}{4} - \frac{2m_\gamma^2}{\mathcal{R}}$ and $\psi(z)$ is the Digamma function. Substituting (16) into (11) we end up with the exact formula for the *chiral condensate for constant curvature*

$$\langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}} = \langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}=0}\cdot\exp\Big[\frac{\pi}{2e^2}m_\gamma^2\{\log(\frac{\mathcal{R}}{2m_\gamma^2}) + \psi(\frac{1}{2}+\alpha) + \psi(\frac{1}{2}-\alpha)\}\Big]. \quad (17)$$

The assymptotic expansions for *large-and small curvatures* are easily worked out inserting the corresponding expansions for the Digamma function [14]. We find

$$\langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}} = \langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}=0}\cdot\exp\Big[-\frac{\pi}{12e^2}\mathcal{R}\Big] \quad \text{for } \frac{\mathcal{R}}{m_\gamma}\to 0 \quad (18)$$

and

$$\langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}} = \langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}=0}\cdot(\frac{\mathcal{R}}{2m_\gamma^2})^{\frac{\pi}{2\pi+g_2^2}}\exp\Big[-\frac{\pi}{4e^2}\mathcal{R}-\frac{\pi m_\gamma^2}{4e^2}\gamma\Big] \quad \text{for } \frac{\mathcal{R}}{m_\gamma}\to\infty. \quad (19)$$

Hence the chiral condensate decays exponentially for large curvature analogous to the high temperature behaviour. However, the pseudo-scalars do not supress the effect of the curvature in contrast to (15). Comparing the exponentials in (19) to (15) we are lead to define the curvature induced effective temperature as

$$T_{eff} = \frac{\mathcal{R}}{4\pi m_\gamma}. \quad (20)$$

In passing we note that if we compare the prefactors, rather than the exponentials, we would write

$$T_{eff} = \frac{\mathcal{R}^{\frac{1}{2}}}{4\pi\sqrt{2}}. \quad (21)$$

The latteridentification actually coincides (up to factor of 2) with the Hawking temperature of free scalars in de Sitter space [15]. The correct identification involves the (dynamical) mass of the gauge field and is therefore not universal. From this observation we learn that the temperature associated with curvature depends on the matter content. Note finally that the non-minimal coupling (g_3) has no effect on the chiral condensate. In Fig. 2 we have plotted the chiral condensate for arbitrary constant values of the curvature.

For gravitational backgrounds with non-constant curvature we have to refer to perturbative methods for the calculation of the massive Green's function. Again we only need the short distance expansion of G_{m_γ}. For geodesic distances s small compared to m_γ^{-1} the massive Green's function may be approximated by the Seeley DeWitt expansion [16]

$$G_m(x,y) \sim \frac{1}{4i}\sum_{j=0}^{\infty} a_j(x,y)(-\frac{\partial}{\partial m^2})^j\, H_0^{(2)}(ms), \tag{22}$$

where $H_0^{(2)}$ is the Hankel function of the second kind and order zero. In particular

$$H_0^{(2)}(z) \to \frac{2}{i\pi}[\log\frac{z}{2}+\gamma] \quad \text{for} \quad z\to 0.$$

Inserting (22) into (11) we end up with the following expansion for the *chiral condensate in an arbitrary background*

$$\langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}} = \langle\psi^\dagger P_+\psi\rangle_{\mathcal{R}=0}\cdot\exp\Big[-\frac{\pi}{2}(\frac{m_\gamma}{e})^2\sum_1^\infty a_j(x)\frac{(j-1)!}{m^{2j}}\Big], \tag{23}$$

where we have used that $a_0(x)=1$. The first order contribution involves $a_1(x)=\frac{1}{6}\mathcal{R}$ and reproduces the assymptotic behaviour (18). Higher order contributions must be taken into account to uncover the effect of variable curvature. For this one has to substitute is the corresponding Seeley DeWitt coefficients a_j into (23). These have been computed up to $j=5$ [17].

4 Summary

We have computed the partition function and the order parameter of the chiral symmetry breaking for a Thirring-like gauge theory. In particular we

find that both, high temperature and high curvature supress the condensate exponentially. Comparing the two results, we defined a curvature induced effective temperature which, unlike the Hawking temperature, depends on the matter content and is therefore not universal. Furthermore we have shown that a non-minimal coupling to gravity affects the Hawking radiation while it has no effect on the chiral symmetry breaking. The non-minimal coupling choosen in our model is however not unique and this result is therefore not general. Finally we obtain that a chemical potential for the electric charge does affect neither the partition function, nor the chiral condensate, in consitency with Gauss's law.

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