Running Coupling Constants from Finite Size Effects

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Abstract

The dependence of effective actions on the finite size of the space-time region $\mathcal{M}$ is investigated in detail. It is shown explicitly that the one-loop effective actions on $\mathcal{M}$ and $\lambda\mathcal{M}$ are the same if the volume and surface coupling constants and fields scale according to the renormalization flow. An efficient algorithm for calculating the beta-functions and anomalous dimensions is derived. The general results are applied to a number of examples, in particular scalar field theories in two, four and six dimensions, $O(N)$-sigma models in two dimensions and gauge field theories with fermions in two and four dimensions.

1. Introduction

The behaviour of quantum systems under a change of the length or energy scale plays an important role in high energy physics [1], statistical mechanics [2] and general relativity [3]. The most simple example is the Casimir effect [4] where the vacuum fluctuations change when the walls enclosing the system are moved. This in turn leads to a change of the vacuum energy and a Casimir force acting on the walls. More recently the study of such *finite size effects* have played an important role in 2-dimensional models, in particular in the conformally invariant ones. For example, one can show that the universal term in the scaling of the free energy is proportional to the central charge [5,6]. This means that the central charge characterizes both the ultraviolet *and* infrared behaviour of such models.

On another front, the behaviour of renormalizable quantum field theories under dilatations has centered on the asymptotic scaling of Green's functions. This scaling exhibits departures from the one suggested by naive dimensional analyses and can be studied on the basis of the Callan-Symanzik equation [7]. For more than 2 dimensions the scale (and conformal) invariance is generically broken by hard anomalies. In [8] it has been shown that the breaking of the Weyl-invariance (or local scale invariance) can be absorbed by changing the (local) couplings and introducing external fields.

It is well known that the perturbation expansion is often plagued with severe infrared divergencies. For gauge theories, the 1-loop correction to the 4-boson vertex depends on the infrared cutoff λ as λ^{4-d} and shows a power divergence in less than 4 dimensions. These infrared divergences are then present in the high temperature regime where the 4-dimensional theory becomes effectively 3-dimensional [9]. One way to solve this problem is to assume that spacetime has a finite volume $|\mathcal{M}| \sim L^d$. Then one averages only over degrees of freedom with momenta $p > 1/L$. Alternatively one could use the average action approach as advocated in [10], which has been successfully applied to determine the running couplings and critical exponents for scalar and gauge theories.

In this paper we investigate how the Green's functions change if the infrared cutoff $|\mathcal{M}|$ is moved to $\lambda^d|\mathcal{M}|$ or if one includes smaller and smaller momenta in the averaging procedure. More precisely, we determine the change of the effective action Γ when the finite space-time region $\mathcal{M}$ is scaled to $\lambda\mathcal{M}$ and the couplings and fields scale naively, that is according to their dimensions. In particular, for classically scale invariant theories the renormalized dimensionless volume- and surface coupling constants are kept fixed.

Since for constant mean fields the effective action is just the effective potential, the minimum of which is the vacuum energy, this change should be interpreted as generalized *Casimir effect*. By using heat kernel techniques we shall derive explicit expressions for the scale-dependence of the 1-loop effective actions when the renormalized couplings and fields scale naively.

Instead of viewing the change of Γ as Casimir effect one may ask whether it is possible to keep it invariant. This can indeed be achieved if we allow the volume and surface couplings and fields to scale differently than suggested by dimensional analysis. We find that the scaling which leaves Γ invariant is the naive one supplemented by the anomalous one following from the renormalization group equation. The energy scale in the Callan-Symanzik equation is thereby replaced by the typical inverse length-scale of space time. Besides this Casimir type interpretation for the beta-functions and anomalous dimensions we obtain a very *efficient algorithm* for computing the Callan-Symanzik coefficients in arbitrary dimensions and for various field theories without calculating any Feynman diagrams.

Related results have been obtained in [11], where interacting scalar field theories in curved spaces without boundaries have been investigated. These works did mainly concentrate on the geometry dependence of the effective potentials and actions and related questions such as symmetry restoration for large curvatures. In [12] a variant of the multiple scattering expansion for the Green's functions has been developed and applied to derive the perturbative expansion for quantum fields in spaces with boundaries. In particular, the additional divergencies present in the loop expansion as a consequence of the presence of boundaries and the 2-loop beta-functions have been calculated. Recently Lüscher et.al [13] have applied finite size techniques to lattice calculations. The scale dependence of the lattice couplings in asymptotically free theories and in particular the interpolation between their perturbative small volume and non-perturbative big volume values has been investigated.

The paper is organized as follows: in the second section we analyse the scaling behaviour of the generating functionals on *spacetimes with boundaries* up to 1 loop with the help of heat kernel techniques. In section three we derive explicit expressions for the anomalous scaling of the fields and coupling constants for scalar field theories. They follow from the requirement that the effective actions are scale invariant. The results are applied to scalar fields in 2, 4 and 6 dimensions. In particular, we obtain the 1-loop renormalization group coefficients for the sine-Gordon and $O(N)$-sigma models in 2 dimensions, the ϕ^4 theory in 4 dimensions and the ϕ^3 theory in 6 dimensions. We also derive the general formula for the trace of the energy-momentum tensor in space-times with boundaries. It is shown that the anomalous trace is proportional to the anomalous dimension and the various volume- and surface beta-functions. In the following section, the program is carried through for gauge theories with fermions. For technical reasons we assume that $\mathcal{M}$ possesses no boundaries. Since all spaces (besides the torus) with finite volume and without boundaries are curved we are lead to consider gauge theories on curved space times. We derive the anomalous scaling of the generating functional for fixed renormalized couplings and fields in the different instanton sectors. It is shown that the renormalization group coefficients are the same in all instanton sectors. In section 5 we apply the general results to realistic 4-dimensional gauge theories in the chiral limit of vanishing quark masses and obtain the beta-functions and anomalous dimensions from demanding that the effective action is scale invariant. In the appendices we collect the relevant heat kernel coefficients and set up the necessary formulae for the semiclassical quantization of sigma models.

2. Scale transformation for scalar-fields in leading logarithm approximation

The action of a (possibly multi-component) scalar field ϕ in d-dimensional Euclidean spacetime $\mathcal{M}$ is given by

$$S[\phi, g] = \int_{\mathcal{M}} d^d x \left\{ \frac{1}{2}(\partial_\mu \phi)^2 + V(\phi) \right\}, \tag{2.1}$$

where $g = \{g_a\}$ denotes the set of coupling constants (including masses) appearing in the classical potential V. A scalar field has length-dimension $d_\phi = \frac{1}{2}(2-d)$ and from that one infers the dimensions of the various coupling constants.

We assume that the volume $|\mathcal{M}|$ of spacetime is finite and that the scalar field obeys certain boundary conditions on the boundary $\partial\mathcal{M}$. For example, if ϕ_0 minimizes the (effective) potential we may impose the condition $\phi|_{\partial\mathcal{M}} = \phi_0$. If there are several minimizing ϕ_0, as it typically happens when a continuous symmetry is spontaneously broken, the boundary values must further be specified. If no external source is applied then these boundary conditions may select the vacuum state which is chosen by the quantum system.

Alternatively we could assume that $\mathcal{M}$ possesses no boundary, e.g. that it is a d-dimensional sphere. For gauge theories (considered in sections 4 and 5) we shall make this assumption, mostly for technical reasons. For scalar theories it is more convenient to assume that space time possesses a boundary, e.g. is a d-dimensional ball. This way we can avoid the problems associated with the zero-modes of the derivative term in (2.1) [14].

The *partition function* which is the generating functional for the Green's functions is formally given by the Euclidean functional integral

$$\mathcal{Z}[\mathcal{M}, j, g] = \frac{1}{\mathcal{N}} \int \mathcal{D}\phi \, \exp\Big[-\frac{1}{\hbar} S[\phi, g] + \frac{1}{\hbar} \int_{\mathcal{M}} d^d x \, j \cdot \phi \Big] , \tag{2.2}$$

where we have made the dependence on the spacetime region explicit. Due to the infrared cutoff we average only over fields with momenta larger than the inverse size of the system. Often it is more convenient to consider the *Schwinger functional* which generates the connected Green's functions

$$\mathcal{W}[\mathcal{M}, j, g] = \hbar \, \log \mathcal{Z}[\mathcal{M}, j, g] \tag{2.3}$$

or its Legendre transform, the *effective action*

$$\Gamma[\mathcal{M}, \varphi, g] = \int_{\mathcal{M}} j \cdot \varphi - \mathcal{W}[\mathcal{M}, j, g], \tag{2.4}$$

where the source solves $\varphi = \delta\mathcal{W}/\delta j$, i.e. is conjugate to the mean field [1] φ. The Schwinger functional can be reconstructed from the effective action by the inverse Legendre transformation

$$\mathcal{W}[\mathcal{M}, j, g] = \int_{\mathcal{M}} j \cdot \varphi - \Gamma[\mathcal{M}, \varphi, g], \tag{2.5}$$

[1] we use the symbols φ for the mean field, i.e. the argument of the effective action, and ϕ for the microscopic field appearing in functional integrals like (2.2).

where the mean field solves $j = \delta\Gamma/\delta\varphi$. In cases where the derivative of $\mathcal{W}(j)$ is not continuous the commonly used transformation (2.4) fails to be applicable. This happens typically when the classical potential is not convex. To handle the general case one uses the transformations

$$\begin{aligned}\Gamma[\mathcal{M},\varphi,g] &= \sup_{j}\Big\{\int\limits_{\mathcal{M}} j\cdot\varphi - \mathcal{W}[\mathcal{M},j,g]\Big\}\\ \mathcal{W}[\mathcal{M},j,g] &= \sup_{\varphi}\Big\{\int\limits_{\mathcal{M}} j\cdot\varphi - \Gamma[\mathcal{M},\varphi,g]\Big\},\end{aligned} \tag{2.6}$$

which coincide with (2.4,5) for differentiable $\mathcal{W}$ [15].

First we evaluate these functionals semiclassically, i.e. include the one-loop corrections by means of the steepest descent approximation. Then we determine how they change if $\mathcal{M}$ is scaled to $\lambda\mathcal{M}$.

To derive the semiclassical expansion on spacetimes with boundaries we set $\phi = \phi_{cl} + \sqrt{\hbar}\,\delta\phi$, where ϕ_{cl} extremizes the exponent in (2.2) and $\delta\phi$ denotes the fluctuation field. To find the equation for the extremizing field ϕ_{cl} we expand the exponent in (2.2) in the fluctuation field:

$$\begin{aligned}-S[\phi] + \int\limits_{\mathcal{M}} j\cdot\phi &= -S[\phi_{cl}] + \int\limits_{\mathcal{M}} j\cdot\phi_{cl}\\ &\quad - \sqrt{\hbar}\oint\limits_{\partial\mathcal{M}} \partial_n\phi_{cl}\,\delta\phi + \sqrt{\hbar}\int\limits_{\mathcal{M}} \big\{\partial^2\phi_{cl} - V'(\phi_{cl}) + j\big\}\delta\phi\\ &\quad - \frac{\hbar}{2}\oint\limits_{\partial\mathcal{M}} \partial_n\delta\phi\,\delta\phi - \frac{\hbar}{2}\int\limits_{\mathcal{M}} \delta\phi\big\{-\partial^2 + V''(\phi_{cl})\big\}\delta\phi + O(\delta\phi^3).\end{aligned} \tag{2.7}$$

Here we encounter surface terms since $\mathcal{M}$ possesses a boundary. However, if we impose the same boundary conditions on ϕ_{cl} as on the fields in the functional integral, that is set $\phi = \phi_{cl} = \phi_0$ on $\partial\mathcal{M}$, then the fluctuations vanish there and both surface integrals in (2.7) vanish. Instead of prescribing the values of ϕ on the boundary we could assume that its normal derivative vanishes. In the semiclassical approximation we would then impose the same condition on ϕ_{cl}. Then the normal derivative of the fluctuations vanishes and again both surface integrals in (2.7) are zero. Thus with both boundary conditions the extremum $\phi_{cl}(j)$ of the exponent in (2.2) is determined by the field equation

$$\frac{\delta S}{\delta\phi}[\phi_{cl}] = -\partial^2\phi_{cl} + V'(\phi_{cl}) = j \tag{2.8}$$

and the imposed boundary conditions. We prefer to prescribe the field on the boundary so that the derivative term in the classical action possesses no zero-mode(s).

Inserting the expansion (2.7) into the functional integral and retaining the terms quadratic in the fluctuations, the resulting Gaussian integral yields

$$
\begin{aligned}
\mathcal{W}^{(1)}[\mathcal{M},j,g] &= \mathcal{W}^{cl}[\mathcal{M},j,g] - \frac{\hbar}{2}\log\det M(j,g), \quad \text{where} \\
\mathcal{W}^{cl}[\mathcal{M},j,g] &= \sup_{\varphi}\Big\{\int_{\mathcal{M}} j\cdot\varphi - S[\varphi,g]\Big\} = \int_{\mathcal{M}} j\cdot\phi_{cl} - S[\phi_{cl},g]
\end{aligned}
\tag{2.9}
$$

is the classical Schwinger functional and $M(j,g)=-\partial^2+V''(\phi_{cl})$ the *fluctuation operator*. ϕ_{cl} depends on the external source and the coupling constants through (2.8) so that $\mathcal{W}^{cl}$ and the determinant are indeed functions of the source. If ϕ has several components then V'' denotes the second derivative matrix at ϕ_{cl}. Both the classical piece and the 1-loop determinant in (2.9) depend on the spacetime $\mathcal{M}$. The spacetime dependence of the determinant enters through the boundary conditions for the fluctuations. The generating functionals depend also on the prescribed boundary field ϕ_0. Actually $\mathcal{Z}[j=0]$ in (2.2) is just the wave functional $\Psi[\phi_0]$ obeying the functional Schrödinger equation with Hamiltonian corresponding to the action in (2.2) [12,16]. But since this aspect is not of important here we shall not make the ϕ_0 dependence explicit.

