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LIOUVILLE AND TODA THEORIES AS CONFORMALLY REDUCED WZNW THEORIES

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Abstract

It is shown that Liouville theory can be regarded as an $SL(2,R)$ Wess-Zumino-Novikov-Witten theory with conformal invariant constraints and that Polyakov's $SL(2,R)$ Kac-Moody symmetry of induced two-dimensional gravity is just one side of the WZNW current algebra. Analogously, Toda field theories can be regarded as conformal-invariantly constrained WZNW theories for appropriate (maximally non-compact) groups. The WZNW formulation shows that the singularities of those Liouville and Toda solutions which are conformally regular are just coordinate singularities.

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Remarkable progress has been made towards solving the two dimensional quantum gravity [1,2] which is induced by string theory in subcritical dimensions. As observed by Polyakov [1] the induced gravity action possesses a hidden chiral $SL(2,R)$ Kac-Moody (KM) symmetry in the light-cone gauge, and this symmetry opened up a new way of solving minimal models coupled to gravity [2]. It is hoped that this KM symmetry will also prove useful in understanding strings in subcritical dimensions. Since in the conformal gauge the induced gravity is described by Liouville theory, it is natural to ask whether there is a 'hidden' KM symmetry associated with Liouville theory as well. In this paper we show there are actually two (a left- and a right-handed) hidden $SL(2,R)$ symmetries associated with Liouville theory. In fact we demonstrate that the Liouville theory is nothing but the $SL(2,R)$ Wess-Zumino-Novikov-Witten (WZNW) theory, reduced in a conformally invariant manner. More generally, we show that Toda field theories [3,4], which are natural, completely integrable generalizations of the Liouville theory can also be obtained by reducing WZNW models in a conformally invariant way.

One important advantage of regarding the Liouville theory as a reduced WZNW system is that the configurations which are singular in the Liouville variables but have regular energy-momentum densities turn out to be regular in the corresponding WZNW variables. For this reason it is hoped that the WZNW description will open up a new way of attacking the hard problem of quantizing the Liouville theory [5-8], namely, by applying the symmetry reduction to the quantized WZNW theory. But in this paper we shall restrict ourselves to purely classical considerations.

The discovery of Polyakov [1] mentioned above is that in the light-cone gauge where

$$ds^2 = d\xi^+ d\xi^- + h(\xi^+, \xi^-)(d\xi^+)^2 \qquad (1)$$

the induced gravity Lagrangian density

$$\mathcal{L} = -\frac{k}{8\pi}\big[\frac{1}{2}R\Delta^{-1}R - M\big]\sqrt{-g}\ ; \qquad\qquad k = \frac{D-26}{6} \qquad (2)$$

where R is the scalar curvature and Δ is the two-dimensional wave operator, is local in the variable f defined as $h = \partial_+ f/\partial_- f$, and is invariant with respect to the

$SL(2,R)$ KM symmetry transformations

$$\tilde{f}(\xi^+,\xi^-) = f\Big(\xi^+, \frac{a(\xi^+)\xi^- + b(\xi^+)}{c(\xi^+)\xi^- + d(\xi^+)}\Big). \tag{3}$$

In the left-right symmetric conformal gauge

$$ds^2 = e^{\phi(\xi^+,\xi^-)} d\xi^+ d\xi^-, \qquad \sqrt{-g} = e^\phi \tag{4}$$

the induced two-dimensional gravitational Lagrangian density reduces to

$$\mathcal{L} = -\frac{k}{8\pi}\big(\frac{1}{2}\partial_+\phi\partial_-\phi - Me^\phi\big) \tag{5}$$

which is the Liouville Lagrangian density (for the value of k given in (2)). We now wish to show that (for any negative value of the parameter k) the Liouville Lagrangian (5) has a hidden two-sided $SL(2,R)$ symmetry, which it inherits from an underlying $SL(2,R)$ WZNW model.

