Confinement and the quark Fermi-surface in SU(2N) QCD-like theories

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(Dated: May 6, 2010)

Yang-Mills theories with a gauge group $SU(N_c \neq 3)$ and quark matter in the fundamental representation share many properties with the theory of strong interactions, QCD with $N_c = 3$. We show that, for N_c even and in the confinement phase, the quark determinant is independent of the boundary conditions, periodic or anti-periodic ones. We then argue that a Fermi sphere of quarks can only exist under extreme conditions when the centre symmetry is spontaneously broken and colour is liberated. Our findings are supported by lattice gauge simulations for $N_c = 2 \ldots 5$ and are illustrated by means of a simple quark model.

PACS numbers: 11.15.Ha, 12.38.Aw, 12.38.Gc

In particle physics, understanding the theory of strong interactions, i.e., QCD, under extreme conditions is key for a successful description of exotic matter which features in the evolution of the universe or cold compact stars. The property of QCD which is most relevant for shaping the QCD phase diagram (if presented as a function of temperature and matter density) is colour confinement. Under normal conditions, confinement implies that quark matter is organised in terms of hadrons only, and quarks and gluons are only part of the particle spectrum under extreme conditions.

By means of lattice gauge simulations, a precise picture of hot QCD matter at small densities has emerged over the last two decades: Centre symmetry is spontaneously broken in the hot de-confinement phase for temperatures above a certain critical value. Colour is liberated and, in a theory with heavy quarks only, the Polyakov line expectation value serves as an order parameter [1, 2].

On the other hand, the situation at high densities and small temperatures is far from being clear. The reason is the lack of first principle QCD results which would help to scrutinise the proposals for the properties of matter in this regime. Lattice gauge simulation techniques cannot be applied because of the severity of the so-called sign or overlap problem. These problems are absent for the so-called two-colour QCD, and intriguing results, even for large quark chemical potential, have been accumulated over the recent years [3–5]. For an investigation of cold and dense matter in $SU(N_c \geq 3)$ QCD(-like) theories, we are still awaiting major conceptual achievements. Promising recent attempts abandon standard lattice Monte-Carlo techniques and are based upon stochastic quantisation [6] or worldline numerics [7].

Perturbative QCD and QCD inspired quark models have been a valuable tool for revealing mechanisms which might operate in the cold and dense phase of QCD. On this basis, the QCD phase diagram has gained a lot of renewed interest when findings suggested that its structure is far more complex than it had been suggested for decades: at the highest densities, it is expected that quark matter is forming a colour-superconductor which carries along a rich phase structure on its own (for a recent review see [8]). Studies of the Gross-Neveu model indicate that dense QCD (at low temperatures) might form an inhomogeneous baryonic crystal [9–11].

Employing arguments based upon the large N_c expansion, it has been recently suggested that the tight relation between confinement and spontaneous chiral symmetry breaking (inherent for zero density QCD) gets alleviated [12, 13]. In the phase diagram as a function of chemical potential and temperature, the phase boundary for chiral restoration might deviate from the boundary for deconfinement. Most interesting, a phase for which confinement is still intact while chiral symmetry is restored has attracted a lot of interest. This so-called *quarkyonic* phase is characterised by a Fermi sphere of quarks while the outer shell of the Fermi sphere necessarily consists of baryons because of confinement [12].

In this paper, we point out that the properties of dense quark matter in $SU(N_c)$ QCD-like theories are vastly different depending on whether the number of colours, N_c, is even or odd. In the confining phase, the quark determinant is averaged over centre-transformed gluonic background fields. Underpinned by lattice gauge simulations, we will show that this averaged quark determinant is insensitive to the boundary conditions of the quarks, periodic or anti-periodic ones. Since the formation of a quark Fermi sphere is crucially linked to anti-periodic boundary conditions (see below for an illustration by a quark model), we will argue that a Fermi sphere of quarks can only exist in the deconfined phase of $SU(N_c)$ QCD-like theories with N_c being even.