We proceed to compute the effective action. From $\varphi=\delta\mathcal{W}/\delta j$ and $\mathcal{W}=\mathcal{W}^{cl}+O(\hbar)$ it follows at once that the mean field is given by the classical one, up to corrections of order $\hbar$. Furthermore, since $\int j\phi - S[\phi]$ is stationary at ϕ_{cl} we see that the effective action is given by

$$
\Gamma^{(1)}[\mathcal{M},\varphi,g] = S[\varphi,g] + \frac{\hbar}{2}\log\det M(\varphi,g) \tag{2.10}
$$

up to terms $O(\hbar^2)$. Note that the fluctuation operator $M(\varphi,g)=-\partial^2+V''(\varphi)$ is now evaluated at φ.

The determinants are to be computed subject to Dirichlet boundary conditions. Then M is selfadjoint and possesses a discrete spectrum. Of course, the fluctuation determinants are ill-defined due to ultraviolet divergences and must be regularized. We shall employ the ζ-function regularization for computing them [17]

$$
\log\det M = -\frac{d}{ds}_{|s=0}\zeta_M(s), \qquad \zeta_M(s) = \operatorname{tr} M^{-s} = \sum_n \lambda_n^{-s}. \tag{2.11}
$$

This is indeed a regularization of the determinant since $\zeta(s)$ is analytic at $s=0$. It includes, up to possible counterterms, the 1-loop normalization $\mathcal{N}$ of the functional integral. This regularization has the nice property that it does not change the coupling constants in the classical potential and hence they may be regarded as renormalized ones. This property is not ment to be obvious but follows from the heat kernel representation for the ζ-function discussed below.

The above definition of the ζ-function does not allow us to take the s-derivative at $s=0$ since the trace in (2.11) is defined only for $Re(s)>d/2$. The analytic continuation can be achieved by taking the Mellin transform of the heat kernel

$$\zeta_M(s)=\frac{1}{\Gamma(s)}\int\limits_0^\infty dt\,t^{s-1}\ \mathrm{tr}\ e^{-tM} \tag{2.12}$$

and this fact will be exploited considerably later on.

Next we consider the *rescaled theory* on the space-time region $\lambda\mathcal{M}$ and the corresponding generating functionals. Under a scale transformation

$$\tilde{x}=\lambda x \tag{2.13}$$

the classical field and source transform as

$$\tilde{\phi}(\tilde{x})=\lambda^{\frac{1}{2}(2-d)}\phi(x)\equiv\lambda^{d_\phi}\,\phi(x),\qquad \tilde{j}(\tilde{x})=\lambda^{-\frac{1}{2}(2+d)}\,j(x), \tag{2.14a}$$

such that the derivative term in (2.1) and the source term in (2.2) are invariant. Let g_a be a coupling constant which appears in the combination $g_a\phi^a$ in V. Classically it scales as

$$\tilde{g}_a=\lambda^{d_a}g_a,\qquad\text{where}\qquad d_a=\frac{1}{2}d(a-2)-a \tag{2.14b}$$

is its length dimension. In particular a mass scales in all dimensions as $\lambda\tilde{m}=m$. Also, for the critical exponent $a_c=2d/(d-2)$ the coupling constant does not scale. Scalar theories with potentials $V=g_{a_c}\phi^{a_c}$ are called classically scale invariant. The point is that we need not assume such a particular form for the potential. By allowing for the (naive) scalings (2.14) of the field, source and coupling constants when we scale $\mathcal{M}$ to $\lambda\mathcal{M}$, the classical action and the source term are both scale invariant for arbitrary scalar field theories

$$\int\limits_{\lambda\mathcal{M}} d^d\tilde{x}\;\tilde{j}\cdot\tilde{\phi}-S[\lambda\mathcal{M},\tilde{\phi},\tilde{g}]=\int\limits_{\mathcal{M}} d^dx\;j\cdot\phi-S[\mathcal{M},\phi,g]. \tag{2.15}$$

Taking the suprema of this equality over all fields proves then the scale invariance of the classical Schwinger functional

$$\mathcal{W}^{cl}[\lambda\mathcal{M},\tilde{j},\tilde{g}]=\mathcal{W}^{cl}[\mathcal{M},j,g]. \tag{2.16}$$

More explicitly, it follows that if ϕ_{cl} solves (2.8) with given source and coupling constants, then $\tilde{\phi}_{cl}$ solves (2.8) with scaled source and scaled constants. Of course for (2.16) to hold one should also check that $\tilde{\phi}_{cl}$ obeys the correct boundary conditions if ϕ_{cl} does.

This means that on the boundary it should mimimize the scaled potential. But since the minimas of $V(\tilde{g})$ scale the same way as the fields this follows at once.

As is well known the scale transformations (2.13,14) cease to be a symmetry of the quantized theory due to the *scale anomaly.* To see that explicitly on the functional level we relate the generating functionals on $\lambda\mathcal{M}$ and $\mathcal{M}$.

From the scale invariance of $\mathcal{W}^{cl}$ it follows from (2.9) and (2.10) that

$$\begin{aligned}\mathcal{W}^{(1)}[\lambda\mathcal{M},\tilde{j},\tilde{g}] &= \mathcal{W}^{cl}[\mathcal{M},j,g] \;-\; \frac{\hbar}{2}\log\det M(\tilde{j},\tilde{g})\\ \Gamma^{(1)}[\lambda\mathcal{M},\tilde{\varphi},\tilde{g}] &= S[\mathcal{M},\varphi,g] \;+\; \frac{\hbar}{2}\log\det M(\tilde{\varphi},\tilde{g}).\end{aligned} \tag{2.17}$$

We focuse on the regulated determinant on the scaled spacetime $\lambda\mathcal{M}$:

$$\log\det M(\tilde{\varphi},\tilde{g}) = -\frac{d}{ds}_{|s=0}\zeta_{M(\tilde{\varphi},\tilde{g})}.$$

Note that the fluctuation operators scale homogeneously under the scale transformations (2.13,14)

$$\tilde{M} \equiv -\tilde{\partial}^2 + V''(\tilde{\varphi},\tilde{g}) = \lambda^{-2}\big[-\partial^2 + V''(\varphi,g)\big] \equiv \lambda^{-2}M. \tag{2.18}$$

It follows from (2.11) that $\zeta_{\tilde{M}}(s) = \lambda^{2s}\,\zeta_M(s)$. Hence the ratio of the scaled to the unscaled determinant becomes

$$\log\frac{\det\tilde{M}}{\det M} = -2\log\lambda\cdot\zeta_M(0) \tag{2.19}$$

and we find the following scaling laws

$$\begin{aligned}\mathcal{W}^{(1)}[\lambda\mathcal{M},\tilde{j},\tilde{g}] &= \mathcal{W}^{(1)}[\mathcal{M},j,g] + \hbar\log\lambda\cdot\zeta_{M(j,g)}(0)\\ \Gamma^{(1)}[\lambda\mathcal{M},\tilde{\varphi},\tilde{g}] &= \Gamma^{(1)}[\mathcal{M},\varphi,g] - \hbar\log\lambda\cdot\zeta_{M(\varphi,g)}(0),\end{aligned} \tag{2.20}$$

and this is the main result of this section. Whereas $\mathcal{W}^{cl}$ and S are both scale invariant, $\mathcal{W}^{(1)}$ and $\Gamma^{(1)}$ are not. The scale anomaly, that is the logarithmic corrections to the scale invariance, has been made explicit in the last terms in (2.20). We emphasize that these scaling laws are correct for arbitrary scalar theories. If all coupling constants are dimensionless then $\tilde{g}=g$ in (2.20). Later we shall see that the formulae (2.20) also hold for gauge theories, up to slight modifications due to zero-modes and gauge fixing.

To determine $\zeta(s)$ for vanishing s we use the representation (2.12). In the limit $s\to 0$ the singular part of the t-integration comes only from the small t region. Using the heat kernel expansion for small t [18]

$$\mathrm{tr}\, e^{-tM}\, f = \frac{1}{(4\pi t)^{\frac{d}{2}}}\sum_{n=0,1,\dots}^{\infty}\Big[\int_{\mathcal{M}} a_{\frac{n}{2}}(f;g) + \oint_{\partial\mathcal{M}} b_{\frac{n}{2}}(f;g)\Big]\, t^{\frac{n}{2}}, \tag{2.21}$$

where $f(x)$ is an arbitrary test-function, one finds [19]

$$\zeta_M(0) = \frac{1}{(4\pi)^{\frac{d}{2}}}\Big[\int a_{\frac{d}{2}}(1;g) + \oint b_{\frac{d}{2}}(1;g)\Big]. \tag{2.22}$$

Here the integral symbol denotes both the integration over spacetime or its boundary and the trace over internal indices if ϕ has several components and thus M is matrix valued. (2.21) contains half-integer powers of t since the trace must be computed with respect to Dirichlet boundary conditions. This leads to boundary contributions to the heat kernel and half-integer powers of t in the small-t expansion.

The volume coefficients a_n vanish for odd n and have length dimension $-2n$. The surface coefficients b_n have length dimension $1-2n$. For general second order fluctuation operators the a_n have been calculated for $n \leq 5$, relevant for 10 and less dimensions [18,20] and the b_n for $n \leq 2$, relevant for 4 and less dimensions [6,21]. In appendix A we have collected the coefficient functions for Dirichlet boundary conditions relevant for theories in 6 or less dimensions.

The physical role of these coefficient functions for $n \leq d$ can be seen more clearly in the proper time or dimensional regularizations which are intimately related to the ζ-function scheme [19,22]. In perturbative calculations of the effective action they are just the divergent terms which must be absorbed by counterterms. For example, $a_{\frac{d}{2}}$ is multiplied by a logarithmically divergent factor, e.g. $\log\epsilon$ in the proper time regularization, and a_0 is multiplied by a factor $\epsilon^{-\frac{d}{2}}$. Thus the most divergent term is $\sim \int a_0 = |\mathcal{M}|$ and such a term can be absorbed by renormalizing the cosmological constant. In the ζ-function regularization these infinite terms are suppressed and thus we may regard the coupling constants in the classical action as renormalized ones. The role of the particular coefficient function $a_{\frac{d}{2}}$ is twofold. It appears as logarithmically divergent contribution in perturbation theory and at the same time determines the universal anomalous scaling of the renormalized generating functionals.

For scalar field theories $A_\mu = 0$ and $C = V''$ [2] in (A1). Thus the a_n and b_n are local polynomials in V'', the extrinsic curvature χ_{ab} of the boundary and their spatial derivatives. Inserting the corresponding coefficients (A2) and (A3) into (2.22) we obtain

[2] For several scalar fields V'' is the second derivative matrix, $V'' = (\frac{\partial^2 V}{\partial\varphi_i \partial\varphi_j})$

in 2, 4 and 6 dimensions:

$$
\begin{aligned}
\zeta_{d=2}(0) &= -\frac{1}{4\pi}\int \operatorname{tr} V'' + \frac{N}{6}\chi_E \\
\zeta_{d=4}(0) &= \frac{1}{2!}\frac{1}{(4\pi)^2}\Big[\int \operatorname{tr}(V'')^2 - \oint \{\partial_n \operatorname{tr} V'' + \frac{2}{3}\operatorname{tr} V''\chi_{aa}\}\Big] \\
&\quad + \frac{N}{180}\chi_E + \frac{N}{280\pi^2}\oint f(\chi) \\
\zeta_{d=6}(0) &= -\frac{1}{3!}\frac{1}{(4\pi)^3}\int \Big[\operatorname{tr}(V'')^3 + \frac{1}{2}\operatorname{tr} V''\Delta V''\Big] \\
&\quad + \oint \mathrm{P}\,(V'', \partial^{(k)}V'', \chi_{ab}).
\end{aligned}
\tag{2.23}
$$

Here N is the number of scalar fields, ∂_n the outward oriented normal derivative and $\int$ ($\oint$) and tr denote integration over $\mathcal{M}$ ($\partial\mathcal{M}$) and trace over internal indices, respectively. χ_E is the Euler number. With our sign convention it is 1 if the boundary is a sphere. It is the winding number of the normal vector field $n(x)$ on $\partial\mathcal{M}$ and thus is a topological invariant. In general it gets a contribution form the metric and extrensic curvature. The function f is the conformally invariant third order polynomial in the extrinsic curvature (see appendix A),

$$
f(\chi) = \operatorname{tr}\ \chi^3 - \operatorname{tr}\ \chi \operatorname{tr}\ \chi^2 + \frac{2}{9}(\operatorname{tr}\chi)^3, \tag{2.24}
$$

and it vanishes if $\partial\mathcal{M}$ is a sphere. The polynomial appearing in the surface integral in 6-dimensions has not yet been calculated.

For *constant fields* $\varphi = \varphi_0$ we have in $d = 2n$ dimensions

$$
\zeta_d(0) = K_d|\mathcal{M}|\operatorname{tr}\big(V''(g,\varphi_0)\big)^n + \oint \mathrm{P}\,(V'', \chi_{ab}), \quad K_d = \frac{(-1)^n}{(4\pi)^n n!} \tag{2.25}
$$

In odd dimensions the a_n vanish and $\zeta_M(0)$ contains no volume terms. From (2.20) it follows then that the generating functionals on $\lambda\mathcal{M}$ and $\mathcal{M}$ are the same, up to surface terms. Thus the scale invariance of the classical theories survives when one includes 1-loop corrections, up to surface terms. For that reason we shall consider theories in even dimensions only in what follows.

2.1 The role of the surface terms

For free massless scalars $V = 0$ and the only contribution to the anomalous scaling comes from the purely geometric surface terms in (2.23). This property holds if $\mathcal{M}$ is flat. In curved spacetimes geometric volume terms are present even for free massless particles [6,11]. However, in this section we shall assume spacetime to be flat.