We start by recalling some facts about the WZNW theory [9]. First, the WZNW action for a group-valued field g is:

$$S(g) = -\frac{k}{8\pi}\int d^2\xi \eta^{\mu\nu}\mathrm{Tr}\big\{(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g)\big\} + \frac{k}{12\pi}\int_{B_3}\mathrm{Tr}\big\{(g^{-1}dg)^3\big\}\ , \tag{6}$$

where the Tr symbol denotes the ordinary matrix trace operation multiplied by a constant to be fixed later and, as usual, B_3 is a three-dimensional manifold whose boundary is Minkowski space-time*. Any Lie algebra element λ gives rise to left- and right KM symmetries of (6) generated by the Noether currents

$$\begin{aligned} J(\lambda) &= \kappa\mathrm{Tr}\big\{\lambda\cdot(\partial_+ g)\cdot g^{-1}\big\} \\ \tilde{J}(\lambda) &= -\kappa\mathrm{Tr}\big\{\lambda\cdot g^{-1}\cdot(\partial_- g)\big\} \end{aligned} \qquad \text{where} \qquad \kappa = -\tfrac{k}{4\pi} \tag{7}$$

and the WZNW field equation is equivalent to current conservation

$$\partial_- J = 0 \qquad ; \qquad \partial_+\tilde{J} = 0. \tag{8}$$

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* Our space-time conventions are the following:
$\eta^{00} = -\eta^{11} = \epsilon^{01} = -\epsilon^{10} = 1;\ \partial_\pm\phi = \partial_0\phi \pm \partial_1\phi = \dot{\phi} \pm \phi'$.

We will make use of the Polyakov-Wiegmann identity [10] that expresses the WZNW action for the product of three matrices A,B,C as the sum of the respective actions for A,B and C, modulo local terms:

$$\begin{aligned} S(ABC) &= S(A) + S(B) + S(C) \\ &\quad + \kappa \int d^2\xi \ \mathrm{Tr}\big\{(A^{-1}\partial_- A)(\partial_+ B)B^{-1} \\ &\quad + (B^{-1}\partial_- B)(\partial_+ C)C^{-1} + (A^{-1}\partial_- A)B(\partial_+ C)C^{-1}B^{-1}\big\}. \end{aligned} \tag{9}$$

After this recapitulation we consider the case of $SL(2,R)$. This group has the property that any group element g in a neighbourhood of the identity admits the Gauss-decomposition $g = ABC$, where

$$\begin{aligned} A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = e^{xE_+} \qquad ; \qquad C = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = e^{yE_-} \\ B = \begin{pmatrix} e^{\frac{1}{2}\phi} & 0 \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix} = e^{\frac{1}{2}\phi H}. \end{aligned} \tag{10a}$$

The neighbourhood in which this parametrization is valid consists of those group elements for which $g_{22} > 0$ and the whole group can be covered by four patches of this type, that is an arbitrary group element g can be parametrized as:

$$g = ABC\omega \qquad \text{where} \qquad \omega = \pm 1 \qquad \text{or} \qquad \omega = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{10b}$$

(Note that for the adjoint group $SL(2,R)/Z_2$ two patches suffice.) Substituting the parametrization (10) into (9) the action takes the simple local form [11]:

$$\begin{aligned} S(g) &= S(x,y,\phi) \\ &= \frac{\kappa}{2} \int d^2\xi \ \{\frac{1}{2}\partial_+\phi\partial_-\phi + 2(\partial_- x)(\partial_+ y)e^{-\phi}\}. \end{aligned} \tag{11}$$

The equations of motion, derived conveniently from (11), read as

$$\partial_-(\partial_+ y e^{-\phi}) = \partial_+(\partial_- x e^{-\phi}) = 0, \tag{12a}$$

$$\partial_+\partial_-\phi + 2(\partial_- x)(\partial_+ y)e^{-\phi} = 0. \tag{12b}$$

Working in the neighbourhood of the identity ($\omega = 1$), let us now consider the following special solutions of (12a):

$$\partial_+ y = \mu e^{\phi} \qquad\qquad \partial_- x = \nu e^{\phi} \ , \tag{13}$$

where μ and ν are arbitrary constants. Using (13), the equation (12b) reduces to the Liouville equation

$$\partial_+\partial_-\phi + Me^{\phi} = 0 \ , \qquad \text{where} \quad M = 2\mu\nu \ . \tag{14}$$

Thus locally at least the Liouville system can be regarded as a reduced WZNW theory. This reduction is a canonical one in the sense that the Poisson brackets of the Liouville phase space variables ϕ and $\partial_0\phi$ can be calculated either in the WZNW theory or using the Liouville Lagrangian (5). Since locally the currents $J(E_+)$ and $\tilde{J}(E_-)$ can be written as

$$J(E_+) = \kappa \ \partial_+ y e^{-\phi} \qquad\qquad \tilde{J}(E_-) = -\kappa \ \partial_- x e^{-\phi}, \tag{15}$$

the solutions (13) correspond to imposing the constraints

$$J(E_+) = \kappa\mu \qquad\qquad \tilde{J}(E_-) = -\kappa\nu \tag{16}$$

which are globally well-defined and therefore can be imposed on any of the patches defined by (10b).

