DERIVATION OF THE S-MATRIX USING SUPERSYMMETRY

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Using supersymmetry and shape invariance the reflection and transmission coefficients for a large class of solvable potentials can be obtained algebraically.

The eigenspectrum and eigenstates of a class of one-dimensional hamiltonians have been derived algebraically using supersymmetry [1-3] and shape invariance [4,5]. In this paper we show that the scattering function can be obtained algebraically as well using these two features.

All one-dimensional Schrödering equations can be factorized [6,7],

$$H_1\Phi_n^{(1)}=\left(-\frac{d^2}{dx^2}+V_1(x,a_1)\right)\Phi_n^{(1)}$$
$$=A^\dagger A\Phi_n^{(1)}=E_n^{(1)}\Phi_n^{(1)}\,, \tag{1}$$

where

$$A^\dagger=-\frac{d}{dx}+W(x,a_1),\quad A=\frac{d}{dx}+W(x,a_1) \tag{2}$$

and

$$V_1=W^2(x,a_1)-\frac{d}{dx}W(x,a_1)\,, \tag{3}$$

and a_1 are some parameters of the potential, n labels the states, $n=0$, 1, ..., and $n=0$ is the lowest state. The superpotential $W(x, a_1)$ is given by

$$W(x,a_1)=-\frac{d}{dx}\ln\Phi_0^{(1)}(x,a_1)\,. \tag{4}$$

The partner hamiltonian $H_2=AA^\dagger=-d^2/dx^2+V_2$,

$$V_2=W^2(x,a_1)+\frac{d}{dx}W(x,a_1), \tag{5}$$

gives the same spectrum as H_1 but with the ground state missing; that is,

$$E_0^{(1)}=0,\quad E_n^{(2)}=E_{n+1}^{(1)}\,. \tag{6a}$$

In addition the eigenfunctions with the same energy are related:

$$\Phi_{n+1}^{(1)}(x)=(E_{n+1}^{(1)})^{-1/2}A^\dagger\Phi_n^{(2)}(x)\,. \tag{6b}$$

This degeneracy in the two spectra is due to a supersymmetry. We define a superhamiltonian H and supercharges Q and $Q^\dagger$:

$$H=\begin{bmatrix}H_1 & 0\\ 0 & H_2\end{bmatrix},\quad Q=\begin{bmatrix}0 & 0\\ A & 0\end{bmatrix},\quad Q^\dagger=\begin{bmatrix}0 & A^\dagger\\ 0 & 0\end{bmatrix}. \tag{7}$$

These operators are the two-dimensional representation of the sl(1/1) super algebra

$$[Q,H]=[Q^\dagger,H]=0,$$
$$\{Q,Q^\dagger\}=H\,,\quad \{Q,Q\}=\{Q^\dagger,Q^\dagger\}=0\,. \tag{8}$$

The fact that the supercharges commute with H gives the energy degeneracy (6).

Clearly this process can be continued; a V_3 can be determined from V_2, etc. This produces a ladder of potentials [4-7], V_n. Furthermore, if the partner potentials are "shape invariant", i.e., V_2 has the same functional form as V_1 but different parameters except for an additive constant,

$$V_2(x,a_1)=V_1(x,a_2)+C(a_1)\,, \tag{9a}$$

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then the full ladder of potentials will be shape invariant,

$$V_n(x, a_1) = V_1(x, a_n) + E_n^{(1)}(a_1) , \tag{9b}$$

and the energy spectrum [4,5] and eigenfunctions [4,8] of the original potential can be determined algebraically:

$$E_n^{(1)} = \sum_{k=1}^{n} C(a_k) , \tag{10a}$$

$$\Phi_n^{(1)}(x, a_1) = \prod_{k=1}^{n} A^\dagger(x, a_k)\Phi_0^{(1)}(x, a_{n+1}) . \tag{10b}$$

In this paper we shall show that, using shape invariance, we can determine the scattering solutions algebraically as well also. There are two types of scattering problems that we shall consider: (1) scattering from a one-dimensional potential well and (2) scattering from a spherically symmetric three-dimensional potential well.