The kown surface coefficients are listed in appendix A. They are relevant when one discusses Casimir type effects for free fields [6]. Inserting b_1 into (2.20,22) immediately leads to the following formula in 2 dimensions

$$\Gamma_{geom} \equiv \Gamma\big[\lambda\mathcal{M},\tilde{\varphi}\big] - \Gamma\big[\mathcal{M},\varphi\big] = -\frac{\hbar N}{6}\,\chi_E\,\log\lambda, \tag{2.26}$$

where N is the number of free massless scalars and χ_E the Euler number of $\mathcal{M}$. The anomalous scaling depends only on the topology of spacetime. Actually one can show that (2.26) holds for any 2-dimensional conformal field theory provided N is replaced by the central charge c [5,6]. This is an exact result and holds beyond perturbation theory.

Similarly, for free massless scalars in 4 dimensions only the coefficient b_2 contributes to $\zeta(0)$ in (2.20) and leads to the following scaling formula [6]

$$\Gamma_{geom} = -\frac{\hbar N\log\lambda}{180}\,\chi_E - \frac{\hbar N\log\lambda}{280\pi^2}\oint f(\chi). \tag{2.27}$$

Contrary to the situation in 2 dimensions the scaling behaviour depends here on the geometry encoded in f and not only on the topology.

The scaling laws (2.26-27) for free scalars are purely geometric and therefore present irrespective of the form of the classical potential. Thus, for an interacting theory there are two sources for the anomalous scaling, namely the presence of the boundary which introduces a geometrical length scale and the interaction between the particles which introduces a dynamical mass scale. We shall not always make the purely geometric contributions (2.26-27) to the anomalous scaling explicit. But they must always be added to the dynamical terms containing powers of the potential and its derivatives.

The geometric surface terms do not change the couplings in V, since those are related to volume integrals. However, their appearance in Γ signals that we should include surface terms in the classical action as possible counterterms [12]. Thus in the presence of boundaries S in (2.1) should be modified to

$$S[\phi,g,h] = \int_{\mathcal{M}} d^dx\left\{\frac{1}{2}(\partial_\mu\phi)^2\,+\,V(\phi)\right\} + \oint_{\partial\mathcal{M}} Q(\chi_{ab},\phi,\partial_n\phi), \tag{2.28}$$

where $h=\{h_a\}$ are the coupling constants appearing in Q. Similarly to the volume couplings they will run due do quantum corrections. We require them to have length dimensions ≤ 0 for the theory to be renormalizable. Since the surface potential Q has length dimension $1-d$ it follows that the surface potential is at most linear in $\partial_n\phi$.

For example, the most general form of Q in 4 dimensions which is invariant under reflection of ϕ reads

$$\begin{aligned}\oint Q =& h_1|\partial\mathcal{M}| + h_2\oint \operatorname{tr}\chi + h_3\chi_E + h_4\oint f(\chi)\\ &+\frac{h_5}{2}\oint\phi^2 + \frac{h_6}{2}\oint\phi^2\operatorname{tr}\chi + \frac{h_7}{2}\oint\partial_n\phi^2,\end{aligned} \tag{2.29}$$

The $h_1,\ldots,h_4$-terms are purely geometrical. Due to the imposed boundary conditions the surface potential Q factorizes in the functional integral (2.2). In the expansion of (2.28) about ϕ_{cl} (similarly to (2.7)) the terms quadratic in the fluctuations $\partial_n\delta\phi$ are always multiplied by $\delta\phi$. Due to the imposed boundary conditions such terms vanish. Thus the 1-loop formulae (2.9,10) still hold with exactly the same determinant but with classical action (2.28). For example, for a free massless field in 4 dimensions with surface potential Q containing only the geometric $h_1,\ldots,h_4$-terms, (2.27) yields

$$\begin{gathered}\Gamma^{(1)}[\lambda\mathcal{M},\tilde{\varphi},\tilde{h}]=\Gamma^{(1)}[\mathcal{M},\varphi,h(\lambda)],\qquad\text{where}\\ h_3(\lambda)=h_3-\frac{\hbar N\log\lambda}{180}\qquad h_4(\lambda)=h_4-\frac{\hbar N\log\lambda}{280\pi^2},\end{gathered}\tag{2.30}$$

and the remaining two coupling constants scale naively. However, in higher orders in a loop expansion these couplings may run as well. We see that if we allow for an anomalous scaling of some constants then the effective action is invariant under scale transformations. The point is that this remains true for interacting theories.

2.2 The 1-loop effective potential from scaling behaviour

Let us see how the general 1-loop scaling behaviour (2.20) relates to more familiar results. We shall derive the 1-loop *effective potential* in even dimensions, that is the effective action density for constant mean field $\varphi=\varphi_0$,

$$U^{(1)}(\mathcal{M},\varphi_0,g)=\frac{1}{|\mathcal{M}|}\,\Gamma^{(1)}[\mathcal{M},\varphi_0,g],\tag{2.31}$$

from scaling arguments. Since the surface terms are not known in $d>4$ dimensions, we shall neglect them for the moment so that our results are correct up to surface terms. From the scaling law (2.20) we obtain

$$\lambda^d\,U^{(1)}(\lambda\mathcal{M},\tilde{\varphi}_0,\tilde{g})=U^{(1)}(\mathcal{M},\varphi_0,g)-\frac{\hbar\log\lambda}{|\mathcal{M}|}\cdot\zeta_{M(\varphi_0,g)}(0),\tag{2.32}$$

where according to (2.17)

$$\begin{aligned}U^{(1)}(\mathcal{M},\varphi_0,g)&=V(\varphi_0,g)+\hbar\Delta U\big(V''(\varphi_0,g)\big)\\ \Delta U(x)&=\frac{1}{2|\mathcal{M}|}\,\log\det(-\partial^2+x).\end{aligned}\tag{2.33}$$

Here we have used that for constant fields the determinant can only depend on $x=V''(\phi_0,g)$. The classical potential cancels in (2.32). Finally, since

$$\tilde{x}=V''(\tilde{\varphi}_0,\tilde{g})=\lambda^{-2}V''(\varphi_0,g)=\lambda^{-2}x$$

we obtain the following equation for the 1-loop contribution to U

$$\lambda^d \Delta U(\lambda^{-2}x) - \Delta U(x) = -\log\lambda \cdot \zeta(0) = -\log\lambda \cdot K_d\, x^{\frac{d}{2}},$$

where we made use of (2.25). The nontrivial solution is just

$$\Delta U(x) = \frac{K_d}{2}\, x^{\frac{d}{2}}\, \log\frac{x}{\mathrm{const}},$$

where one takes a convenient normalization in the logarithm. Adding this 1-loop result to the classical term we end up with

$$U^{(1)}(\varphi_0) = V(\varphi_0) + \frac{\hbar K_d}{2}\, \big(V''(\varphi_0)\big)^{\frac{d}{2}}\, \log\frac{V''(\varphi_0)}{\mathrm{const}}. \tag{2.34}$$

In 4 dimensions the surface contributions to the anomalous scaling are known and we can go further (2 dimensions are too trivial, since the surface scale-anomalies are purely geometrical). Using the result (2.23) in 4 dimensions yields the effective potential

$$\begin{aligned} U^{(1)}(\varphi_0) =& V(\varphi_0) + \frac{\hbar}{32\pi^2}\, \big(V''(\varphi_0)\big)^2\, \log\frac{V''(\varphi_0)}{\mathrm{const}} \\ & - \frac{\hbar}{48\pi^2|\mathcal{M}|}\, V''\, \log\frac{V''}{\mathrm{const}} \cdot \oint \mathrm{tr}\,\chi, \end{aligned} \tag{2.35}$$

up to purely geometric and thus φ_0-independent terms. Note that for 'reasonable' boundaries the finite volume effective potentials (2.35) tend to the infinite volume result (2.34) for $d=4$ as required.

3. The running coupling in scalar theories

In this section we apply the general results (2.20) to a class of interacting renormalizable scalar field theories in various dimensions. We recover the scaling behaviour of the different volume and surface couplings together with the 1-loop $\beta-$functions, anomalous dimensions γ and finally the trace anomaly of the energy-momentum tensor. Recall that in *odd dimensions* $\zeta(0)$ has no volume terms. Hence the wave functions and the volume couplings are not renormalized in the 1-loop approximation. In the chosen regularization scheme we cannot see any running of the volume coupling constants in odd dimensions. However, the surface couplings do run since $b_{\frac{d}{2}}$ does not necessarily vanish in odd dimensions (see A3). Although this is interesting in its own right, we shall concentrate here on the commonly considered volume terms and therefore consider even dimensions only.

2 dimensions.

We start with the general action for a one-component scalar field

$$S[\phi,g,h]=\int\Big\{\frac{1}{2}(\partial_\mu\phi)^2+\sum_{a=0}^{\infty}\frac{g_a}{a!}\,\phi^a\Big\}+h\chi_E, \tag{3.1}$$

where we added a topological surface term proportional to the Euler number. In 2 dimensions we could add infinitely many relevant and marginal surface terms since ϕ is dimensionless. But besides the non-universal $|\partial\mathcal{M}|$ and the universal χ_E none of them is needed as counterterm. Hence everything what we say holds also if we add other surface terms.

From (2.20) and (2.23) we derive the following scaling behaviour for the effective action

$$\begin{aligned}\Gamma^{(1)}[\lambda\mathcal{M},\varphi,\tilde{g},h]&=S[\varphi,g,h]+\frac{\hbar}{2}\log\det M(\varphi,g)\\&+\frac{\hbar\log\lambda}{4\pi}\sum_{a=2}^{\infty}\frac{g_a}{(a-2)!}\int\varphi^{a-2}\;-\;\frac{\hbar\log\lambda}{6}\chi_E,\end{aligned}$$

where we have used that h and φ are both dimensionless so that $\tilde{h}=h$ and $\tilde{\varphi}=\varphi$ and that the determinant does not depend on the surface coupling constant h. We rearrange the different terms with the result

$$\Gamma^{(1)}[\lambda\mathcal{M},\varphi,\tilde{g},h]=S[\mathcal{M},\varphi,g(\lambda),h(\lambda)]+\frac{\hbar}{2}\log\det M(\varphi,g), \tag{3.2}$$

where we introduced the *running coupling constants*

$$g_a(\lambda)=g_a+\frac{\hbar g_{a+2}}{4\pi}\log\lambda\quad\text{and}\quad h(\lambda)=h-\frac{\hbar}{6}\log\lambda. \tag{3.3a}$$

Since the replacement $g\to g(\lambda)=g+O(\hbar)$ in $\log\det M$ changes the right hand side of eq. (3.2) only in $O(\hbar^2)$, which does not affect the 1-loop equation, we can relate the scaled and unscaled functionals as

$$\Gamma^{(1)}[\lambda\mathcal{M},\varphi,\tilde{g},h]=\Gamma^{(1)}[\mathcal{M},\varphi,g(\lambda),h(\lambda)]. \tag{3.4}$$

In other words, we can restore the invariance of the effecive action if we allow for an anomalous scaling of the coupling constants. Thus, if the naive dimensional scaling (2.14) is supplemented by the anomalous one (3.3a), then the 1-loop generating functional is scale invariant.

To compare our results with the more conventional renormalization group results in momentum space we note that (2.13) implies

$$\tilde{p}=\lambda^{-1}p\equiv\mu p. \tag{3.5}$$

So we find the following leading logarithm expression for the running couplings in 2 dimensions:

$$g_a(\mu) = g_a - \frac{\hbar g_{a+2}}{4\pi}\log\mu \qquad \text{and} \qquad h(\mu) = h + \frac{\hbar}{6}\log\mu. \tag{3.3b}$$

From that we immediately recover the $\beta-$functions to leading order

$$\begin{aligned} \beta(g_a) &= \mu\frac{\partial}{\partial\mu}g_a(\mu) \Longrightarrow \beta^{(1)}(g_a) = -\frac{\hbar g_{a+2}}{4\pi} \\ \beta(h) &= \mu\frac{\partial}{\partial\mu}h(\mu) \Longrightarrow \beta^{(1)}(h) = \frac{\hbar}{6}. \end{aligned} \tag{3.6}$$

As expected in these models there is no wave function renormalization.

As an application we calculate the anomalous scaling behaviour of the perturbative mass in the *sine-Gordon* model. We parametrize the classical potential as [23]

$$V(\phi) = \gamma - \frac{m^2}{\beta^2}\cos(\beta\phi), \tag{3.7}$$

where β is dimensionless and m the perturbative mass. It is now easy to calculate the rescaled effective action from (3.4,6) (or directly from (2.20,23)) and one finds that it is scale invariant,

$$\Gamma^{(1)}[\lambda\mathcal{M},\phi,\tilde{\gamma},\beta,\tilde{m}^2] = \Gamma^{(1)}[\mathcal{M},\phi,\gamma(\lambda),\beta,m^2(\lambda)], \tag{3.8}$$

provided the mass runs as

$$m^2(\lambda) = m^2\big(1 - \frac{\hbar}{4\pi}\beta^2\log\lambda\big). \tag{3.9a}$$

and the cosmological constant γ as

$$\gamma(\lambda) = \gamma - \frac{\hbar}{6|\mathcal{M}|}\chi_E\log\lambda. \tag{3.9b}$$

The equation (3.8) is an exact 1-loop relation including surface terms.

For models with *polynomial interactions* the coupling of the highest power is not renormalized. This is of course related to the fact that these models are superrenormalizable.

Note that the anomalous scale dependence of the surface coupling constant h is insensitive to the details of the model. Since it scales the same way for all 2-dimensional models we shall ignore it in the following sigma-model calculations.

We conclude this section with a discussion of the scaling behaviour of the $O(N)-$sigma models. In terms of the constrained field $n_a; a=1,..,N$ the action reads

$$S[n_a,g] = \frac{1}{2g^2}\int \partial^\mu n^a\cdot\partial_\mu n_a, \qquad n^2 = 1. \tag{3.10}$$

First we shall evaluate the partition function, where attention must be paid to the constraint on the field. In (2.2) we integrate over fields with fixed length $n^2 = 1$.