An important question is how the conformal symmetry of the Liouville theory appears in the WZNW context. The left- and right Virasoro algebras of the Liouville theory are given by the improved (traceless) energy-momentum tensor:

$$T_{0\pm} = (\frac{\kappa}{2})\big[\frac{1}{2}(\partial_\pm\phi)^2 + Me^{\phi} \mp 2(\partial_\pm\phi)'\big]. \tag{17}$$

Due to the improvement term $(\partial_\pm\phi)'$, there is a classical centre $c = -6k$ in this Virasoro algebra. We have to look for a Virasoro algebra in the WZNW model which, upon imposing the constraint (16), yields the Liouville Virasoro (17). It is easy to see that the Sugawara Virasoro density L of the WZNW theory

$$L = (-\frac{2\pi}{k})\big[\frac{1}{2}J(H)^2 + 2J(E_+)J(E_-)\big] \tag{18}$$

(and similarly for $\tilde{L}$) does not commute with the constraint (16). However, there is a whole family of Virasoro subalgebras in the semidirect product formed by the KM algebra and its associated Sugawara Virasoro algebra, namely, those generated by the densities

$$l = L + a_\lambda J(\lambda) + b_\lambda J'(\lambda), \tag{19}$$

(and similarly for $\tilde{l}$) where $\lambda \in sl(2,R)$ and a_λ and b_λ are real numbers. Within this family it is possible to find a Virasoro algebra that does weakly commute with the constraint. Indeed, by imposing the requirement that the Virasoro weakly commutes with (16), one is led to the solution

$$l = L - J'(H) \qquad\qquad \tilde{l} = \tilde{L} + \tilde{J}'(H), \tag{20}$$

which is unique up to terms which are proportional to the constraint (16) and do not contribute in Liouville theory. As a consequence of the relation

$$\{J(H,\sigma),\ J(H,\sigma')\} = \frac{k}{2\pi}\ \mathrm{Tr}H^2\ \delta'(\sigma-\sigma'), \tag{21}$$

(20) has the same centre as (17), namely

$$c = -3k\ \mathrm{Tr}H^2 = -6k. \tag{22}$$

Since there is only one Virasoro symmetry in Liouville theory, T_{0+} and l are expected to be the same functionals of the Liouville phase space data ϕ and $\partial_0\phi$. In local coordinates the density l takes the form

$$\begin{aligned} l = L - J(H)' = \frac{\kappa}{2}\{[\frac{1}{2}(\partial_+\phi)^2 + 2(\partial_+x)(\partial_+y)e^{-\phi}] \\ -[2\partial_+\phi + 4x(\partial_+y)e^{-\phi}]'\} \end{aligned} \tag{23}$$

and this expression does indeed reduce to T_{0+} on imposing the constraint (16). The term $(\partial_+\phi)^2$ comes entirely from the Sugawara density L, while the improvement term comes from $J'(H)$. On the other hand, the exponential interaction potential in (17) arises as a combination of two terms , one of them coming from L, the other from $J'(H)$.

One can use the WZNW $\longrightarrow$ Liouville reduction to obtain the general solution of the Liouville equation from that of the WZNW model. As is well-known, the former is given as

$$e^{\phi(\xi^+,\xi^-)} = \frac{2}{M}\frac{F'(\xi^+)\tilde{F}'(\xi^-)}{[1+F(\xi^+)\tilde{F}(\xi^-)]^2}\ , \tag{24}$$

where F and $\tilde{F}$ are arbitrary functions satisfying $F' > 0$, $\tilde{F}' > 0$ and the prime means derivative taken with respect to the argument. The general WZNW solution is described by the simpler formula

$$g(\xi^+, \xi^-) = g_L(\xi^+) \cdot g_R(\xi^-) \ , \tag{25}$$

where g_L and g_R are arbitrary group-valued functions. Now, if one assumes that g, g_L and g_R are all Gauss-decomposable then trivial matrix multiplication yields (in an obvious notation)

$$e^{\phi(\xi^+,\xi^-)} \equiv \frac{e^{\phi_L(\xi^+)} e^{\phi_R(\xi^-)}}{[1 + y_L(\xi^+) x_R(\xi^-)]^2} \ . \tag{26}$$