The asymptotic wavefunction for scattering from a one-dimensional potential well V_2 is given by

$$\Phi^{(2)}(k, x\to -\infty) \to e^{ikx} + R_2 e^{-ikx} , \tag{11}$$

$$\Phi^{(2)}(k, x\to \infty) \to T_2 e^{ikx} . \tag{12}$$

Using (6b) we can determine the asymptotic behaviour of $\Phi^{(1)}(k, x)$ in terms of the asymptotic behaviour of $\Phi^{(2)}(k, x)$ and derive the following relation between the two transmission and reflection coefficients [9–11]:

$$T_1 = \frac{W_+ - ik}{W_- - ik} T_2 , \tag{13a}$$

$$R_1 = \frac{W_- + ik}{W_- - ik} R_2 , \tag{13b}$$

where $W_\pm = W(x\to \pm\infty)$. We have assumed that $W_+^2 = W_-^2$ for simplicity of exposition only; this assumption implies that the potential is symmetrical asymptotically. If we also assume shape invariance, (9), then we get

$$T(k, a_1) = \frac{W_+(a_1) - ik}{W_-(a_1) - ik} T(k, a_2) , \tag{14a}$$

$$R(k, a_1) = \frac{W_-(a_1) + ik}{W_-(a_1) - ik} R(k, a_2) . \tag{14b}$$

Hence we find a recursion relation with W being the same function but with a different value of the parameters. In particular if there is some parameter set a_N such that the transmission and reflection coefficient is known, then we have

$$T(k, a_1) = \prod_{n=1}^{N-1} \frac{W_+(a_n) - ik}{W_-(a_n) - ik} T(k, a_N), \tag{15}$$

$$R(k, a_1) = \prod_{n=1}^{N-1} \frac{W_-(a_n) + ik}{W_-(a_n) - ik} R(k, a_N). \tag{16}$$

As an example we take the Pöschl–Teller potential; $W(x, a_1) = \alpha \tanh(x)$,

$$V(x, a_n) = a_n^2 - \frac{a_n(a_n+1)}{\cosh^2(x)} , \tag{17a}$$

where

$$a_n = \alpha - n + 1 . \tag{17b}$$

If $\alpha = N-1$, then $a_N = 0$ and the potential vanishes: hence the transmision coefficient is unity. Thus (15) gives

$$T(k, \alpha) = \frac{\Gamma(-ik-\alpha)\,\Gamma(-ik+\alpha+1)}{\Gamma(-ik)\,\Gamma(1-ik)} . \tag{18}$$

When the potential vanishes, the reflection coefficient vanishes. From (16) we see that the reflection coefficient vanishes for all integer values of α. This means that

$$R(k, \alpha) = \sin(\pi\alpha)\, R_0(k, \alpha) . \tag{19}$$

For α small the potential will be small and $R_0(k, 0) = (1/\pi)\dot{R}(k, 0)$, where $\dot{R}$ is the derivative of R with respect to α. At small α we can use the Born approximation

$$\dot{R}(k, 0) = \int e^{2iky}\dot{W}(y, 0)\, dy \tag{20}$$

and we finally get

$$R(k, \alpha) = \frac{i\sin(\pi\alpha)}{\sinh(\pi k)} T(k, \alpha) . \tag{21}$$

Next, we consider the spherically symmetric three-dimensional Schrödinger equation:

$$H_1 = -\frac{d}{dr^2} + \frac{l(l+1)}{r^2} + V_1 = A^\dagger A , \tag{22}$$

$$H_2 = -\frac{d}{dr^2} + \frac{(l+1)(l+2)}{r^2} + V_2 = AA^\dagger , \tag{23}$$

where

$$A^\dagger = -\frac{d}{dr} - \frac{l+1}{r} + W(r, a_1) . \tag{24}$$

We see from the above that the partner potential will be a solution for angular momentum increased by one.

The asymptotic radial wavefunction for partial wave l is

$$\Psi_2(r, l, a) \to \frac{1}{2k} [S_{l+1}(k, a)\, e^{ikr} - (-1)^{l+1} e^{-ikr}] . \tag{25}$$

where S_l is the scattering function for the lth partial wave.

Using supersymmetry and shape invariance as before, we find

$$S_l(k, a_1) = \frac{ik - W(\infty, a_1)}{ik + W(\infty, a_1)} S_{l+1}(k, a_2). \tag{26}$$

For the Coulomb potential $W = (2l+2)^{-1}$ which is shape invariant. Solving the recursion relation in (26) we derive the well-known result

$$S_{l+1}(k) = \frac{\Gamma(l+1+(2ik)^{-1})\, \Gamma(1-(2ik)^{-1})}{\Gamma(l+1+(2ik)^{-1})\, \Gamma(1+(2ik)^{-1})} S_0(k) , \tag{27}$$

which gives the scattering function for all l in terms of that for $l=0$. The term S_0 does not contribute to the angular distribution since $|S_0|^2 = 1$. Although the Coulomb scattering does not have the form (25) asymptotically, the result (27) still follows.

If we identify the operators A and $A^\dagger$ with ladder operators of a potential group, then the recursion relations we have obtained can be given a group algebraic interpretation [12,13].

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