The classical field N_a, which extremizes the exponent in (2.2) and fulfils the constraint is determined by

$$\partial^2 N_a \; - \; (N^b \cdot \partial^2 N_b) N_a \; = \; g^2 \{ (N^b \cdot j_b) N_a \; - \; j_a \}. \tag{3.11}$$

Note that the solution N_a for a given source j_a is also a solution for the locally transformed source $j_a(x) + f(x) N_a(x)$. Such an ambiguity is to be expected from counting degrees of freedom. Hence there is no one-to-one correspondence between fields and sources. This means that the effective action or Legendre transform of $\mathcal{W}$ cannot be defined in the n-variables. If we would introduce unconstrained variables, for example by a stereographic projection, this problem could be overcome. But it is more convenient to use the n-field for which we must deal with $\mathcal{W}$ rather than Γ.

In Appendix B we review the calculation of $\mathcal{W}^{(1)}$. One finds

$$\mathcal{W}^{(1)}[\mathcal{M}, j, g] \; = \; \mathcal{W}^{cl}[\mathcal{M}, j, g] \; - \; \frac{\hbar}{2} \log \det D,$$

where the fluctuation operator D is given in appendix B. Now we apply the general scaling formula (2.20). Inserting $\zeta_D(0)$ from (B6) we obtain

$$\begin{aligned} \mathcal{W}^{(1)}[\lambda \mathcal{M}, \tilde{j}, \tilde{g}] \; &= \; \mathcal{W}^{cl}[\mathcal{M}, j, g] \; - \; \frac{\hbar}{2} \log \det D \\ + \frac{\hbar (N-2)}{4\pi} \log \lambda \int \partial^\mu N^a \cdot \partial_\mu N_a \; &- \; \frac{\hbar g^2 (N-1)}{4\pi} \log \lambda \int j^a \cdot N_a, \end{aligned} \tag{3.12}$$

where we omitted the trivial boundary terms. We restore scale invariance by supplementing the naive scaling (2.13,14) with the anomalous one for g and the source j_a. We find

$$\mathcal{W}^{(1)}[\lambda \mathcal{M}, \tilde{j}, \tilde{g}] = \mathcal{W}^{(1)}[\mathcal{M}, j(\lambda), g(\lambda)],$$

where

$$g^2(\lambda) = \frac{g^2 \log \lambda}{1 - \frac{\hbar}{2\pi} g^2 (N-2)}, \quad j_a(\lambda) = \Big(1 - \frac{\hbar g^2 (N-1)}{4\pi} \log \lambda \Big) j_a. \tag{3.13}$$

Note that in contrast to the other models considered it is now the anomalous scaling of the source j_a which yields the anomalous dimension. Translating the above result to momentum space we obtain the β-function

$$\beta^{(1)}(g^2) = \mu \frac{\partial}{\partial \mu} g^2 = - \frac{\hbar (N-2)}{2\pi} g^4 \tag{3.14a}$$

in agreement with the literature [24]. Since β is negative the coupling becomes weaker if $\mathcal{M}$ shrinks, as expected for an asymptotically free theory. The anomalous dimension is related to the transformation behaviour of the source. We find it to be

$$\gamma(g^2) = \frac{\hbar(N-1)}{2\pi} g^2. \tag{3.14b}$$

4 dimensions.

Here we consider the perturbatively renormalizable Higgs model with quartic self-interaction

$$V(\phi) = g_0 + \frac{g_2}{2!}\phi^2 + \frac{g_4}{4!}\phi^4 \tag{3.15}$$

and the general surface-interaction (2.29). For $g_0 = 0$ and $g_2 = m^2$ this corresponds to the unbroken theory with perturbative mass m, and for $g_0 = gv^4/4$, $g_2 = -gv^2/6$ and $g_4 = g$ we obtain the Higgs model with perturbative Higgs mass $m_H = gv^2/3$. Applying (2.20) one obtains the following explicit form for the scaled effective action

$$\begin{aligned}\Gamma^{(1)}[\lambda\mathcal{M}, \tilde{\varphi}, \tilde{g}, \tilde{h}] &= \Gamma^{(1)}[\mathcal{M}, \varphi, g, h] - \frac{\hbar \log\lambda}{32\pi^2} \int \{g_2^2 + g_2 g_4 \varphi^2 + \frac{g_4^2}{4}\varphi^4\} \\ &+ \frac{\hbar\log\lambda}{4\pi^2}\Big[\oint \{\frac{g_4}{8}\partial_n\varphi^2 + (\frac{g_2}{6} + \frac{g_4}{12}\varphi^2)\operatorname{tr}\chi\} - \frac{\pi^2\chi_E}{45} - \frac{f(\chi)}{70}\Big].\end{aligned} \tag{3.16}$$

As in 2 dimensions the 1-loop contributions can be absorbed in the classical action (2.28) if the constants in the potentials (2.29) and (3.10) are rescaled. Hence the effective action is scale invariant,

$$\Gamma^{(1)}[\lambda\mathcal{M}, \tilde{\varphi}, \tilde{g}, \tilde{h}] = \Gamma^{(1)}[\mathcal{M}, \varphi, g(\lambda), h(\lambda)], \tag{3.17}$$

provided the volume couplings run as

$$g_0(\lambda) = g_0 - \frac{\alpha}{2} g_2^2, \quad g_2(\lambda) = g_2\big(1 - \alpha g_4\big), \quad g_4(\lambda) = g_4\big(1 - 3\alpha g_4\big) \tag{3.18}$$

and the surface constants as

$$h_2(\lambda) = h_2 + \frac{\alpha}{3} g_2, \quad h_6(\lambda) = h_6 + \frac{\alpha}{3} g_4, \quad h_7(\lambda) = h_7 + \frac{\alpha}{2} g_4, \tag{3.19}$$

where we have introduced $\alpha = \hbar \log\lambda / 16\pi^2$. The constant h_5 does not scale and the remaining geometrical constants h_3, h_4 scale as in (2.30). To derive (3.17) we replaced the couplings g in the 1-loop contribution to the effective action by the scaled ones $g(\lambda) = g + O(\hbar)$. Since this changes Γ only in order $O(\hbar^2)$ this does not affect the one loop result.

To compare our result with the momentum space renormalization [25] we identify the inverse length scale $1/\lambda$ of space time with the energy scale μ as in (3.5). This immediately yields the running coupling constants in momentum space and the corresponding 1-loop β-functions for the volume coefficients

$$\beta^{(1)}(g_0) = \frac{\hbar}{32\pi^2} g_2^2, \quad \beta^{(1)}(g_2) = \frac{\hbar}{16\pi^2} g_2 g_4, \quad \beta^{(1)}(g_4) = \frac{3\hbar}{16\pi^2} g_4^2, \tag{3.20}$$

and for the surface coefficients

$$\beta^{(1)}(h_2) = -\frac{\hbar}{48\pi^2} g_2, \quad \beta^{(1)}(h_6) = -\frac{\hbar}{48\pi^2} g_4, \quad \beta^{(1)}(h_7) = -\frac{\hbar}{32\pi^2} g_4^2. \tag{3.21}$$

The β-functions for the mass and quartic coupling coincide with the ones calculated with the more commonly used Green's function method in momentum space [25]. The running of the cosmological constant g_0 is usually not considered in the literature, since one requires the normalization condition $\mathcal{W}(j=0)=0$ for the Schwinger functional. This condition removes a cosmological constant and terms containing the Casimir effect. Also, surface terms are not present on the whole Minkowski (Euclidean) spacetime so that their scale dependence cannot be studied in the conventional perturbation expansion.

Note that the 1-loop corrections do not lead to a wave function renormalization in 4-dimensional one-component ϕ^4 theories. Again this agrees with the more widely used dimensional regularization.

6 dimensions.
We consider the renormalizable ϕ^3-theory with general potential

$$V(\phi) = g_0 + g_1\phi + \frac{g_2}{2!}\phi^2 + \frac{g_3}{3!}\phi^3. \tag{3.22}$$

The surface contributions to the coefficient a_3, which enters the scaling law in 6 dimensions, has not been calculated yet. For that reason we focuse on the scaling of the volume couplings. For the cubic potential the general formula (2.20) with $\zeta(0)$ from (2.23) yields the following scaling law for the effective action:

$$\begin{aligned} \Gamma^{(1)}[\lambda\mathcal{M},\tilde{\varphi},\tilde{g}] &= S[\mathcal{M},\varphi,g] + \frac{\hbar}{2}\log\det M(\varphi,g) \\ &+\frac{\hbar\log\lambda}{3!(4\pi)^3}\int\Big\{\frac{1}{2}g_3^2\varphi\Delta\varphi + g_2^3 + 3g_2^2 g_3\varphi + 3g_2 g_3^2\varphi^2 + g_3^3\varphi^3\Big\}. \end{aligned} \tag{3.23}$$

Now the 1-loop corrections contain a derivative term of the same form as in the classical action. When we try to absorb it in the classical action we change the coefficient $\frac{1}{2}$ of $(\partial_\mu\phi)^2$. To restore it we must renormalize the field. This multiplicative renormalization of φ further rescales the coupling constants. The point is that nevertheless we can restore the invariance of the effecive action

$$\Gamma^{(1)}[\lambda\mathcal{M},\tilde{\varphi},\tilde{g}] = \Gamma^{(1)}[\mathcal{M},\sqrt{Z_3}\varphi,g(\lambda)], \tag{3.24}$$

where the explicit wave function renormalization

$$Z_3 = 1 - \frac{\alpha}{3!}\, g_3^2, \quad \text{where} \quad \alpha = \frac{\hbar \log \lambda}{(4\pi)^3} \tag{3.25}$$

and the running of the coupling constants

$$\begin{aligned} g_0(\lambda) &= g_0 + \frac{\alpha}{3!} g_2^3 \quad , \quad g_1(\lambda) = g_1 \big[1 + \frac{\alpha}{2} g_3 (\frac{g_3}{3!} + \frac{g_2^2}{g_1})\big] \\ g_2(\lambda) &= g_2 \big[1 + \frac{7\alpha}{6} g_3^2\big] \quad , \quad g_3(\lambda) = g_3 \big[1 + \frac{5\alpha}{4} g_3^2\big] \end{aligned} \tag{3.26}$$

can be read off from (3.23). The corresponding β-functions read

$$\begin{aligned} \beta(g_0) &= -\frac{\hbar}{3!(4\pi)^3} g_2^2 \quad , \quad \beta(g_1) = -\frac{\hbar}{2(4\pi)^3} \big[g_2^2 + \frac{g_1 g_3}{3!}\big] \\ \beta(g_2) &= -\frac{7}{6} \frac{\hbar}{(4\pi)^3} g_2 g_3^2 \quad , \quad \beta(g_3) = -\frac{5}{4} \frac{\hbar}{(4\pi)^3} g_3^3. \end{aligned} \tag{3.27}$$

Contrary to the ϕ^4-coupling in 4 dimensions the ϕ^3 coupling in 6 dimensions gets stronger when $\mathcal{M}$ expands. Thus the theory is asymptotically free. The main difference to 2 and 4 dimensions is that here the wave function is affected by an anomalous scaling already in the 1-loop approximation. The anomalous dimension of the field is

$$\gamma(g) = \mu \frac{\partial}{\partial \mu} \log Z_3 = \frac{\hbar}{3!(4\pi)^3} g_3^2. \tag{3.28}$$

The anomalous trace of T.

To relate the scale anomaly to the trace of the energy momentum tensor it is convenient to couple the dynamical fields covariantly to an external gravitational field. Then both the classical and quantum mechanical energy momentum tensor can be derived by variation with respect to the metric as

$$T^{cl}_{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{\mu\nu}} \qquad \text{and} \qquad \langle T_{\mu\nu} \rangle = \frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g^{\mu\nu}}. \tag{3.29}$$

The so defined T^{cl} needs no further improvement [26]. For a theory containing only dimensionless coupling constants its trace vanishes automatically if the fields are conformally coupled to gravity.

If we scale the metric as $g_{\mu\nu} \to \lambda^2 g_{\mu\nu}$ the formula (3.29) reduces to

$$\frac{d\Gamma}{d\lambda}|_{\lambda=1} = -2 \int \frac{\delta \Gamma}{\delta g^{\mu\nu}}\, g^{\mu\nu} = -\int \sqrt{g} \langle T_{\mu\nu} \rangle\, g^{\mu\nu}. \tag{3.30}$$

Instead of scaling the metric with fixed coordinates we can scale the coordinates and leave the metric invariant. For diffeomorphism invariant Γ's both transformations must have the same effect. In other words, if we scale the coordinates according to (2.13), keep $g_{\mu\nu}=\delta_{\mu\nu}$ and do not scale the fields and coupling constants, then this variation is related to the trace of the energy momentum. This may now be exploited by differentiating the scaling formula

$$\Gamma[\lambda\mathcal{M},\lambda^{d_\varphi}\varphi,\lambda^{d_a}g_a,\lambda^{\tilde{d}_a}h_a]=\Gamma[\mathcal{M},Z_3^{\frac{1}{2}}\varphi,g_a(\lambda),h_a(\lambda)] \tag{3.31}$$

with respect to the scale parameter. Here we have inserted the naive scalings of the fields and volume couplings from (2.14). Similarly the $\tilde{d}_a$ are the length-dimensions of the running surface couplings h_a which one must introduce to guarantee (3.31).

