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Consistency of String Propagation on Curved Spacetimes: An SU(1,1) based Counterexample

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ABSTRACT

String propagation on non-compact group manifolds is studied as an exactly solvable example of propagation on more general curved space-times. It is shown that for the only viable group $SU(1,1) \times G_c$ string propagation is consistent classically but not quantum mechanically (unitarity is violated). This shows that conformal invariance of the corresponding σ-model (vanishing of the β-functions) is not sufficient to guarantee unitarity.

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1. Introduction

Classical two dimensional conformally invariant sigma models can be interpreted as string propagation on the target space. It is generally believed that if the target space has one time-like direction and the corresponding beta functions vanish then the quantized version of the sigma model is consistent (unitary) and describe the quantum- mechanical string propagation [1]. To investigate the problem of unitarity it is very convenient to use exactly solvable models and as is known, principal sigma-models (with a Wess-Zumino-Witten term) are of this kind and lead to a realization of a Kac-Moody algebra [2]. The $\mathcal{M} = M_4 \times G_c$ sigma model, where M_4 is 4-dimensional Minkowski space and G_c is a compact group, has of course been shown to be unitary and describe string propagation on $\mathcal{M}$. It would be desirable to extend this manifold to more general Einstein-spaces, but in seeking solvable models one is limited to the case of $\mathcal{M}$ being a group manifold. Actually $\mathcal{G} = SU(1,1) \times G_c$ is the only group manifold with a single, nontrivially embedded time-like direction. So in this paper we should like to consider the $\mathcal{G}$–model as a first step beyond $\mathcal{M} = M_4 \times G_c$ in the study of string propagation on general curved spaces. This model constitutes a rather small step in generalizing $\mathcal{M}$ but it already raises serious questions for more general curved spaces, since as we shall show the $\mathcal{G}$–model does not admit consistent (unitary) quantum-mechanical string propagation.

We recall that the most general renormalizable Weyl invariant 2-dimensional sigma model is given as

$$S = \frac{1}{2\pi\alpha'} \int d^2\xi \Big(\frac{1}{2}\sqrt{\gamma}\gamma^{ab} G_{\mu\nu}(X)\, \partial_a X^\mu \partial_b X^\nu + \frac{1}{2}\epsilon^{ab} B_{\mu\nu}(X)\, \partial_a X^\mu \partial_b X^\nu \Big), \quad (1.1)$$

where $G_{\mu\nu}$ and $B_{\mu\nu}$ can be identified as the background spacetime metric and an antisymmetric tensor field, respectively.

Weyl invariance on the world sheet, which is crucial for the consistency of the string theory is not automatically guaranteed. To account for this anomaly one adds the non Weyl-invariant piece

$$S_d = \frac{1}{4\pi} \int d^2\xi \sqrt{\gamma}\, R^{(2)} \Phi(X) \quad (1.2)$$

to the action (1.1), where $R^{(2)}$ denotes the scalar curvature of the world sheet and $\Phi(X)$ is the background dilaton field. Weyl-invariance can be expressed as the tracelessness of the energy-momentum tensor. This leads to demanding the vanishing of the beta-functions β^{Φ}, β^{G}, β^{B}. Perturbative calculations of the beta-functions using background field expansion to one-loop yield the Einstein equations and the equations of motion for the dilaton field and the $B_{\mu\nu}$ field. Higher loop calculations seem not to spoil this interpretation.

For the case of group-manifolds it has been observed [3] that the Wess-Zumino-Witten (WZW) term in (1.1) has the interpretation of introducing a parallelizing torsion field, that is to add to the standard Riemannian connection $\Gamma^{\mu}_{\nu\rho}$ an $H^{\mu}_{\nu\rho}$ term coming from $\nabla^{\mu}B_{\nu\rho}$ which makes the generalized Riemann tensor zero. For any group manifold such a parallelizing torsion exists. Furthermore, since the generalized Einstein equations are expressed as polynomials in this generalized Riemann tensor and its covariant derivatives, they are automatically satisfied if the curvature tensor identically vanishes [4].

Motivated by the above considerations we consider a string moving on the manifold of a noncompact simple group, G. Just as in the compact case [2] we consider the standard action

$$\begin{aligned} S = &\frac{k}{16\pi}\int d\sigma d\tau \eta^{ab}\,\mathrm{tr}\Big((g^{-1}\partial_a g)(g^{-1}\partial_b g)\Big) \\ &+\frac{k}{24\pi}\int d^3\xi \epsilon^{abc}\,\mathrm{tr}\Big((g^{-1}\partial_a g)(g^{-1}\partial_b g)(g^{-1}\partial_c g)\Big) \end{aligned} \tag{1.3}$$

together with the closed string boundary conditions

$$g(0,\tau) = g(2\pi,\tau). \tag{1.4}$$

Introducing the complex variables $z = \exp(i(\tau+\sigma))$ and $\bar{z} = \exp(i(\tau-\sigma))$ the action has a noncompact Kac-Moody symmetry generated by the left and right currents

$$J(z) = -\frac{k}{2}\,\partial_z g\, g^{-1} \qquad \bar{J}(\bar{z}) = -\frac{k}{2}\,\partial_{\bar{z}} g^{-1}\, g. \tag{1.5}$$