QCD-like theories are $SU(N_c)$ Yang-Mills theories coupled to fermions ("quarks") in the fundamental representation. We here adopt the lattice regularisation based upon a toroidal space-time lattice with lattice spacing a and extension $N_t \times N_s^3$. The gluonic degrees of freedom $U_\mu(x) \in SU(N_c)$ satisfy periodic boundary conditions, e.g., $U_\mu(x_0 + N_t a, \vec{x}) = U_\mu(x_0, \vec{x})$. Quark fields $q(x)$ are associated with the lattice sites. Because of the Fermi statistics of the bare quark fields, these fields satisfy antiperiodic boundary conditions:

$$q(x_0 + N_t a, \vec{x}) \;=\; (-1)\, q(x_0, \vec{x}).$$

Using the Wilson action for the gluonic fields, the partition function is given by

$$\mathcal{Z} \;=\; \int \mathcal{D}U_\mu \; \mathrm{Det}_A M[U] \; \mathrm{e}^{\frac{\beta}{N_c} \sum_{x,\mu\nu} \; \mathrm{Re\,tr}\, P_{\mu\nu}(x)}, \quad (1)$$

where $\beta = 2N_c/g^2$ is given in terms of the Yang-Mills gauge coupling g, and $P_{\mu\nu}$ is the standard plaquette. The determinant in (1) arises from the integration over the quark fields. The subscript "A" indicates that the quarks were subjected to antiperiodic boundary conditions. Here, we work with Wilson quarks where

$$\begin{aligned} M[U] \;&=\; (m+4)\delta_{xy} - \frac{1}{2} \sum_{\mu=1}^{4} \Big[(1-\gamma_\mu)\, U_\mu(x)\, \delta_{x+\mu,y} \\ &+\; (1+\gamma_\mu)\, U^\dagger_\mu(x-\mu)\, \delta_{x-\mu,y} \Big] \,, \end{aligned} \quad (2)$$

where γ_μ are the Hermitian Dirac matrices and m is the current quark mass in units of the lattice spacing. For an investigation of the quark determinant, we insert

$$1 \;=\; \int dQ \; \delta\Big(Q - \mathrm{Det}_A M[U] \Big)$$

into (1) to write the partition function as

$$\mathcal{Z} \;=\; \int dQ \, Q \, P_A(Q) \,, \quad (3)$$

$$P_A(Q) \;=\; \int \mathcal{D}U_\mu \, \delta\Big(Q - \mathrm{Det}_A M[U] \Big) \, \mathrm{e}^{S_{\mathrm{YM}}[U]}, \quad (4)$$

where $P_A(Q)$ is the probability distribution of the quark determinant. Note that $P_A(Q)$ can be calculated using Monte-Carlo techniques for *pure* Yang-Mills theory. Note that the probability distribution of the quark determinant of full QCD, $P_{\mathrm{full}}(Q)$, is related to that of pure Yang-Mills theory, i.e., $P_A(Q)$, by $P_{\mathrm{full}}(Q) = Q P_A(Q)$. In particular for large lattices, little statistics is expected for the large Q regime which is more relevant for the simulation of Yang-Mills theory with dynamical quarks included. If this regime is under considerations, it is advisable to include the determinant in the simulation [14] and to use refined simulation techniques such as those in [15, 16] when finite temperatures and densities are addressed. The intermediate Q regime will turn out to be sufficient to illustrate our findings below, and thus only quenched simulations are used throughout this paper.

Let us now consider a Roberge-Weiss transformation [17] in the gluonic functional integral in (4). For a fixed $x_0 = t$, we consider

$$U_0(t, \vec{x}) \;\to\; z_n \, U_0(t, \vec{x}) \qquad \forall \vec{x}\,, \quad (5)$$

$$z_n \;=\; \exp\Big\{ \frac{2\pi i}{N_c} n \Big\}, \quad 0 \le n \le N_c - 1. \quad (6)$$

Given the invariance of the gluonic action and of the Haar measure $\mathcal{D}U_\mu$, we find:

$$P_A(Q) = \int \mathcal{D}U_\mu \, \delta\Big(Q - \mathrm{Det}_A M[z_n U] \Big) \, \mathrm{e}^{S_{\mathrm{YM}}[U]}. \quad (7)$$