On the other hand, the constraints (16) in this case reduce to

$$y_L'(\xi^+) = \mu e^{\phi_L(\xi^+)} \qquad\qquad x_R'(\xi^-) = \nu e^{\phi_R(\xi^-)} \ , \tag{27}$$

and if these are imposed and one makes the identification

$$F = y_L \qquad , \qquad \tilde{F} = x_R \tag{28}$$

then (26) reduces to (24), as required. Conversely, if a Liouville solution e^ϕ is given in terms of F and $\tilde{F}$ as in (24) then one can build a family of corresponding local WZNW solutions by reversing the above procedure. This family is parametrized by the arbitrary functions y_R and x_L which describe the gauge freedom

$$g \ \longrightarrow \ \pm A(x_L) \cdot g \cdot C(y_R) \tag{29}$$

of the constraint (16). (The discrete freedom $g \longrightarrow -g$ disappears, of course, for the adjoint group.)

It is known [6] that the Liouville equation admits 'singular' solutions with perfectly regular energy-momentum density and that these configurations play an important role in the quantized version of the theory. One of the main advantages of the WZNW description of Liouville theory is that these configurations are represented by globally regular WZNW fields. For example, the Liouville solution

$$e^\phi = \frac{1}{\cos^2 \alpha} \qquad \text{where} \qquad \alpha = \mu \xi^+ - \nu \xi^- \tag{30}$$

is apparently singular, but the corresponding Virasoro densities are not (they are simply constants). On the other hand, using the reduction procedure, the configuration (30) is obtained from the following solution of WZNW theory:

$$g = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} , \tag{31}$$

which is indeed perfectly regular.

To see that this holds for any Liouville solution with regular Virasoro densities, let us consider two independent real solutions Ψ_1, Ψ_2 of the Schrödinger equation with periodic potential $V \equiv \frac{\kappa}{2} T_{0+}$ (and similarly for $\tilde{V} = \frac{\kappa}{2} T_{0-}$),

$$\Psi''(\xi) = \frac{1}{2} V(\xi)\Psi(\xi). \tag{32}$$

Normalizing the pair of solutions by the Wronskian condition

$$\Psi_2'\Psi_1 - \Psi_1'\Psi_2 = \mu , \tag{33}$$

it is easy to see that the left moving $SL(2,R)$ matrix

$$g_L = \begin{pmatrix} \Psi_1/N & -\Psi_2/N \\ \Psi_2 & \Psi_1 \end{pmatrix} \qquad \text{where} \qquad N = \Psi_1^2 + \Psi_2^2 \tag{34}$$

satisfies the left-handed part of the constraint (16). Moreover, since Ψ_1 and Ψ_2 are two independent solutions of the same Schrödinger equation and thus can never vanish simultanously, the matrix (34) is always regular. It is known that V is the Schwarzian derivative of F and it follows from (32) and (33) that V is also the Schwarzian derivative of the ratio Ψ_2/Ψ_1. Therefore, using the $SL(2,R)/Z_2$ freedom of the Schwarzian derivative, we can choose Ψ_1 and Ψ_2 so that they reproduce F:

$$F = \frac{\Psi_2}{\Psi_1} . \tag{35}$$

Applying the analogous procedure to the right hand side as well one finds that the Liouville solution e^ϕ is recovered as

$$e^\phi = \big((g_L \cdot g_R)_{22}\big)^{-2} = \frac{1}{(\Psi_1\tilde{\Psi}_1 + \Psi_2\tilde{\Psi}_2)^2} , \tag{36}$$

where the matrix g_R is built out of Ψ_1 and Ψ_2 according to the matrix transpose of the formula (34). To summarize, we have shown that it is indeed possible to construct a globally regular WZNW representation for any Liouville configuration with regular Virasoro densities. From (36) one sees that the only singularities of the Liouville function e^{ϕ} occur at the zeros of the matrix element g_{22} of the corresponding WZNW solution, and from (10) one sees that these are only coordinate singularities associated with the patching of $SL(2,R)$ (or, more precisely, of $SL(2,R)/Z_2$).