Since J is an analytic function of z, expanding into a Laurent series

$$J(z) = \sum_{n=-\infty}^{\infty} J_n\, z^{-n-1} \tag{1.6}$$

the Laurent coefficients J_n^a form a Kac-Moody (KM) algebra (and similarly for $\bar{J}(\bar{z})$ and $\bar{J}_n^a$).

The energy momentum tensor is just the Sugawara one:

$$T(z) = \frac{1}{2\kappa} g_{ab} : J^a(z) J^b(z) : \tag{1.7}$$

where κ is a constant. The Laurent coefficients of $T(z)$, L_n, are the generators of the Virasoro algebra, (similarly for $\bar{T}(\bar{z})$, $\bar{L}_n$). Since the Virasoro generators correspond to reparametrization of the string we impose the usual constraints

$$\begin{aligned} L_n \,|\,phys\rangle = \bar{L}_n \,|\,phys\rangle &= 0 \qquad \text{for} \qquad n > 0 \\ (L_0 - \bar{L}_0)\,|\,phys\rangle &= 0 \\ (L_0 + \bar{L}_0 - 2)\,|\,phys\rangle &= 0. \end{aligned} \tag{1.9}$$

We will discuss the structure of the physical Hilbert space, i.e. the set of states satisfying the constraints (1.9) in sections 3 and 4 in detail. The main subject of this paper is to show that for completely consistent string propagation it is not sufficient that the resulting sigma model is conformally invariant. It is also necessary that the quantum mechanical propagation of the string be unitary. In fact, we will see that for the very special and simple compactification corresponding to string propagation on a non-compact group manifold, while the beta functions vanish, the quantum theory is never unitary.

2. Classical String Propagation on $\widetilde{SU(1,1)}$

In this section we study the classical string motion and show that it is completely consistent (causal). This is not a priori guaranteed since space-time is curved and strings are extended objects. Indeed we shall see that we have to work on the universal covering group $\widetilde{SU(1,1)}$ of $SU(1,1)$.

The group $SU(1,1)$ can be parametrized as

$$g = x^0 + x^i\tau_i \quad \text{where} \quad \vec{\tau} = (\sigma_1, \sigma_2, i\sigma_3) \tag{2.1}$$

and the coordinates x^μ lie on the hyperboloid

$$(x^0)^2 + (x^3)^2 - (x^1)^2 - (x^2)^2 = 1. \tag{2.2}$$

It follows, in particular, that $(x^0)^2+(x^3)^2 \geq 1$, so there are non-contractible loops in the $x^0 - x^3$ plane. We see that $SU(1,1)$ is topologically equivalent to $R^2 \times S^1$. This also follows from the Cartan-Mal'cev-Iwasawa theorem, stating that a noncompact group is topologically the same as $R^n \times G_c$, where G_c is its maximal compact subgroup.

The constraint (2.2) can be satisfied by introducing the coordinates

$$(x^0, x^1, x^2, x^3) = (\cosh\rho\ \cos t, \sinh\rho\ \cos\phi, \sinh\rho\ \sin\phi, \cosh\rho\ \sin t) \tag{2.3}$$

so that

$$g = \begin{pmatrix} \cosh\rho\ e^{it} & \sinh\rho\ e^{-i\phi} \\ \sinh\rho\ e^{i\phi} & \cosh\rho\ e^{-it} \end{pmatrix}. \tag{2.4}$$

In these coordinates the natural metric of the group reads

$$ds^2 = -\frac{1}{2}\mathrm{tr}\,(g^{-1}dg)^2 = \cosh^2\rho\ dt^2 - \left(d\rho^2 + \sinh^2\rho\ d\phi^2\right). \tag{2.5}$$

Note that the 'time' coordinate t is periodic with period 2π. Since the group-manifold is to be interpreted as our space-time it is therefore more natural to consider the universal covering group $\widetilde{SU(1,1)}$ which is topologically $R^2 \times R^1$. Then the periodicity in t is lost and time extends to the whole real line.