The variation of the effective action due to the change $\mathcal{M}\to\lambda\mathcal{M}$ yields the integrated trace of $\langle T_{\mu\nu}\rangle$, so that we find

$$\int\langle T^\mu_{\ \mu}\rangle=\big[d_\varphi-\frac{dZ_3^{\frac{1}{2}}}{d\lambda}\big]\int\varphi\,\frac{\delta\Gamma}{\delta\varphi}+\sum_a\Big\{\big[d_ag_a-\frac{dg_a}{d\lambda}\big]\frac{\delta\Gamma}{\delta g_a}+\big[\tilde{d}_ah_a-\frac{dh_a}{d\lambda}\big]\frac{\delta\Gamma}{\delta h_a}\Big\}, \tag{3.32}$$

where the derivatives are evaluated at $\lambda=1$. Now we replace λ by the momentum scale μ and the derivatives of the wave function renormalization and coupling constants by the anomalous dimensions and beta-functions. We obtain

$$\int\langle T^\mu_{\ \mu}\rangle=\big[d_\varphi+\frac{1}{2}\gamma\big]\int\varphi\cdot j+\sum_a\Big\{\big[d_ag_a+\beta(g_a)\big]\frac{\delta\Gamma}{\delta g_a}+\big[\tilde{d}_ah_a+\beta(h_a)\big]\frac{\delta\Gamma}{\delta h_a}\Big\}, \tag{3.33}$$

where we have inserted $\delta\Gamma/\delta\varphi=j$.

Note that we only used the general relation (3.31) in deriving (3.33), which is a deep consequence of renormalizability and valid order by order in perturbation theory. *Therefore (3.33) is also valid order by order in perturbation theory and represents the general structure of the trace anomaly of the energy-momentum tensor.*

In *two dimensions* the first term vanishes if $\gamma=0$ and only the two sums contribute in the 1-loop approximation. However the d_a cannot vanish so that only free theories possess a traceless energy momentum tensor [3], up to surface terms. For free theories only the surface beta-function term constributes in (3.33). In the 1-loop approximation we may replace Γ on the right in (3.33) by S. Varying now the action (3.1) with all $g_a=0$ with respect to h and inserting the beta-function $\beta(h)$ from (3.6) we find

$$\int\langle T^\mu_{\ \mu}\rangle=N\frac{\hbar}{6}\chi_E \tag{3.34}$$

[3] Besides the free theories only models containing a Liouville mode possess an improved traceless tensor. But these models would need a separate discussion.

for N free fields in 2 dimensions. The finite volume regulates the theory in the infrared but at the same time introduces a length scale into the theory. This is the reason why quantum fluctuations lead to a non-zero trace and this trace is very much related to the Casimir effect.

In *higher dimensions* $d_\varphi \neq 0$ and the source must vanish for the trace to be zero. This is to be expected since already the classical improved tensor has trace zero only on shell and the classical on shell condition is exactly the condition $j=0$. On shell the trace is given by the two sums in (3.33) which contain classical and anomalous contributions. For the classically scale invariant theories

$$S[\phi,g,h] = \int \Big\{\frac{1}{2}(\partial_\mu\phi)^2 + \frac{g}{a!}\phi^a\Big\} + \oint Q(\chi_{ab},\phi), \quad a = \frac{2d}{d-2} \tag{3.35}$$

with a surface potential containing only dimensionless couplings, the $O(1)$ terms in (3.33) vanish and on shell only the anomalous part remains in the trace. To $O(\hbar)$ we can replace Γ by the classical action to compute this trace. For the theories (3.35) the volume potential and action are related on shell as

$$S = b\int V + \frac{1}{2}\oint \phi\partial_n\phi + \oint Q, \qquad b = \frac{2}{2-d}, \tag{3.36}$$

so that (3.33) can be written as

$$\int \langle T^\mu_\mu\rangle = \frac{\beta(g)}{bg}S + \big[\sum \beta(h_i)\frac{\partial}{\partial h_i} - \frac{\beta(g)}{bg}\big]\oint Q - \frac{\beta(g)}{2bg}\oint \varphi\partial_n\varphi. \tag{3.35}$$

We see that the 1-loop anomalous trace is completely determined by the volume and surface beta functions and the scale invariant classical action. This formula holds for arbitrary scale invariant scalar theories in d dimensions. Let us now consider the 4 and 6-dimensional cases in turn.

In *4 dimensions* $b=-1$ in (3.35) and scale invariance requires that $h_1=h_2=h_5=0$ in the surface potential (2.29). Thus we find

$$\begin{aligned}\int \langle T^\mu_\mu\rangle = -\frac{\beta(g)}{g}S + \frac{\beta(g)}{g}\oint Q + \beta(h_3)\chi_E + \beta(h_4)\oint f(\chi)\\ + \beta(h_6)\frac{1}{2}\oint \varphi^2\,\mathrm{tr}\,\chi + \big[\frac{\beta(g)}{2g} + \beta(h_7)\big]\frac{1}{2}\oint \partial_n\varphi^2,\end{aligned} \tag{3.38}$$

with beta functions from (3.18,3.19) and the ones following from (2.30). When the volume $|\mathcal{M}|$ tends to infinity we may neglect the surface terms and we conclude that

$$\int \langle T^\mu_\mu\rangle = -\frac{3\hbar}{16\pi^2}g\,S(\varphi). \tag{3.39}$$

In *6 dimensions* $b=-\frac{1}{2}$ and the analogous result reads

$$\int \langle T^\mu_\mu \rangle = -2\frac{\beta(g)}{g} S(\varphi) = \frac{5}{2}\frac{\hbar}{(4\pi)^3} g_3^2\, S(\varphi), \tag{3.40}$$

where we inserted the beta-function from (3.27).

4. The scaling behaviour of gauge theories in the 1-loop approximation

In this section we consider the scaling behaviour of abelian and non-abelian gauge theories coupled to one fermion flavour enclosed in finite spacetimes $\mathcal{M}$ of dimension d. For fermions there are only two types of consistent boundary conditions, namely the non-local ones introduced by Atiyah, Patodi and Singer (APS) [27] and the local bag boundary conditions [28]. For both the surface Seeley-deWitt coefficients are not known in more than 2 dimensions. One can nevertheless find the scaling law for the fermionic determinant with respect to bag boundary conditions by indirect means, up to purely geometric terms. However, here we prefer to assume that $\mathcal{M}$ possesses no boundary, that is it may be a d-dimensional sphere, torus or some other compact spacetime without boundaries. We assume that $\mathcal{M}$ is imbedded in a flat space such that the scaling $\mathcal{M} \to \lambda\mathcal{M}$ makes sense. The price we pay for getting rid of the surface terms is that now $\mathcal{M}$ (if it is not a torus) is curved. Furthermore, the configuration space of fields becomes topologically non-trivial and the different topological sectors are characterized by the instanton numbers. Because of the index theorem there are fermionic zero modes and this leads to some technical subtleties.

We start with the classical action for the gauge fields and massless fermions [4] in a d-dimensional Euclidean manifold

$$S = S[A,g] \,+\, S[\psi,g],$$

where

$$S[A,g] = \frac{1}{4}\int \sqrt{g}\, G^{\mu\nu\, a} G^a_{\mu\nu} \quad , \quad S[\psi,g] = -\int \sqrt{g}\, \psi^\dagger i\gamma^\mu D_\mu \psi\,, \tag{4.1}$$

and $D_\mu = \partial_\mu + \omega_\mu + A_\mu$ is the covariant derivative. Here $\omega_\mu = \frac{i}{2}\omega_\mu^{AB}\,\Sigma_{AB}$ denotes the connection, $\gamma^\mu = e^\mu_A \gamma^A$ the Dirac matrices in curved spacetime, e^μ_A the vierbein related to the metric through $g^{\mu\nu} = e^\mu_A e^\nu_B \delta^{AB}$ and Σ_{AB} the generators of the quantum mechanical $SO(4)-$rotation. The gauge potential may be expanded as $A_\mu = -igA^a_\mu T^a$, where the

[4] Only for simplicity we assume the fermions to be massless. There is no major obstacle assuming the fermions to be massive.

$SU(N)-$ generators T^a obey the algebra $[T^a, T^b] = i\, f^{ab}_{\ \ c} T^c$ together with the normalization condition $\operatorname{tr} T_a T_b = \frac{1}{2}\delta_{ab}$. The Yang-Mills field strength and the curvature are defined by

$$
\begin{aligned}
[D_\mu, D_\nu] =& - ig\big(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu\big) + \big(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu]\big) \\
\equiv& -ig G^a_{\mu\nu} T_a + \frac{i}{2} R_{\mu\nu}^{\ \ AB} \Sigma_{AB}.
\end{aligned}
$$

Although we use the same symbol g for the gauge coupling constant and the determinant of the metric its actual meaning should be clear from the context.

As for the scalar theories we extract the running coupling constants from the change of the generating functional

$$
\mathcal{Z}[\mathcal{M}, j, \eta, g] = \sum_{q=-\infty}^{\infty} e^{iq\theta} \mathcal{Z}_q[\mathcal{M}, j, \eta, g], \tag{4.2}
$$

when $\mathcal{M}$ scales into $\lambda\mathcal{M}$. Here j is an external bosonic current which couples to the gauge field and η a Grassmann-valued source coupled to the Dirac fermions. $\mathcal{Z}$ is a sum over different topological sectors labelled by an integer q, the instanton number. For $\mathcal{Z}_q$ we have the formal path integral representation

$$
\begin{aligned}
\mathcal{Z}_q[\mathcal{M}, j, \eta, g] =& \frac{1}{\mathcal{N}} \int \mathcal{D}A^{(q)}_\mu \, \mathcal{D}\psi^\dagger \, \mathcal{D}\psi \\
& \cdot \exp\Big\{ -\frac{1}{\hbar} S + \frac{1}{\hbar} \int \sqrt{g}\, \big[j \cdot A + \psi^\dagger \eta + \eta^\dagger \psi \big] \Big\},
\end{aligned} \tag{4.3}
$$

where the integration in $\mathcal{Z}_q$ is restricted to gauge fields with fixed instanton number

$$
q = \frac{1}{n!} \frac{i^n}{(4\pi)^n} \int d^d x \, \sqrt{g}\, \epsilon^{\mu_1 \mu_2 \cdots \mu_{2n}} \operatorname{tr}\big[G_{\mu_1\mu_2} G_{\mu_3\mu_4} \cdot \ldots \cdot G_{\mu_{2n-1}\mu_{2n}} \big] \tag{4.4}
$$

in $d = 2n$ dimensions. Note that perturbation theory for $\mathcal{Z}_q$ is not yet applicable and we are forced to recast it in a gauge fixed form.

As in the previous case of scalar theories we evaluate $\mathcal{Z}_q$ semiclassically, i.e. up to one-loop-corrections by means of a steepest descent approximation. The extremum of the exponent is fixed by the classical equations of motion

$$
-\left(D_\mu[A_{cl}] G^{\mu\nu}[A_{cl}] \right)^a = j^{\nu a} \tag{4.5}
$$

and the condition that $A_{cl} \to A^0_{cl}$ for vanishing external current $j \to 0$. Here A^0_{cl} is an instanton solution with topological charge q. The fermionic fields remain infinitesimal fluctuations in our approximation.

We now expand the exponent about A_{cl} and retain only the terms quadratic in the fluctuations. Writing $A^{\mu a}=A^{\mu a}_{cl}+\sqrt{\hbar}B^{\mu a}$ we find for the exponent in (4.3)

$$\begin{aligned} S^{(1)}_{eff} =& \mathcal{W}^{cl}(j,g)+\int\sqrt{g}\,\psi^\dagger i\not{D}\,\psi+\int\sqrt{g}\,\left[\psi^\dagger\eta+\eta^\dagger\psi\right] \\ +\frac{\hbar}{2}&\int\sqrt{g}\;B^a_\mu\big[(D^2)^{\mu\nu\,ab}-(D^\mu D^\nu)^{ab}+2gf^{acb}G^{\mu\nu c}_{cl}-\delta^{ab}R^{\mu\nu}\big]B^b_\nu \end{aligned} \tag{4.6}$$

where

$$\mathcal{W}^{cl}(j,g)=-\frac{1}{4}\int\sqrt{g}\,G^{\mu\nu\,a}_{cl}G^a_{\mu\nu\,cl}+\int\sqrt{g}\,j\cdot A_{cl} \tag{4.7}$$

is the classical Schwinger functional. It is understood that $A_{cl}=A_{cl}(j)$ depends on the external current via the classical field equation (4.5). In these expressions D^μ contains the background field A^μ_{cl} and the connection ω^μ.

Now we are ready to evaluate the semiclassical functional $\mathcal{Z}^{(1)}_q$. We have in mind the computation of gauge invariant correlators in instanton backgrounds. The perturbative expansion about instantons is subtle due to the occurence of various zero-modes [29,30]. Therefore we shall discuss the problems due to zero-modes rather carefully.

For fixing the gauge we apply the wellknown Faddeev-Popov procedure [31]. In topologically non-trivial backgrounds the Faddeev-Popov operator possesses zero-modes. They are due to constant infinitesimal background gauge transformations

$$B_\mu\to B_\mu-[A_{cl\,\mu},\Theta]$$

which leave B invariant. If h denotes the little (or stable) algebra of the instanton, that is the subalgebra commuting with A_{cl}, then these are just the gauge transformations with $\Theta=-i\Theta^aT^a$, where the T^a lie in h. More explicitly, h is the maximal subalgebra which commutes with the $su(2)$-subalgebra defined by the instanton [30]. To eliminate these constant gauge transformations one inserts

$$1=\Delta[A_\mu]\int\mathcal{D}g\int\mathcal{D}h\,\delta\big(F[A_\mu]\big) \tag{4.8}$$

into the functional integral. Here $\int\mathcal{D}g$ denotes the measure on $SU(N)/H$ and $\int\mathcal{D}h$ the measure on the stability group H. Now one proceeds in the usual way and calculates

$$\Delta[A_\mu]=\det{}'\frac{\delta F}{\delta g}=\det{}'M_{gh}\,, \tag{4.9}$$

where the prime indicates that the zero modes of M_{gh} must be omitted. After absorbing all the volume-independent terms in the normalization of the Gaussian integral we find

$$\mathcal{Z}^{(1)}_q[\mathcal{M},j,\eta,g]=\frac{1}{\mathcal{N}^{(1)}}\int\mathcal{D}h\int\mathcal{D}B^{(q)}_\mu\,\mathcal{D}\psi^\dagger\,\mathcal{D}\psi\,\det{}'M_{gh}\,\delta\big(F[A_\mu]\big)\,e^{-\frac{1}{\hbar}S^{(1)}_{eff}}. \tag{4.10}$$

We cannot absorb $\int \mathcal{D}h$ in the normalization because it exhibits an explicit dependence on the volume of $\mathcal{M}$ as discussed below.