Reintroducing the quark fields for a moment,

$$\mathrm{Det}_A M[z_n U] = \int \mathcal{D}q \, \mathcal{D}\bar{q} \; \exp\Big\{ \sum_{xy} \bar{q}(x) M_{xy} q(y) \Big\}. \quad (8)$$

we explore the virtue of the transformation:

$$U^\Omega_\mu(x) \;=\; \Omega(x)\, U_\mu(x)\, \Omega^\dagger(x+\mu), \quad (9)$$

$$q^\Omega(x) \;=\; \Omega(x)\, q(x) \,, \quad (10)$$

$$\Omega(x_0, \vec{x}) \;=\; z_n \text{ for } x_0 > t \,, \quad \Omega(x_0, \vec{x}) = 1 \text{ else.} \quad (11)$$

Note that t has been defined above (5). We point out that $\Omega(x)$ satisfies the boundary condition $\Omega(x_0 + N_t a, \vec{x}) = z_n \Omega(x_0, \vec{x})$. The transformation (9) does not change the boundary conditions of the link fields and leaves the Yang-Mills action invariant. It can be therefore considered as a gauge transformation of pure $SU(N_c)$ Yang-Mills theory. Note, however, that the transformation (10) does change the boundary conditions of the quark fields:

$$q^\Omega(x_0 + N_t a, \vec{x}) \;=\; (-1)\, z_n \, q^\Omega(x_0, \vec{x}). \quad (12)$$

It therefore does not qualify as a gauge transformation of the full theory. It, however, reveals an important property of the distribution $P_A(Q)$. Specialising to an *even* number of colours and choosing $n = N_c/2$, $z_n = -1$, we obtain from (8) using (9-11):

$$\mathrm{Det}_A M[z_n U] \;=\; \mathrm{Det}_P M[U] \,, \quad (13)$$

where the subscript "P" signals that the quark fields now obey *periodic* boundary conditions. Inserting the last result (13) into (7) and using the gauge invariance of Yang-Mills action and Haar measure, we finally obtain:

$$P_A(Q) \;=\; \int \mathcal{D}U_\mu \, \delta\Big(Q - \mathrm{Det}_P M[U] \Big) \, \mathrm{e}^{S_{\mathrm{YM}}[U]} = P_P(Q).$$

In QCD-like theories for an even number of colours and, at least, for a finite volume (see comment below), the partition function is independent of the choice of the quark boundary conditions, periodic or antiperiodic ones. In the infinite volume limit, the centre symmetry corresponding to the transformation (6) is not always realised: it has been known for a long time [1, 2] that this symmetry is spontaneously broken under extreme conditions, high temperature and/or fermion densities. While periodic boundary conditions are associated with an instability of the partition function for vanishing temperature corresponding to Bose-Einstein condensation, antiperiodic boundary conditions are the essential ingredient for building up a Fermi sphere in the dense phase. Hence, we argue that in the confining phase of $SU(2N)$ QCD-like theories, a quark Fermi surface is unlikely to exist. We