The connection with Polyakov's light-cone gauge theory mentioned earlier is obtained by making the WZNW $\longrightarrow$ Liouville reduction in two steps. In the first step we impose only the right-moving part of the constraint (16). Then the equations of motion yield

$$\partial_-(e^{-\phi}\partial_+\partial_-\phi) = 0 \quad , \quad \text{where} \quad \phi \equiv \ln(\frac{1}{\nu}\partial_- x). \tag{37}$$

This equation can be derived from the effective Lagrangian

$$\mathcal{L} = (-\frac{k}{16\pi})\frac{(\partial_+\partial_- x)(\partial_-\partial_- x)}{(\partial_- x)^2} \ , \tag{38}$$

which is the same as Polyakov's Lagrangian for the two-dimensional induced gravity in the light-cone gauge (with $k = \frac{D-26}{6}$) if we identify x with Polyakov's variable f through the relation

$$x(\xi^+, f(\xi^+,\xi^-)) = \xi^- \ . \tag{39}$$

The effective Lagrangian (38) is invariant under the residual left-moving $SL(2,R)$ transformations

$$\tilde{x}(\xi^+,\xi^-) = \frac{a(\xi^+)x(\xi^+,\xi^-) + b(\xi^+)}{c(\xi^+)x(\xi^+,\xi^-) + d(\xi^+)} \qquad ; \quad ad - bc = 1, \tag{40}$$

which, in terms of the variable f, are just Polyakov's left-moving $SL(2,R)$ KM transformations. If we now break this residual symmetry by imposing the left-moving part of (16), then we reach the Liouville theory again, since the E_+ component of the $SL(2,R)$ Noether current corresponding to the symmetry transformation (40) of the Lagrangian (38) turns out to be

$$J(E_+) = -\frac{\kappa}{2\nu}e^{-\phi}\partial_+\partial_-\phi \ . \tag{41}$$

Let us now consider the Liouville $\longrightarrow$ Toda generalization of the above results. Let $\mathcal{G}$ be any complex simple Lie algebra, Φ the set of roots with respect to some Cartan subalgebra $\mathcal{H}$ and Δ a set of simple roots. We fix a Cartan-Weyl basis, consisting of root vectors E_α, $\alpha \in \Phi$ and Cartan generators $H_\alpha \equiv [E_\alpha, E_{-\alpha}]$, $\alpha \in \Delta$, with respect to which all the structure constants of $\mathcal{G}$ are real numbers. The real span of the Cartan-Weyl basis yields a particular real form $\mathcal{G}_R$ of $\mathcal{G}$. (This 'maximally non-compact' real form of $\mathcal{G}$ is well-defined by the Cartan-Weyl basis up to isomorphism and for the classical Lie algebras A_n, C_n, B_n and D_n is in fact provided by the real Lie algebras $sl(n+1,R)$, $sp(2n,R)$ and $so(p,q,R)$ for $p-q=1,0$, respectively.)

A property that distinguishes $\mathcal{G}_R$ from all the other real forms of $\mathcal{G}$, and that will be crucial for our purposes, is that it is the only real form for which any connected Lie-group G_R with $\mathcal{G}_R$ as its Lie-algebra admits a local, unique, group-valued Gauss-decomposition similar to (10),

$$g = ABC \tag{42a}$$

where

$$\begin{gathered}A = \exp\Big\{\sum_{\alpha\in\Phi^+} x^\alpha E_\alpha\Big\} \qquad ; \qquad C = \exp\Big\{\sum_{\alpha\in\Phi^-} y^\alpha E_\alpha\Big\}\ , \\ B = \exp\Big\{\frac{1}{2}\sum_{\alpha\in\Delta}\phi^\alpha H_\alpha\Big\}\ .\end{gathered} \tag{42b}$$

(Here $\Phi^\pm$ denotes the set of positive (negative) roots, respectively.) This property makes the WZNW models based on the non-compact groups G_R the natural generalizations of the $SL(2,R)$ WZNW model and these are the models that we shall consider.

We need to recall the following results and conventions from the theory of Lie algebras [12]:

$$\begin{aligned} K_{\alpha,\beta} &= \alpha(H_\beta) = \frac{2\alpha\cdot\beta}{|\alpha|^2} \qquad \alpha,\beta\in\Delta; \qquad\qquad |\alpha_{\rm long}|^2 = 2 \\ \mathrm{Tr}\ (H_\alpha\cdot H_\beta) &= \frac{2}{|\alpha|^2}K_{\alpha,\beta} = C_{\alpha,\beta} \\ \mathrm{Tr}\ (E_\alpha\cdot E_\beta) &= \frac{2}{|\alpha|^2}\delta_{\alpha,-\beta}, \quad \alpha,\beta\in\Phi, \quad \mathrm{Tr}\ (E_\alpha\cdot H_\beta) = 0\ , \end{aligned} \tag{43}$$

which are valid in any finite dimensional representation of $\mathcal{G}$ where Tr is the usual matrix trace multiplied by an appropriate constant.