At this point we choose the background gauge $F^a(A_{cl},B) = (D^\mu B_\mu)^a$ such that that the middle term in the second line in (4.6) vanishes. The corresponding Faddeev-Popov operator is

$$M_{gh}(A_{cl},g)^{ab} = \left(-D^2\right)^{ab}, \tag{4.11a}$$

where D^2 is a matrix in the adjoint representation. The remaining quadratic operators acting on the bosonic and fermionic fluctuations are

$$M_B(A_{cl},g)^{ab}_{\mu\nu} = \left(-D^2\right)^{ab}_{\mu\nu} - 2gf^{acb}G^c_{\mu\nu cl} + \delta^{ab}R_{\mu\nu} \tag{4.11b}$$

and

$$M_\psi(A_{cl},g)^{ab}_{kl} = \left(-D^2\right)^{ab}_{kl} + \frac{ig}{2}G^{\mu\nu}_{cl\,m}T^{ab}_m(\gamma_\mu\gamma_\nu)_{kl} + \frac{1}{2}(R^{\mu\nu}\Sigma_{\mu\nu})_{kl}, \tag{4.11c}$$

where $R_{\mu\nu} = R^{AB}_{\mu\nu}\Sigma_{AB}$. For later purposes we displayed all the relevant indices: a,b,c belong to the gauge algebra; i,k are Dirac and μ,ν Lorentz indices. For notational simplicity we shall skip the indices in what follows. Note that A_{cl} plays the same role as ϕ_{cl} in the scalar theories.

From the discussion of the gauge fixing we conclude that the fluctuation operators may possess zero modes. As far as the pure gauge sector is concerned, that is the fluctuation operators M_B and M_{gh}, this problem has been lucidly discussed in [30]. Here we give only the results. One obtains for $\int \mathcal{D}h$

$$\int \mathcal{D}h = [\frac{\sqrt{2\pi}g}{|\mathcal{M}|^{\frac{1}{2}}}]^{d_H}\frac{1}{V_H}, \tag{4.12}$$

where d_H and V_H are the dimension and volume of the stability group H, respectively. The fluctuation operator M_B may possess additional p zero modes arising from the variation of the collective parameters $\{\gamma_r\}$. Expanding the fluctuations B_μ in terms of eigenfunctions of M_B one may convert the integration over the expansion parameters $\{\alpha_r\}; r = 1,..,p$ belonging to the zero modes to an integration over $\{\gamma_r\}$

$$\prod_1^p d\alpha_r = \prod_1^p d\gamma_r\,(\det J)^{\frac{1}{2}}, \tag{4.13}$$

where J denotes the corresponding Jacobian.

Finally since in the sector with instanton charge q the Dirac operator $i\not{D}$ and hence M_ψ has $|q|$ zero modes of definite chirality, one must be cautious in evaluating the fermionic path integral [32]. Let $\psi_n(x)$ denote the orthonormal zero modes of $i\not{D}$, $G'(x,y)$ the 'excited' Green's function belonging to $i\not{D}$ and $\det'^{\frac{1}{2}}M_\psi$ the fermionic determinant with zero eigenvalues excluded.

With all the notations fixed we finally obtain for the one-loop functional

$$\mathcal{Z}_q^{(1)}[\mathcal{M},j,\eta,g] = e^{\frac{1}{\hbar}\mathcal{W}^{cl}(\mathcal{M},j,g)}\,[\frac{\sqrt{2\pi}g}{|\mathcal{M}|^{\frac{1}{2}}}]^{d_H}\,\frac{1}{V_H}\int\prod_1^p d\gamma_r\,(\det J)^{\frac{1}{2}}$$
$$\cdot{\det}'^{-\frac{1}{2}} M_B(A_{cl},g)\,{\det}' M_{gh}(A_{cl},g)\,{\det}'^{\frac{1}{2}} M_\psi(A_{cl},g) \tag{4.14}$$
$$\cdot\prod_n(\eta^\dagger,\psi_n)(\psi_n^\dagger,\eta)\,e^{-\int\sqrt{g}\,\eta^\dagger G'\eta}.$$

The product of divergent determinants will be gauge invariantly regularized by the $\zeta-$function method. This regularization seems to be the most convenient one when dealing with different topological sectors [32].

Let us now consider the rescaled theory and construct its generating functional $\mathcal{Z}[\lambda\mathcal{M},\tilde{j},\tilde{\eta},\tilde{g}]$. Under a scale transformation (2.13) the fields transform as

$$\begin{aligned} \tilde{A}_{cl}(\tilde{x}) &= \lambda^{\frac{1}{2}(2-d)}A_{cl}(x) & \qquad \tilde{\psi}(\tilde{x}) &= \lambda^{\frac{1}{2}(1-d)}\psi(x) \\ \tilde{j}(\tilde{x}) &= \lambda^{-\frac{1}{2}(2+d)}j(x) & \qquad \tilde{\eta}(\tilde{x}) &= \lambda^{-\frac{1}{2}(1+d)}\eta(x), \end{aligned} \tag{4.15a}$$

and the gauge coupling constant as

$$\tilde{g} = \lambda^{\frac{1}{2}(d-4)}\,g. \tag{4.15b}$$

Note that the metric $g^{\mu\nu}$ is not scaled so that the connection ω_μ transforms like a derivative ∂_μ. The classical action and hence $\mathcal{W}^{cl}$ are both invariant under the scale transformations (2.13), (4.15). However, only in 4 dimensions is the gauge coupling dimensionsless and thus the energy-momentum tensor traceless (see 3.33). The topological charge (4.4) is scale invariant. Note that if A_{cl} solves (4.5) with coupling g and current j then $\tilde{A}_{cl}$ is a solution with rescaled current and coupling constant.

As in the scalar case the classical scale invariance is broken by quantum corrections . To see that more explicitly we establish the connection between the generating functional on $\lambda\mathcal{M}$

$$\mathcal{Z}_q^{(1)}[\lambda\mathcal{M},\tilde{j},\tilde{\eta},\tilde{g}] = e^{\frac{1}{\hbar}\mathcal{W}^{cl}(\lambda\mathcal{M},\tilde{j},\tilde{g})}\,[\frac{2\pi\tilde{g}^2}{|\tilde{\mathcal{M}}|}]^{d_H/2}\frac{1}{V_H}\int\prod_1^p d\tilde{\gamma}_r\,(\det J)^{\frac{1}{2}}$$
$$\cdot{\det}'^{-\frac{1}{2}} M_B(\tilde{A}_{cl},\tilde{g})\,{\det}' M_{gh}(\tilde{A}_{cl},\tilde{g})\,{\det}'^{\frac{1}{2}} M_\psi(\tilde{A}_{cl},\tilde{g}) \tag{4.16}$$
$$\cdot\prod_n(\tilde{\eta}^\dagger,\tilde{\psi}_n)(\tilde{\psi}_n^\dagger,\tilde{\eta})\,e^{-\int\sqrt{g}\,\tilde{\eta}^\dagger\tilde{G}'\tilde{\eta}}$$

and the one on $\mathcal{M}$ (4.14). As for scalars (see above (2.19)) we have $\zeta_{\tilde{M}}(s) = \lambda^{2s}\zeta_M(s)$ for all fluctuation operators. Hence the product of the rescaled primed determinants can be related to the unscaled ones as in (2.19) and one finds

$$\log\frac{{\det}'^{-\frac{1}{2}}\tilde{M}_B\,{\det}'\tilde{M}_{gh}\,{\det}'^{\frac{1}{2}}\tilde{M}_\psi}{{\det}'^{-\frac{1}{2}}M_B\,{\det}'M_{gh}\,{\det}'^{\frac{1}{2}}M_\psi} = -\log\lambda\cdot\left\{2\zeta_{M_{gh}}+\zeta_{M_\psi}-\zeta_{M_B}\right\}_{s=0}. \tag{4.17}$$

Here we used the abbreviation $\tilde{M} = M(\tilde{A}_{cl}, \tilde{g})$ for the fluctuation operators on $\lambda\mathcal{M}$. Note that the ζ-functions belonging to the bosonic and fermionic fluctuation operators contribute with different signs in the last bracket.

Before calculating the ζ-functions in (4.17) we discuss the scaling behaviour of the other terms in $\mathcal{Z}_q^{(1)}$. All the *normalized* zero-modes of the operators M_B, M_{gh} and M_ψ scale as

$$\tilde{\varphi}(\tilde{x}) = \lambda^{-\frac{1}{2}d}\varphi(x),$$

that is transform differently as the fluctuation fields in the functional integral. Hence

$$\frac{\tilde{g}^{d_H}}{|\tilde{\mathcal{M}}|^{d_H/2}} = \frac{\lambda^{-2d_H} g^{d_H}}{|\mathcal{M}|^{d_H/2}}, \qquad \prod_1^p d\tilde{\gamma}_r = \lambda^p \prod_1^p d\gamma_r, \qquad (\tilde{\eta}^\dagger, \tilde{\psi}_n) = \lambda^{-\frac{1}{2}}(\eta^\dagger, \psi_n).$$

Altogether the different zero-modes contribute a factor $\lambda^{p-2d_H-|q|}$ in the scaling of $\mathcal{Z}_q^{(1)}$. The fermionic Green's function scales as

$$\tilde{G}'(\tilde{x}, \tilde{y}) = \lambda^{1-d} G'(x, y)$$

so that $\int \eta^\dagger G' \eta$ is scale invariant. Thus we find the following scaling law

$$\mathcal{Z}_q^{(1)}[\lambda\mathcal{M}, \tilde{j}, \tilde{\eta}, \tilde{g}] = \lambda^{p-2d_H-|q|} \mathcal{Z}_q^{(1)}[\mathcal{M}, j, \eta, g] \cdot e^{-\log\lambda \cdot \left\{2\zeta_{M_{gh}} + \zeta_{M_\psi} - \zeta_{M_B}\right\}_{|s=0}}. \tag{4.18}$$

In deriving (4.18) we have used the scale invariance of the classical Schwinger functional.

Let us now explicitly evaluate the ζ-functions at $s = 0$ for the various fluctuation operators. Thereby we must be careful to project out the zero modes of the operators M in the various $\zeta-$functions in order to finally get the desired primed determinants [19,32]. We get

$$\zeta_M(0) = \lim_{s\to 0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1}\, \mathrm{Tr}\, e^{-tM}(1-P), \tag{4.19}$$

where $P(x, y) = \sum_n \varphi_n(x)\varphi_n^\dagger(y)$ is the projector onto the zero modes φ_n of M. One sees at once that

$$\zeta_{M_B(g)}(0) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int d^d x \sqrt{g}\ \mathrm{tr}\, a^B_{\frac{d}{2}}(x; g) \,-\, p\,, \tag{4.20a}$$

$$\zeta_{M_{gh}(g)}(0) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int d^d x \sqrt{g}\ \mathrm{tr}\, a^{gh}_{\frac{d}{2}}(x; g) \,-\, d_H \tag{4.20b}$$

and

$$\zeta_{M_\psi(g)}(0) = \frac{1}{(4\pi)^{\frac{d}{2}}} \int d^d x \sqrt{g}\ \mathrm{tr}\, a^{\psi}_{\frac{d}{2}}(x; g) \,-\, |q| \tag{4.20c}$$

instead of (2.22). Here tr denotes the trace over gauge, Dirac and Lorentz indices. The constants p, d_H and $|q|$ lead to a factor $\lambda^{-p+2d_H+|q|}$ in (4.18), which exactly chancels the factor $\lambda^{p-2d_H-|q|}$ coming from the zero-modes. Thus we obtain the following scaling law for the Schwinger functional

$$\begin{aligned}\mathcal{W}_q^{(1)}[\lambda\mathcal{M},\tilde{j},\tilde{\eta},\tilde{g}] &= \mathcal{W}_q^{(1)}[\mathcal{M},j,\eta,g]\\ &+\frac{\hbar\log\lambda}{(4\pi)^{\frac{d}{2}}}\int\sqrt{g}\,\mathrm{tr}\,\big\{a^B_{\frac{d}{2}}(x,g)-2a^{gh}_{\frac{d}{2}}(x,g)-a^{\psi}_{\frac{d}{2}}(x,g)\big\}.\end{aligned}\tag{4.21}$$

This scaling law should be compared with the corresponding one for scalar fields (2.20,2.22). There the surface of $\mathcal{M}$ affects the scaling of the Schwinger functional whereas here the topology of the classical gauge field configurations and fermionic zero modes do not. *The scaling is the same in all instanton sectors.*

The Seeley-de-Witt coefficient $a_n(x;g)$ for gauge covariant operators of the form

$$M=-D^2+C(x),\quad \text{where}\quad D_\mu=\partial_\mu+A_{cl\,\mu}+\omega_\mu,\tag{4.22}$$

where $C(x)$ denotes a general field of hermitean matrices, have been calculated for $n\le 5$ and those for $n\le 3$ are given in the appendix. We evaluate now the gauge field, fermionic and ghost contributions in 2 and 4 dimensions in turn.