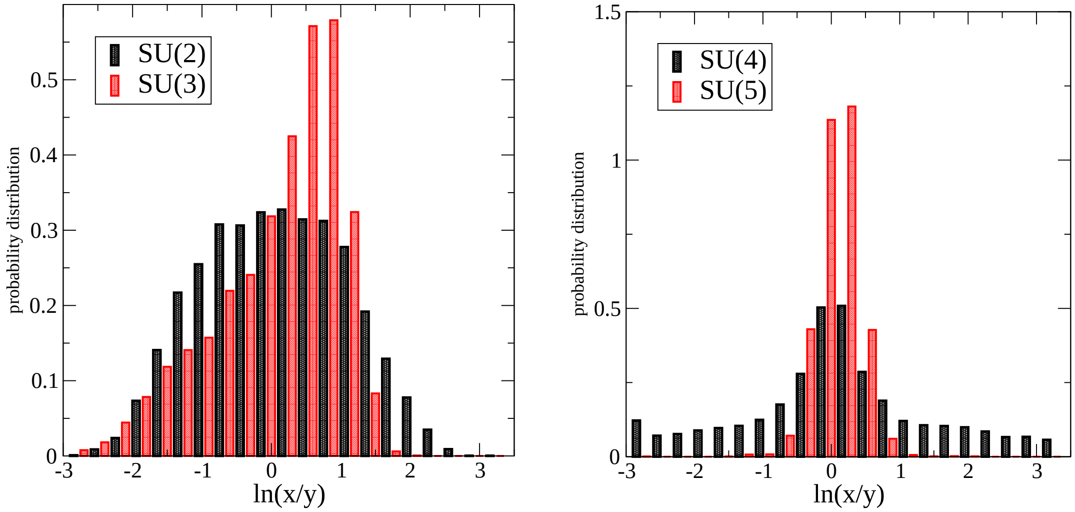

FIG. 1: Probability distribution for $\ln r$ in comparison for SU(2) and SU(3) (left) and for SU(4) and SU(5) (right).

stress, however, that the deconfinement phase at high densities (and low temperatures) might well feature a Fermi sphere of quarks.

We are now going to illustrate our findings using lattice gauge simulations as well as a simple quark model. We have carried out simulations for SU(N_c), $N_c = 2, 3, 4, 5$ using a 4^4 space time lattice and a Dirac mass $m = 0.01$. The Wilson parameters β have been chosen such that in all cases the lattice spacing a in units of string tension σ was roughly constant, i.e., $\sigma a^2 = 0.467(10)$. Keeping σa^2 fixed when the number of colours N_c is increased also complies with the so-called 't Hooft limit where $g^2 N_c$ =constant. The simulation parameters are summarised in table I.

group	SU(2)	SU(3)	SU(4)	SU(5)
β	2.2	5.63	10.99	16.57
$g^2_{\text{naive}} N_c$	3.63	3.2	2.9	3.0

TABLE I: Simulation parameters; 9902 configurations were used for SU(2) and SU(3), 8000 for SU(4) and 4500 for SU(5) to estimate the expectation values.

The determinants have been calculated exactly using the standard LU-decomposition. In order to explore the sensitivity of the quark determinants to the boundary condition, we define

$$x = \frac{\text{Det}_P M[U]}{\text{Det}_A M[U=1]}, \quad y = \frac{\text{Det}_A M[U]}{\text{Det}_A M[U=1]}, \qquad (14)$$

and consider the ratio $r[U] := x/y$ for a given lattice configuration. For N_c *even*, we find with the results above that

$$r\,[z_n U] = 1/r[U] \quad \Rightarrow \quad \langle r \rangle = \langle 1/r \rangle \qquad (15)$$

if the centre-symmetry is realised. Our numerical findings for these expectation values are summarised in table II.

	SU(2)	SU(3)	SU(4)	SU(5)
$\langle r \rangle$	1.66 ± 0.02	1.365 ± 0.009	2.9977 ± 0.07	1.0469 ± 0.005
$\langle 1/r \rangle$	1.65 ± 0.02	1.587 ± 0.02	2.87 ± 0.07	1.0571 ± 0.007

TABLE II: Sensitivity parameters $\langle r \rangle$ and $\langle 1/r \rangle$ for several gauge groups.

We here find a clear coincidence between $\langle r \rangle$ and $\langle 1/r \rangle$ for the gauge group SU(2) while the corresponding parameters for SU(3) are significantly different. We also observe a tendency that the difference fades away for increasing number of colours N_c. We finally present the probability distribution for the variable $\ln r$ in figure 1. For gauge groups with N_c even, we expect that the corresponding histograms are symmetric with respect to $\ln r \to -\ln r$ (see 15). This expectation is nicely confirmed for the gauge groups SU(2) and SU(4). A clear asymmetry is observed for SU(3) while the asymmetry is very small for SU(5) if present at all. We also empirically observe that the width of the probability distribution decreases for an increasing number of colours.