To understand the classical string propagation it is useful to consider first the motion of free point particles on $\widetilde{SU(1,1)}$. Free particles move along time- or light-like geodesics

$$g(\tau) = g(0)\exp(v(0)\,\tau) \quad \text{where} \quad v(\tau) = v^i(\tau)\tau_i = g^{-1}(\tau)\dot{g}(\tau), \tag{2.6}$$

and the initial velocity $v(0)$ is time- or light-like, $v(0) \cdot v(0) = m^2 \geq 0$. Here and in what follows the norms are with respect to the lorentzian metric $(-,-,+)$. In particular, the geodesics passing through the identity are given by

$$
\begin{aligned}
t &= \arctan\left(\frac{\alpha}{m}\tan(m\tau)\right) \\
\cosh^2\rho &= \cos^2(m\tau) + \frac{\alpha^2}{m^2}\sin^2(m\tau) \\
\phi &= \phi_0 = const,
\end{aligned}
\tag{2.7}
$$

where (t,ρ,ϕ) are the coordinates introduced in (2.3). Again one sees that one must use the covering group to avoid periodicity in time. Then $t(\tau)$ can be defined so that it covers the whole real line. In the massless limit $m \to 0$ the geodesics become light-like, describing the motion of point-like strings.

The propagation of extended strings is determined by the action (1.3). The general solution of the field equations is

$$
g(\sigma,\tau) = A(\sigma+\tau)B^{-1}(\sigma-\tau) \tag{2.8}
$$

where A and B are group elements determined by the initial conditions and are such that g is periodic in σ. Due to reparametrization invariance the energy-momentum tensor must vanish

$$
T_{--} = \mathrm{tr}(A^{-1}\dot{A}) = u \cdot u = 0 \qquad T_{++} = \mathrm{tr}(B^{-1}\dot{B}) = v \cdot v = 0, \tag{2.9a}
$$

where we have introduced the velocities

$$
A^{-1}\dot{A} = u^i\tau_i \qquad B^{-1}\dot{B} = v^i\tau_i. \tag{2.9b}
$$

These constraints further restrict the allowed initial conditions.

Clearly for the string propagation to be consistent, (causal), any point of the string should move forward in time and not faster than light. In other words, if

$$
-\mathrm{tr}\Big((g^{-1}\dot{g})^2\Big) \geq 0 \quad \text{and} \quad -\mathrm{tr}\Big(\tau_3\, g^{-1}\dot{g}\Big) > 0 \tag{2.10}
$$

are true at $\tau = 0$ then they must remain true at all later times τ. By using the light-like velocities u and v introduced in (2.9) one can show that the conditions (2.10) are equivalent to

$$
u \cdot v \geq 0 \quad \text{and} \quad u^3 + v^3 > 0. \tag{2.11}
$$

Note that the two velocities u and v depend only on $\sigma+\tau$ and $\sigma-\tau$ respectively. Using (2.11) and (2.9a) at $\tau=0$ together with Schwarz's inequality, one can show that u^3 and v^3 must be separately positive at $\tau=0$

$$u^3(\sigma)\geq 0 \quad \text{and} \quad v^3(\sigma)\geq 0. \tag{2.12}$$

Since u and v are both functions of one variable, (2.12) implies that they remain separately positive at all later times, which is sufficient to guarantee that (2.11) and (2.10) are satisfied at all τ. What we have shown then is that classically both point particles and extended strings propagate consistently on the 'space-time' $\widetilde{SU(1,1)}$, i.e. in accordance with relativity and causality. The rest of this paper is devoted to the related quantum mechanical problem.

3. Kac-Moody Algebras for Non-Compact Groups

In this section we first summarize the algebraic structure underlying the problem of string propagation on non-compact group manifolds. This is a rather straigthforward generalization of the similar problem for the case of compact groups discussed in detail in [2]. Here we recall the construction of highest weight representations of KM algebras and associated Virasoro algebras mainly to establish our notations and conventions. In this section we concentrate on the left-moving part of the theory only since left- and right-moving parts are identical and they are completely decoupled because of conformal invariance.

Our starting point is the KM algebra [5]

$$[J_n^a, J_m^b] = if^{ab}{}_c J_{n+m}^c - Kg^{ab} n\delta_{n+m,0} \qquad (n,m\in Z). \tag{3.1}$$

Here the $f^{ab}{}_c$ are the real structure constants of the underlying D-dimensional simple Lie-algebra

$$[J^a, J^b] = if^{ab}{}_c J^c\,, \qquad a,b,c=1,2,...,D \tag{3.2}$$

i.e. the subalgebra generated by J_0^a, and g^{ab} is the Cartan-Killing metric defined by:

$$f^{ax}{}_y f^{yb}{}_x = Qg^{ab}. \tag{3.3}$$

where

$$g=\text{diag} < \pm 1 >, \tag{3.4}$$

and the diagonal elements are +1 (−1) for compact (non-compact) directions respectively. With this convention Q is positive, and is actually the same for all real forms of a given Lie algebra.

We use g^{ab} (and its inverse g_{ab} which is of the same form) to raise and lower indices in the usual way. The metric also enters the definition of the quadratic Casimir-operator

$$\mathbf{C} = \frac{2}{Q} g_{ab} J^a J^b. \tag{3.5}$$

$\mathbf{C}$ commutes with the generators of the Lie-algebra, and if the generators are hermitian so is $\mathbf{C}$.