5. The running coupling constant in four dimensions in leading log approximation

Before dealing with the realistic 4-dimensional case we comment on the quite trivial situation in *two dimensions*, that is on the scaling behaviour of the Schwinger model. In 2 dimensions the (would be) anomalous contributions to the effective action are given by the coefficients a_1 of the various fluctuation operators, and hence by the trace of C. For the ghosts $C=0$, for the gauge fields the Lorentz-trace of C vanishes and for the fermions its Dirac trace is zero. Thus we have

$$\mathcal{Z}_q^{(1)}[\lambda\mathcal{M},\tilde{j},\tilde{\eta},\tilde{g}]=\mathcal{Z}_q^{(1)}[\mathcal{M},j,\eta,g]\tag{5.1}$$

in 2 spacetime dimensions. We see that the 1-loop generating functional is invariant with respect to the naive scalings of the coupling constants and fields. A detailed computation shows that even the full generating functional of the Schwinger model is in the naive sense scale invariant. Of course this follows from the fact that in QED_2 the gauge coupling and fields are not renormalized.

Now we apply the results of the previous section to QCD, or more generally to *four dimensional nonabelian gauge theories* with gauge group $SU(N)$. In four dimensions the gauge coupling is dimensionless and its scale-dependence is a quantum effect. Let us collect the different contributions to the anomalous scaling (see Appendix A). We thereby omit all the purely geometric terms noting already here that there is no mixing between them and the gauge contributions. These geometric terms lead to running cosmological and gravitational constants. In addition, the dimensionless constants associated with dimension four operators in the gravitation field, e.g R^2, D^2R, become scale dependent [11]. In the following the integrations are performed with respect to the invariant measure.

Gauge fields.

For the gauge field fluctuations the C-field in (4.16) is $C_B = -2G + R$ and calculating the Lorentz trace in (A2) yields

$$\mathrm{tr}_L a_2(x;g) = 4\frac{1}{12}G^{\mu\nu}G_{\mu\nu} \; - \; 4\frac{1}{2}G^{\mu\nu}G_{\mu\nu} + \ldots = -\frac{5}{3}G^{\mu\nu}G_{\mu\nu} + \ldots,$$

where we have used that $\mathrm{tr}_L C_B = 0$. Thus we get

$$\int \sqrt{g}\ \mathrm{tr}\, a_2(x;g) = -\frac{5}{3}\int \sqrt{g}\ \mathrm{tr}_A G^{\mu\nu}G_{\mu\nu} + \ldots, \tag{5.2a}$$

where the trace has to be taken in the adjoint representation, denoted by A.

Ghosts.

For the ghosts $C_{gh} = 0$ and

$$\int \sqrt{g}\ \mathrm{tr}\, a_2(x;g) = \frac{1}{12}\int \sqrt{g}\ \mathrm{tr}_A G^{\mu\nu}G_{\mu\nu} + \ldots \tag{5.2b}$$

Fermions.

For Dirac fermions $C_\psi = -\frac{1}{2}G^{\mu\nu}\gamma_\mu\gamma_\nu - \frac{1}{2}R^{\mu\nu}\gamma_\mu\gamma_\nu$ and the Dirac trace yields

$$\mathrm{tr}_D a_2(x;g) = 4\frac{1}{12}G^{\mu\nu}G_{\mu\nu} \; - \; 2\frac{1}{2}G^{\mu\nu}G_{\mu\nu} + \ldots = -\frac{2}{3}G^{\mu\nu}G_{\mu\nu} + \ldots,$$

where have we used that $\mathrm{tr}_D C_\psi = 0$. Thus we remain with

$$\int \sqrt{g}\ \mathrm{tr}\, a_2(x;g) = -\frac{2}{3}\int \sqrt{g}\ \mathrm{tr}_F G^{\mu\nu}G_{\mu\nu} + \ldots, \tag{5.2c}$$

where the fermions transform according to some representation denoted by F. Let us define the second order Casimir T_R of a representation R as $\mathrm{tr}\, T^aT^b = \delta^{ab}T_R$. Above, $\ldots$ stand for all the purely geometrical contributions which we omited. Collecting the results we find the anomalous term in (4.15) to be

$$\frac{g^2}{16\pi^2}\, X \,\log\lambda \int \sqrt{g}\, G^{\mu\nu\, a}G^a_{\mu\nu}\,, \quad \text{where} \quad X = \frac{11}{6}T_A - \frac{2}{3}T_F. \tag{5.3}$$

At this stage of the calculation it is again convenient to introduce the effective action, that is the (partial) Legendre transform of the Schwinger function,

$$\Gamma_q^{(1)}[\mathcal{M}, a, \eta, g] = \int \sqrt{g}\, j \cdot a - \mathcal{W}_q^{(1)}[\mathcal{M}, j, \eta, g], \tag{5.4}$$

where the mean field a and the current are related by $a = \delta\mathcal{W}/\delta j$. By using the same arguments as for scalar fields one finds from (4.15) the following scaling law for the effective action

$$\Gamma_q^{(1)}[\lambda\mathcal{M}, \tilde{a}, \tilde{\eta}, g] = \frac{1}{4}(1 - \frac{\hbar \log\lambda}{4\pi^2} X g^2) \int d^4x \sqrt{g} G^{\mu\nu\, a}(a) G^a_{\mu\nu}(a) + \dots, \tag{5.5}$$

where we inserted (5.2) for the anomalous term in (4.15) and the classical field has been replaced by the mean field a. Here $\dots$ stand for those fluctuation terms, which we do not need at this stage. In order to read off the scaling of the field and coupling constant from (5.5) we write the integral in (5.5) in terms of the mean field

$$\frac{1}{4}(1 - \frac{\log\lambda}{4\pi^2} X\, g^2) \int \sqrt{g}\, G^{\mu\nu\, a} G^a_{\mu\nu} = \frac{1}{4}(1 - \frac{\log\lambda}{4\pi^2} X\, g^2) \int \sqrt{g}$$
$$\left\{ a^a_\mu (-\partial^2 \delta^{\mu\nu} + \partial^\mu \partial^\nu) a^a_\nu - 4g f^{abc} \partial^\mu a^{\nu a} \cdot a^b_\mu a^c_\nu + g^2 f^{abc} f^{ade} a^b_\mu a^{\mu d} a^c_\nu a^{\nu e} \right\}.$$

Now it is clear that the wave function has to be scaled, and the scale factor Z_3 is given by the coefficient multiplying the second derivative term above. To restore the invariance of the effective action we need to rescale the field as

$$a_\mu \longrightarrow \sqrt{Z_3} a_\mu, \quad \text{where} \quad Z_3 = 1 - \frac{\hbar \log\lambda}{4\pi^2} X\, g^2. \tag{5.7}$$

We reexpress Γ in terms of the scaled field and the running coupling constant

$$g^2(\lambda) = \frac{g^2}{Z_3} = g^2 \frac{1}{1 - \frac{\log\lambda}{4\pi^2} X\, g^2} \tag{5.8}$$

and obtain this way the invariance of the effective action

$$\Gamma_q^{(1)}[\lambda\mathcal{M}, \tilde{a}, \tilde{\eta}, g] = \Gamma_q^{(1)}[\mathcal{M}, \sqrt{Z_3} a, \eta, g(\lambda)]. \tag{5.9}$$

Again we have used the fact that in all 1-loop contributions to the effective action (the dots in (5.5)) g and a can be replaced by the scaled coupling constant and field without affecting the 1-loop scaling result.

Note that $1/g$ and a scale the same way as required in a theory having one coupling constant in different interaction terms. The result (5.9) shows that we can restore scale invariance if we supplement the naive classical scale transformations with the above anomalous ones. Also note that the coupling g *runs the same way in all the topological sectors.*

This result is usually obtained using the backgroundfield method [33,34 and references therein]. In the 1-loop approximation considered here there is a simple relation between $\Gamma_q^{(1)}[a]$ and the effective action $\Gamma_{q,bg}^{(1)}[a_{bg}, B]$ in the presence of the background field B evaluated with the usual Feynman graph technique. One finds $\Gamma_q^{(1)}[a] = \Gamma_{q,bg}^{(1)}[a_{bg}=0, B=a]$ so that in our calculation the mean field a simply replaces the field B introduced in the backgroundfield method to restore manifest gauge invariance.

To compare with the momentum space renormalization one expresses (5.7,8) in terms of μ to find the leading log expression for the β-function [25]

$$\beta^{(1)}(g) = \mu \frac{\partial}{\partial \mu} g(\mu) = -\frac{\hbar X}{8\pi^2}\, g^3(\mu) \tag{5.10}$$

and the anomalous dimension of the gauge fields

$$\gamma_A^{(1)}(g) = \mu \frac{\partial}{\partial \mu} \log Z_3 = \frac{\hbar X}{4\pi^2}\, g^2(\mu). \tag{5.11}$$

Concering the trace of the symmetric energy-momentum tensor $T_{\mu\nu}$ for gauge theories the same arguments as those leading to (3.18) lead now to the anomlous divergence of the energy momentum tensor

$$\int \langle T^\mu_{\ \mu} \rangle = \frac{2\beta^{(1)}(g)}{g(\mu)}\, S, \tag{5.14}$$

a result originally due to Collins, Duncan and Joglecar [35].

6. Conclusions

In this paper we have studied the 1-loop anomalous scaling laws for the effective actions of scalar and gauge theories on finite spacetimes in various dimensions. We showed *explicitly* that if we allow the coupling constants and fields to scale differently as suggested by dimensional analysis we can get rid of the scale dependence of the effective action Γ. The anomalous scaling of the couplings and fields is just the one belonging to the 1-loop beta functions and anomalous dimensions if the relative size λ of spacetime is related to the energy scale as $\mu = 1/\lambda$. However, if spacetime possesses a boundary we need to add surface terms to the classical action and the surface coupling constants must scale as well for Γ to stay invariant. From the invariance of Γ we derived a general formula for the (integrated) trace of the energy momentum tensor for space-times with boundaries.

In our explicit calculations we have included the 1-loop corrections to the classical results. It would be interesting to see how effectively one can derive the β-functions and anomalous dimensions from finite size effects in higher orders. Finite size calculations are infrared finite and already that could be a good reason for extending our methods beyond 1-loop. Since the ζ-function method naturally extends to the operator regularization which is applicable to higher order calculations [36] this should indeed be possible.

As already pointed out in the introduction, our investigation of field theories on space-times with boundaries have implication for the Casimir effect. To see that more clearly assume that space-time is a cylinder, $\mathcal{M} = [0,\beta]\times\mathcal{S}$. Then $-\beta\mathcal{W}[\beta,\mathcal{S},0,g]$ is the free energy at temperature $T=1/\beta$ and

$$E_0(\mathcal{S},g) = \lim_{\beta\to\infty}\frac{1}{\beta}\ \inf_{\varphi}\Gamma[\beta,\mathcal{S},\varphi,g]$$

the ground state energy of the system with renormalized couplings g and confined in the space region $\mathcal{S}$. From the scale invariance of Γ one derives then the following scaling law for the ground state energy

$$E_0[\lambda\mathcal{S},\tilde{g}] = \frac{1}{\lambda}E_0[\mathcal{S},g(\lambda)]. \tag{6.1}$$

In particular, if the system is confined between two infinite and parallel plates, then the energy per surface element scales as

$$\frac{1}{A}E_0[\lambda\mathcal{S},\tilde{g}] = \frac{1}{\lambda^3}\frac{1}{A}E_0[\mathcal{S},g(\lambda)]. \tag{6.2}$$

For a classically scale invariant theory this implies that the Casimir energy

$$E_{Cas} \equiv E_0[\lambda\mathcal{S},g] - E_0[\mathcal{S},g] \tag{6.4}$$

can be recast into

$$E_{Cas} = \frac{1}{\lambda}E_0[\mathcal{S},g(\lambda)] - E_0[\mathcal{S},g]. \tag{6.5}$$

This identity has the following interpretation: when we enclose a quantum system into a space $\mathcal{S}$ and (in a Gedanken experiment) change the renormalized coupling constants according to the renormalization flow, $g\to g(\lambda)$, then E_0 changes by the same amount as when we leave g fixed and move the walls so that $\mathcal{S}\to\lambda\mathcal{S}$. This statement holds up to the trivial $1/\lambda$ factor in (6.5).

Let us finally comment on a apparently different problem which has been our motivation for studying the finite size scaling of the generating functionals, namely the chiral symmetry breaking in QCD. When one imposes the one-parameter chirality breaking bag boundary conditions on the fermions as in [28] then the parameter θ appearing in the boundary conditions can be interpreted as θ-parameter in QCD [37]. Furthermore, it can be argued that the chiral condensate does not vanish for small volumes. This may not come as a surprise since the boundary conditions explicitly break the axial $SU(N)$. We have seen that even in the presence of boundaries the scaling of correlators is governed by the renormalization group coefficients. Now the task would be to derive bounds on the QCD-beta-function and anomalous dimensions such that the condensate survives when the volume increases and one leaves the perturbative, small volume regime and enters the non-perturbative large volume sector.