Our main result is that the constraints

$$J(E_\alpha) = \kappa\mu^\alpha \qquad \tilde{J}(E_{-\alpha}) = -\kappa\nu^\alpha \qquad (\alpha \in \Phi^+), \tag{44}$$

where μ^α and ν^α are arbitrary nonzero real numbers for the l primitive roots $\alpha \in \Delta$ and zero for all other positive roots (which are natural generalizations of the SL(2,R) constraints (16)) reduce the G_R WZNW theory to the Toda theory, defined for any simple Lie algebra by the Lagrangian

$$\mathcal{L} = -\frac{k}{8\pi}\Big(\frac{1}{4}C_{\alpha,\beta}\partial_+\phi^\alpha\ \partial_-\phi^\beta - \sum_{\alpha\in\Delta} M^\alpha \exp\{\frac{1}{2}K_{\alpha,\beta}\ \phi^\beta\}\Big). \tag{45}$$

Because μ^α and ν^α are zero for all but the primitive roots the constraints (44) may be written for $g = ABC$ as

$$\begin{aligned} A^{-1}\partial_- A &= B\big[\sum_{\alpha\in\Delta}\frac{|\alpha|^2}{2}\nu^\alpha E_\alpha\big]B^{-1} = \sum_{\alpha\in\Delta}\frac{|\alpha|^2}{2}\nu^\alpha E_\alpha \exp\{\frac{1}{2}K_{\alpha,\beta}\phi^\beta\} \\ (\partial_+ C)C^{-1} &= B^{-1}\big[\sum_{\alpha\in\Delta}\frac{|\alpha|^2}{2}\mu^\alpha E_{-\alpha}\big]B = \sum_{\alpha\in\Delta}\frac{|\alpha|^2}{2}\mu^\alpha E_{-\alpha} \exp\{\frac{1}{2}K_{\alpha,\beta}\phi^\beta\}\ . \end{aligned} \tag{46}$$

Since the matrices A and C occur in the WZNW equation of motion (8) only in the combinations shown in (46), they can be eliminated and the equation then reduces to an equation for B (i.e. for the ϕ^α's) alone. A little algebra shows that this equation is just the Toda equation

$$\partial_+\partial_-\phi^\alpha\ +\ \frac{1}{2}|\alpha|^2 M^\alpha \exp\{\frac{1}{2}K_{\alpha,\beta}\phi^\beta\} = 0 \quad , \text{ where } \qquad M^\alpha \equiv |\alpha|^2\mu^\alpha\nu^\alpha. \tag{47}$$

This shows that the constraints (44) reduce the G_R WZNW theory to the Toda theory. As in the Liouville case, the reduction is canonical in the sense that the Poisson brackets of the Toda variables are preserved by the reduction. (Note that, as far as they are positive, the actual values of the constants M^α in (47) are irrelevant since they can be redefined simply by shifting the fields ϕ^α.)

At this point it is worth mentioning that the general solution of the Toda field equation (47) can be immediately generated from that of the corresponding WZNW model, (25). Applying the local Gauss-decomposition (42) for g and also for g_L and g_R, g can be written as

$$\begin{aligned} g(\xi^+,\xi^-) &= A\exp\{\tfrac{1}{2}\sum_{\alpha\in\Delta}\phi^\alpha H_\alpha\}C \\ &= g_L(\xi^+)g_R(\xi^-) = A_L\exp\{\tfrac{1}{2}\sum_{\alpha\in\Delta}\phi^\alpha_L H_\alpha\}C_L A_R\exp\{\tfrac{1}{2}\sum_{\alpha\in\Delta}\phi^\alpha_R H_\alpha\}C_R. \end{aligned} \tag{48}$$

The problem of projecting out the the matrix elements $e^{\phi^\alpha(\xi^+,\xi^-)}$ of B can be elegantly solved [3] by introducing the l normalized lowest weight states $\mid\lambda_\alpha\rangle$ of the l (finite dimensional) fundamental representations of $\mathcal{G}$, so that

$$H_\alpha \mid \lambda_\beta\rangle = -\delta_{\alpha,\beta}\mid\lambda_\beta\rangle \qquad\qquad \alpha,\beta\in\Delta. \tag{49}$$