Let us finally illustrate by means of a quark model how centre sector tunneling eliminates the quark Fermi surface. The theory which we are proposing is essentially a free theory of quarks which, however, interact with a constant centre background field. We here work in the ab initio continuum formulation. Parameterizing the centre background field by (H from the Cartan algebra)

$$A_n = 2\pi n \, T \, H, \quad 0 \le n \le N_c - 1, \qquad (16)$$

(where T is the temperature) the partition function of our model is in Euclidean space

$$Z = \sum_n p_n \, \frac{\int \mathcal{D}q\mathcal{D}\bar{q} \, \exp\{\bar{q}(i\partial\!\!\!/ + (A_n + i\mu)\gamma_0 + im)q\}}{\int \mathcal{D}q\mathcal{D}\bar{q} \, \exp\{\bar{q}(i\partial\!\!\!/ + im)q\}}, \qquad (17)$$

where m is the quark mass and μ is quark chemical potential. Thereby, p_n is the probability that the centre sector

n is attained. In the quenched approximation, all centre sectors occur with equal probability, i.e., $p_n = 1/N_c$. Note, however, that in a more QCD relevant setting dynamical quarks induce a bias towards the trivial centre sector, i.e., $p_0 > p_{n\neq 0}$. Furthermore, A_n acts as constant gauge field which can be eliminated from the action at the expense of changing the boundary conditions for the quark fields. The partition function can be calculated in closed form:

$$Z = \sum_n p_n \prod_{\vec{p},\vec{q}} \left(1 + z_n \mathrm{e}^{-[E(\vec{p})-\mu]/T}\right) \left(1 + z_n^\dagger \mathrm{e}^{-[E(\vec{q})+\mu]/T}\right),$$

where z_n is related to the trace of the Polyakov line P line by

$$\frac{1}{N_c}\,\mathrm{tr}P = \exp\{iA_n/T\} = z_n.$$

Moreover, the product extends over spatial momenta $\vec{p}$ and $E(\vec{p}) = (m^2+\vec{p})^{1/2}$. Expanding the brackets, because of the sum over the centre elements only states with vanishing N-ality contribute to the partition function asserting confinement. We here focus on the BEC instability. We consider temperatures T which are small compared to the mass gap implying that we may neglect the contribution of antiquarks to the free energy for $\mu \gtrsim m$. The free energy is then given by $\ln Z \approx \ln \sum_n p_n\, \rho_n$ with

$$\rho_n \;=\; \exp\Big\{\frac{V}{\pi^2}\int_m^\infty dE\, E\,\sqrt{E^2-m^2}\,\ln\left(1 + z_n \mathrm{e}^{-\frac{E-\mu}{T}}\right)\Big\},$$

where V is the spatial volume. Because of the spin of the quarks, we have used two states per Matsubara mode. For the search for Fermi surface effects, it is most instructive to study the baryon number density:

$$b \;=\; \frac{T}{V}\frac{\partial \ln Z}{\partial \mu} = \frac{1}{\pi^2}\int_m^\infty dE\; E\;\sqrt{E^2-m^2}\;\rho(E,T,\mu),$$

where ρ is defined by

$$\rho(E,T,\mu) \;=\; \sum_n \frac{z_n}{\mathrm{e}^{[E-\mu]/T} + z_n}\; w_n\;, \qquad (18)$$

with the weights $w_n = p_n\,\rho_n / \sum_i p_i\,\rho_i$. In the deconfined phase, tunneling between centre sectors stops because of spontaneous symmetry breaking (on top of the explicit breaking by the quark determinant). The sum over the centre elements collapses to the trivial centre element $p_0 = 1$, $p_{n\neq 0} = 0$ implying $w_0 = 1$, $w_{n\neq 0} = 0$. In this case, $\rho(E,T,\mu)$ can be interpreted as spectral density. This is given by the familiar Fermi function $\rho_{\mathrm{decon}}(E,T,\mu) = [\mathrm{e}^{[E-\mu]/T} + 1]^{-1}$, which features a Fermi surface for $E \approx \mu$.

Let us now consider the confinement phase of a SU(2) gauge theory. Despite of the bias towards the trivial centre sector, we have $p_0 < 1$ because of centre sector tunneling. The crucial observation is that while increasing the chemical potential μ to approach the mass m from below, ρ_1 develops a singularity while ρ_0 is perfectly finite. This implies that the weights are given by $w_0 = 0$, $w_1 = 1$ for $\mu \to m$. Thus, $\rho(E,T,\mu)$ is approximately given by

$$\rho(E,T,\mu) \approx -\frac{1}{\mathrm{e}^{[E-\mu]/T} - 1}$$

and, hence, signals the BEC instability for $\mu \to m$. Note that the sign of ρ is dictated by the centre element $z_1 = -1$. Because of this sign, the contribution to the baryon number density is negative and arises from centre dressed quarks. This contribution is genuinely different from the contribution from bare anti-quarks which can be neglected for the present choice of parameters.