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Appendix A

In this appendix we list the Seeley-deWitt coefficients a_n and b_n for operators of the form

$$M = -D^2 + C(x), \quad \text{where} \quad D_\mu = \partial_\mu + A_\mu + \omega_\mu . \tag{A1}$$

The relevant volume coefficients a_n are [11,18,19,32]

$$\begin{aligned}
\int \sqrt{g}\, a_0(x) =& \ \mathrm{Tr}\,|\mathcal{M}| \\
\int \sqrt{g}\, a_1(x) =& \int \sqrt{g}\ \mathrm{tr}\,\big[\frac{1}{6}R - C\big] \\
\int \sqrt{g}\, a_2(x) =& \ \frac{1}{2!}\int \sqrt{g}\ \mathrm{tr}\,\big[\frac{1}{15}D^2R + \frac{1}{36}R^2 - \frac{1}{90}R^{\mu\nu}R_{\mu\nu} + \frac{1}{90}R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\
&\quad + \frac{1}{6}G^{\mu\nu}G_{\mu\nu} - \frac{1}{3}D^2C - \frac{1}{3}RC + C^2\big] \\
\int \sqrt{g}\, a_3(x) =& \ \frac{1}{3!}\int \sqrt{g}\ \mathrm{tr}\,\big[- C^3 - \frac{1}{2}CD^2C - \frac{1}{2}CG^{\mu\nu}G_{\mu\nu} \\
&+ \frac{1}{10}(D_\mu G^{\mu\nu})(D_\rho G^\rho_\nu) - \frac{1}{15}G_{\mu\nu}G^{\nu\rho}G_{\rho\mu}\big] + \text{geom.terms}\,.
\end{aligned} \tag{A2}$$

For Dirichlet boundary conditions and $g_{\mu\nu} = \delta_{\mu\nu}$ the surface coefficients b_n are given by

[6,21]

$$
\begin{aligned}
\int b_{\frac{1}{2}}(x) &= -\frac{\sqrt{\pi}}{2}\,\mathrm{Tr}\,|\partial\mathcal{M}| \\
\int b_1(x) &= \frac{1}{3}\oint\ \mathrm{Tr}\ \mathrm{tr}\,\chi = \frac{2\pi}{3}\chi_E \\
\int b_{\frac{3}{2}}(x) &= -\frac{\sqrt{\pi}}{192}\oint\ \mathrm{Tr}\,\big[-96C + 7(\,\mathrm{tr}\,\chi)^2 - 10\,\mathrm{tr}\,\chi^2\big] \\
\int b_2(x) &= \frac{1}{945}\oint\ \mathrm{Tr}\,\big[+40\,\mathrm{tr}\,\chi^3 - 33\,\mathrm{tr}\,\chi\,\mathrm{tr}\,(\chi)^2 + 5(\,\mathrm{tr}\,\chi)^3\big] \\
&\qquad -\frac{1}{3}\oint\ \mathrm{Tr}\,\big[C\,\mathrm{tr}\,\chi - \partial_n C\big] \\
&= (4\pi)^2\big[\frac{1}{180}\chi_E + \frac{1}{280\pi^2}\oint f(\chi)\big] \\
&\qquad -\frac{1}{3}\oint\ \mathrm{Tr}\,\big[C\,\mathrm{tr}\,\chi - \partial_n C\big],
\end{aligned}
\tag{A3}
$$

where we have introduced the topological Euler number for 3-surfaces imbedded in flat spaces,

$$
\chi_E = -\frac{1}{12\pi^2}\oint\big[2\,\mathrm{Tr}\,\chi^3 - 3\,\mathrm{Tr}\,\chi\,\mathrm{Tr}\,\chi^2 + (\,\mathrm{Tr}\,\chi)^3\big] \tag{A4a}
$$

and the conformally invariant third-order-polynomial [21]

$$
f(\chi) = \,\mathrm{Tr}\,\chi^3 - \,\mathrm{Tr}\,\chi\,\mathrm{Tr}\,\chi^2 + \frac{2}{9}(\,\mathrm{Tr}\,\chi)^3. \tag{A4b}
$$

The surface integrals are performed with respect to the induced metric, i.e. $\oint\cdots \equiv \oint\sqrt{\tilde{g}}d^{d-1}du\cdots$, where the boundary is (locally) parametrized through functions $x^a = x^a(u^i)$, and

$$
\tilde{g}_{ij} \equiv g_{ab}\frac{\partial x^a}{\partial u^i}\frac{\partial x^b}{\partial u^j} \tag{A5}
$$

is the induced metric on $\partial\mathcal{M}$. The second fundamental form

$$
K_{ij} \equiv -\frac{\partial x^a}{\partial u^i}\frac{\partial x^b}{\partial u^j}n_{a;b} \tag{A6}
$$

or more precisely its extrinsic form

$$
\chi^{ab} = \frac{\partial x^a}{\partial u^i}\frac{\partial x^b}{\partial u^j}K^{ij} \tag{A7}
$$

enter the above expression for the Seeley-deWitt coefficients. In the body of the paper we assume that $\mathcal{M}$ is flat in which case $g_{ab}=\delta_{ab}$ and the covariant derivative in (A6) becomes an ordinary one.

Appendix B

We set up the 1-loop approximation to the partition function for the $O(N)-$sigma models [38]. The main point is to use an expansion of the n_a-fields about the classical background which respects the constraint $n^2 = 1$, up to cubic and higher order terms in the fluctuation fields ξ_α. We set

$$n_a \ = \ N_a + \xi^\beta \epsilon_{\beta a} - \frac{1}{2}\xi^\beta \xi_\beta N_a, \tag{B1}$$

where the $\epsilon_{\beta a}; \beta = 2,..,N$ together with N_a form a orthonormal system in the space of fields

$$\begin{aligned} \epsilon_\beta^a N_a &= 0 \\ \epsilon_\alpha^a \epsilon_{\beta a} &= \delta_{\alpha\beta} \\ N_a N_b + \epsilon_a^\beta \epsilon_{\beta b} &= \delta_{ab}. \end{aligned} \tag{B2}$$

It is clear that $n^2 = 1+$ O (ξ^3). Expanding S to second order yields

$$S[n,g] = S[N,g] \ + \ \frac{1}{2g^2}\int \xi^\alpha D_{\alpha\beta}\xi^\beta,$$

where

$$\begin{aligned} D_{\alpha\beta} &= -D_{\alpha\beta}^2 + \sigma_{\alpha\beta}, \\ D_\mu^{\alpha\beta} &= \partial_\mu \delta^{\alpha\beta} + \epsilon^{\alpha a}(\partial_\mu \epsilon_a^\beta) \\ \sigma^{\alpha\beta} &= (\epsilon^{\alpha a}\partial^\mu N_a)\,(\epsilon^{\beta b}\partial_\mu N_b) \ - \ \delta_{\alpha\beta}(\partial^\mu N^a \cdot \partial_\mu N_a - g^2 j^a \cdot N_a). \end{aligned} \tag{B3}$$

For the functional measure we find

$$\prod_x d^N n(x)\,\delta(n^2-1) \ = \ \prod_x d^{N-1}\xi(x)\,(1+O(\xi^2)),$$

where we used that the Jacobian from the coordinate change exactly cancels the contribution from integrating out the δ-distribution. Shifting $\xi \to \sqrt{\hbar}\xi$ we find the 1-loop partition function

$$\mathcal{Z}^{(1)}[\mathcal{M},j,g] = e^{\frac{1}{\hbar}\mathcal{W}^{cl}[\mathcal{M},j,g]}\,\det{}^{-\frac{1}{2}}\,D \tag{B4}$$

and

$$\mathcal{W}^{(1)}[\mathcal{M},j,g] = \mathcal{W}^{cl}[\mathcal{M},j,g] \ - \ \frac{\hbar}{2}\log\det\, D. \tag{B5}$$

For the ζ-function we find with the help of Appendix A

$$\begin{aligned} \zeta_D(0) &= -\frac{1}{4\pi}\int \operatorname{tr}\sigma \\ &= \frac{N-2}{4\pi}\int \partial^\mu N^a \cdot \partial_\mu N_a \ - \ \frac{g^2(N-1)}{4\pi}\int j^a N_a. \end{aligned} \tag{B6}$$

References

1. J. Wess, Nuovo Cim. **18** (1960) 1086; G. Mack and A. Salam, Ann. Phys. **53** (1969) 174.

2. A.M. Polyakov, JETP Lett. **12** (1970) 381; A.A. Migdal, Phys. Lett. **37B** (1971) 386.

3. N.D. Birell and P.C.W. Davies, Quantum fields in curved space (Cambridge University Press, Cambridge, 1982).

4. H.B.G. Casimir, Proc. Kon. Ned. Aka. Wet. **51** (1948) 793; H.B.G. Casimir and D. Polder, Phys. Rev. **73** (1948) 360; S. Blau, M. Visser and A. Wipf, Nucl. Phys. **310** (1990) 163; E. Elizalde and A. Romeo, Int. J. Mod. Phys. **A5** (1990) 1653.

5. J.L. Cardy and I. Peschel, Nucl. Phys. **B300** (1988) 377;
J.L. Cardy and D.C. Lewellen, Phys. Lett. **259B** (1991) 274; I. Affleck, Phys. Rev. Lett. **56** (1986) 746; H. Blöte, J. Cardy and M. Nightingale, Phys. Rev. Lett.**56** (1986) 742.

6. A. Dettki and A. Wipf, Nucl. Phys. **B377** (1992) 252.

7. C.G. Callan, Phys. Rev. **D2** (1970) 1541; K. Symanzik, Comm. Math. Phys. **18** (1970) 227.

8. E. Kraus and K. Sibold, Nucl. Phys. **B372** (1992) 113; preprint MPI-Ph/92-79.

9. L. Dolan and R. Jackiw, Phys. Rev. **D9** (1974) 3320; A.D. Linde, Phys. Lett. **96B** (1980) 289.

10. C. Wetterich, Nucl. Phys. **B352** (1991) 529; M. Reuter and C. Wetterich, preprint HD-THEP-92-62.

11. D.J. O'Connors, B.L. Hu and T.C. Shen, Phys. Lett. **130B** (1983) 31; D.J. Toms, Phys. Rev. **D21** (1980) 2805; S. Blau, M. Visser and A. Wipf, Phys. Lett. **209B** (1988) 209; L. Berkin, Phys. Rev. **D46** (1992) 1551; G. Cognola, K. Kirsten and S. Zerbini, Trento preprint, U.T.F. 284.

12. D.M. McAvity and H. Osborn, preprint DAMTP/92-31.

13. U.Wolff, Nucl. Phys. **B 265** [FF 15] (1986) 506, 537; M. Lüscher, P. Weisz and U. Wolff, Nucl. Phys. **B359** (1991) 221; M. Lüscher, R. Narayanan, P. Weisz and U. Wolff, Desy preprint, DESY 92-025.

14. A. Dettki, I. Sachs and A. Wipf, preprint ETH-TH/93-14.

15. A.Wightman, Introduction to convexity in the theory of lattice gases, ed. R.B. Israel (Princeton Univ. Press, 1979); L. O'Raifeartaigh and A. Wipf, Nucl. Phys. **B271** (1986) 653.

16. K. Symanzik, Nucl. Phys. **B190** [FS3] (1981) 1; M. Lüscher, Nucl. Phys. **B254** (1985) 52; M. Lüscher, K. Symanzik and P. Weisz, Nucl. Phys. **B173** (1980) 365.

17. J.S. Dowker and R. Critchley, Phys. Rev. **D13** (1976) 3224; S.W. Hawking, Commun. Math. Phys. **55** (1977) 133.

18. P. Gilkey, Invariance theory, the heat equation and the Athiyah Singer index theorem (Publish or Perish, 1984).

19. S. Blau, M. Visser and A. Wipf, Int. J. Mod. Phys. **A4** (1989) 1467.

20. A.E.M. van de Ven, Nucl. Phys. **B250** (1985) 593.

21. G. Kennedy, J. Phys. **A11** (1978) L173; T.P. Branson and P.B. Gilkey, Commun. Part. Diff. Eq **15** (1990) 2; J. Melmed, J. Phys. **A21** (1989) L113; J.S. Dowker and J.P. Schofield, J. Math. Phys. **31** (1990) 808; I.G. Moss and J.S. Dowker, Phys. Lett. **B229** (1990) 2; D.M. McAvity and H. Osborn, Class. Quantum Grav. **8** (1991) 603.

22. D. Lohiya, Mod. Phys. Lett. **A4** (1989) 863.

23. S. Coleman, Phys. Rev. **D11** (1975) 2088.

24. S. Hikami and E. Brezin, J. Phys. **A11** (1978) 1141; W. Bernreuther and F.J. Wegner, Phys. Rev. Lett. **57** (1986) 1383; A.A. Tseytlin, Nucl. Phys. **B294** (1987) 399; H. Osborn, Nucl. Phys. **B308** (1988) 629; K. Behrndt and H. Dorn, Int. J. Mod. Phys. **A7** (1992) 1375.

25. see e.g. Kerson Huang, Quarks, Leptons and Gauge Fields, (World Scientific 1982)

26. C.G. Callan, S. Coleman and R. Jackiw, Ann. Phys. **59** (1970) 42.

27. M.F. Atiyah, V.K. Patodi and I.M. Singer, Math. Proc. Camb. Phil. Soc. **77** (1975) 43; P. Forgacs, L. O'Raifeartaigh and A. Wipf, Nucl. Phys. **B293** (1987) 559.

28. A. Chodos, R.L. Jaffe, I. Johnson, C.B. Thorn and V.F. Weisskopf, Phys. Rev. **D9** (1974) 3471; J. Baacke and Y. Igarashi, Phys. Rev. **D27** (1983) 460; J. Balog and P. Hrasko, Nucl. Phys. **B245** (l984) 118.

29. G. t'Hooft, Phys. Rev. **D14** (1976) 3432; R.D. Peccei and H.R. Quinn, Phys. Rev. **D16** (1977) 1791.

30. H. Osborn, Annals of Phys. **135** (1981) 373.

31. L. Fadeev and V.N. Popov, Phys. Lett. **25B** (1967) 29.

32. I. Sachs and A. Wipf, Helv. Physica Acta **65** (1992) 652.

33. M. Lüscher, Ann. Phys. **142** (1982) 359.

34. L.F. Abbott, Acta Phys. Pol. **B13** (1982) 33.

35. J.C. Collins, A. Duncan and S.D. Joglekar, Phys. Rev. **D16** (1977) 438.

36. L. Cullumovic, M. Leblanc, R.B. Mann, D.G.C. McKeon and T.N. Sherry, Phys. Rev. **D41** (1990) 514; D.G.C. McKeon and T.N. Sherry, Ann. Phys. **218** (1992) 325.

37. S. Dürr and A. Wipf, to appear; see also: H. Leutwyler and A. Smilga, preprint BUTP-92/10.

38. J. Gasser and H. Leutwyler, Ann. Phys. **158** (1984) 142; D. Friedan, Ann. Phys. **163** (1985) 318; L. Alvarez Gaume, D. Freedman and S. Mukhi, Ann. Phys. **134** (1981) 85.