Now, by calculating the matrix element $\langle\lambda_\alpha\mid g\mid\lambda_\alpha\rangle$ of (48) we obtain

$$e^{-\frac{1}{2}\phi^\alpha(\xi^+,\xi^-)} = e^{-\frac{1}{2}\phi^\alpha_L-\frac{1}{2}\phi^\alpha_R}\langle\lambda_\alpha\mid C_L A_R\mid\lambda_\alpha\rangle. \tag{50}$$

(50) is the general solution of the Toda field equations provided g_L and g_R satisfy the constraints (44). Following [3] we choose the set of functions $\{\phi^\alpha_L(\xi^+),\phi^\alpha_R(\xi^-)\}$ as our independent variables. Then the constraints (44) can be solved for the matrices C_L and A_R in terms of these functions. (Alternatively, one could start with a set of l matrix elements of C_L and A_R each and try to solve the constraints for ϕ^α_L, ϕ^α_R and the remaining matrix elements of C_L and A_R.) To get the solution in the form given in [3] * we have to introduce

$$\frac{1}{2}\phi^\alpha_{L,R} = \sum_\beta G^{\alpha\beta}\ln f^\pm_\beta \tag{51}$$

where $f^\pm_\alpha$ are arbitrary functions and $G^{\alpha\beta}$ is the inverse of the Cartan matrix. So finally we find

$$e^{-\frac{1}{2}\phi^\alpha} = \langle\lambda_\alpha\mid \frac{C_L(\xi^+)}{\prod_\beta (f^+_\beta)^{G^{\alpha\beta}}}\frac{A_R(\xi^-)}{\prod_\beta (f^-_\beta)^{G^{\alpha\beta}}}\mid\lambda_\alpha\rangle. \tag{52}$$

Note, in particular, that $e^{-\frac{1}{2}\phi^\alpha}$ always decomposes into a sum of products of chiral factors.

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* Note that our ϕ^α's differ by a factor of 2 from those of ref. [3].

As in the $SL(2,R)$ case, there is a family of Virasoro generators in the WZNW model given by (19). The fact that the constraints (44) reduce the G_R WZNW theories to the respective Toda theories and that the latter are conformally invariant shows that although the constraints break the KM symmetry completely, they preserve at least one member of the family (19). To see this, and to place our results in a more abstract context, we now show that our reduction procedure is a field theoretical realization of a general mechanism for breaking KM symmetries without breaking the corresponding conformal symmetries. The analogues of the constraints (44) for an abstract $\mathcal{G}_R$ KM algebra are

$$J(E^\alpha) = \mu^\alpha \qquad\qquad \alpha \in \Phi^+ \ , \tag{53}$$

where $\mu^\alpha \neq 0$ for the primitive roots and $\mu^\alpha = 0$ for the other positive roots. If one now looks for the normalizer $\mathcal{N}$ of the constraints (53) in the semi-direct sum of the KM and its associated Sugawara Virasoro algebra one finds that $\mathcal{N}$ is, in analogy to (20), generated by the Virasoro operators

$$l = L - J'(H) \ , \tag{54}$$

where the element H of the Cartan subalgebra is determined by the condition

$$\alpha(H) = 2 \qquad \text{for all} \qquad \alpha \in \Delta \ . \tag{55}$$

The unique solution of (55) is given by

$$H = 2\hat{\delta}, \tag{56}$$

where $\hat{\delta}$ is the sum of the l fundamental co-weights (or equivalently half the sum of the positive co-roots). Note that (55) could not be satisfied for any system of positive roots larger than Δ and this is why μ^α must be zero for all non-primitive positive roots. From these considerations it is clear that the reduction which was applied in this paper to break the KM-symmetry of the WZNW model actually depends only on the algebraic structure of the KM algebra and could be applied to any system with a KM symmetry. The KM equation (21) holds in general, and hence, from (22) the classical centre of the Virasoro (54) is

$$c = -12k|\hat{\delta}|^2. \tag{57}$$

Since the Virasoro algebra of Toda theory is obtained by the above reduction its centre must be given by (57), and indeed (57) agrees with the Toda result of Gervais and Bilal [4].

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