In conclusions, we have shown that the quark determinant of $SU(2N)$ gauge theories does not depend on the type of boundary conditions for the quark fields as long as the centre symmetry is realised. Lattice gauge simulations for the gauge groups $SU(2\ldots5)$ corroborate these findings. Our results supplement the recent findings of [18] where SU(2) deconfinement was brought in line with the sensitivity of the quark determinant to the boundary conditions. Our results may have far reaching phenomenological implications: (i) they exclude the (pre)formation of a quark Fermi sphere at finite densities in the confinement phase; (ii) since quarks are dressed with centre fields of the gluonic background they escape the spin-statistics connection. This is in analogy to the anyons in solid state physics [19]. While anyons only occur in 2+1 dimensions, our model is the first one of its kind which evades the spin-statistics connection in 3+1 dimensions. (iii) Exotic states of matter, such as a Bose-Einstein condensate of quarks, might exist prior to deconfinement induced by density.

Acknowledgments: Helpful discussions with G. Dunne, H. Gies, K.Ya. Glozman and M. Rho are gratefully acknowledged.

[1] B. Svetitsky and L. G. Yaffe, Nucl. Phys. B **210**, 423 (1982).
[2] B. Svetitsky, Phys. Rept. **132**, 1 (1986).
[3] J. B. Kogut, D. K. Sinclair, S. J. Hands and S. E. Morrison, Phys. Rev. D **64**, 094505 (2001) [arXiv:hep-lat/0105026].
[4] S. Conradi, A. D'Alessandro and M. D'Elia, Phys. Rev. D **76**, 054504 (2007) [arXiv:0705.3698 [hep-lat]].
[5] S. Hands, S. Kim and J. I. Skullerud, PoS **LAT2005**, 149 (2006) [arXiv:hep-lat/0508027].

[6] G. Aarts and I. O. Stamatescu, JHEP **0809**, 018 (2008) [arXiv:0807.1597 [hep-lat]].
[7] G. Dunne, H. Gies, K. Klingmuller and K. Langfeld, arXiv:0903.4421 [hep-th].
[8] M. G. Alford, A. Schmitt, K. Rajagopal and T. Schafer, Rev. Mod. Phys. **80**, 1455 (2008) [arXiv:0709.4635 [hep-ph]].
[9] O. Schnetz, M. Thies and K. Urlichs, Annals Phys. **314**, 425 (2004) [arXiv:hep-th/0402014].
[10] O. Schnetz, M. Thies and K. Urlichs, Annals Phys. **321**, 2604 (2006) [arXiv:hep-th/0511206].
[11] C. Boehmer, M. Thies and K. Urlichs, Phys. Rev. D **75**, 105017 (2007) [arXiv:hep-th/0702201].
[12] L. McLerran and R. D. Pisarski, Nucl. Phys. A **796**, 83 (2007) [arXiv:0706.2191 [hep-ph]].
[13] L. McLerran, K. Redlich and C. Sasaki, arXiv:0812.3585 [hep-ph].
[14] S. Duane, A. D. Kennedy, B. J. Pendleton and D. Roweth, Phys. Lett. B **195**, 216 (1987).
[15] Z. Fodor and S. D. Katz, JHEP **0203**, 014 (2002) [arXiv:hep-lat/0106002].
[16] Z. Fodor and S. D. Katz, JHEP **0404**, 050 (2004) [arXiv:hep-lat/0402006].
[17] A. Roberge and N. Weiss, Nucl. Phys. B **275**, 734 (1986).
[18] E. Bilgici, C. Gattringer, E. M. Ilgenfritz and A. Maas, arXiv:0904.3450 [hep-lat].
[19] Frank Wilczek, *Fractional Statistics and Anyon Superconductivity*, World Scientific, (1 Mar 1990).