Since we are interested in unitary representations of the KM algebra (3.1), we require the generators to be hermitian;

$$(J^a_n)^\dagger = J^a_{-n}, \tag{3.6}$$

which is compatible with (3.1) provided the parameter K is real.

Given the generators in (3.1) it is natural to construct the Virasoro generators:

$$L_n = \frac{1}{2\kappa} \sum_r g_{ab} : J^a_r J^b_{n-r} : \tag{3.7}$$

where $2\kappa = Q - 2K$ and the normal ordering is defined with respect to the lower indices n in the usual way. As usual the relationship between the Virasoro and KM algebra (3.1) is:

$$\begin{aligned} [L_n, J^a_m] &= -m\, J^a_{n+m} \\ [L_n, L_m] &= (n-m)\, L_{n+m} + \frac{1}{12} c_v n(n^2-1)\delta_{n+m,0}\ , \end{aligned} \tag{3.8}$$

where the central charge of the Virasoro algebra is given by:

$$c_v = \frac{2K}{2K-Q} D. \tag{3.9}$$

The simplest examples, and also the ones we will extensively use in our analysis, are the KM algebras based on the Lie-algebras $SU(2)$ and $SU(1,1)$. In both cases $Q = 2$ and the totally antisymmetric structure constants can be chosen to be the same for both cases

$$f^{abc} = \epsilon^{abc} \qquad a, b, c = 1, 2, 3. \tag{3.10}$$

But the metrics are different:

$$g = \text{diag} < \epsilon, \epsilon, 1 > , \tag{3.11}$$

where $\epsilon = 1$ and -1 for $SU(2)$ and $SU(1,1)$ respectively. From (3.9) we see that for both of these groups the central charge of the Virasoro algebra is

$$c_v = \frac{3K}{K-1} . \tag{3.12}$$

It is useful to introduce the complex basis $\{J^\pm, J^3\}$, where

$$J^\pm = J^1 \pm iJ^2 \quad , \qquad (J^\pm)^\dagger = J^\mp. \tag{3.13}$$

In this basis the algebras take the form

$$\begin{aligned} [J^3, J^\pm] &= \pm J^\pm \\ [J^+, J^-] &= 2\epsilon\, J^3. \end{aligned} \tag{3.14}$$

The metric and the Casimir-operator are given by

$$g^{33} = 1 \, , \qquad g^{++} = g^{--} = 0 \, , \qquad g^{+-} = g^{-+} = 2\epsilon \tag{3.15}$$

and

$$\mathbf{C} = J^3(J^3 - 1) + \, \epsilon J^+ J^- = J^3(J^3 + 1) + \, \epsilon J^- J^+ \tag{3.16}$$

respectively.

Let us now turn to the representations of the KM algebra (3.1). The simplest representations and also the ones relevant for string theory are the highest weight (Fock space) representations. These are constructed as follows. One first postulates the existence of the 'vacuum' representation $\{\,|\,\alpha\rangle\}$. This set of states is annihilated by all generators with positive lower index n and constitute a representation of the finite-dimensional subalgebra generated by $\{J_0^a\}$:

$$\begin{aligned} &J_n^a \mid \alpha\rangle = 0 \qquad n > 0 \\ &J_0^a \mid \alpha\rangle = \sum_\beta \tau^a_{\beta\alpha} \mid \beta\rangle \qquad\qquad [\tau^a, \tau^b] = i f^{ab}{}_c \tau^c. \end{aligned} \tag{3.17}$$

For non-compact groups the nontrivial unitary representations are of course infinite dimensional. For irreducible highest weight representations the Casimir-operator (3.5) takes the same eigenvalue on all vacuum states $\mid \alpha\rangle$

$$\mathbf{C} \mid \alpha\rangle = C \mid \alpha\rangle \qquad \text{where} \qquad C = \frac{2}{Q} g_{ab} \tau^a \tau^b. \tag{3.18}$$

Note that due to normal ordering, the positive index Virasoro generators also annihilate the vacuum states and L_0 reduces to a constant on them:

$$\begin{aligned} L_n \mid \alpha\rangle &= 0 \qquad\qquad n > 0 \\ L_0 \mid \alpha\rangle &= \frac{CQ}{2(Q-2K)} \mid \alpha\rangle. \end{aligned} \tag{3.19}$$

Starting from the vacuum representation $\{\mid \alpha\rangle\}$, the vectors of the representation can be obtained by repeatedly acting with negative lower index generators on them. The vectors of the representation can be characterized by their L_0 eigenvalue (grade) and it is easy to see that a basis for the vectors of grade N is provided by the states of the form

$$\mid \Psi_N\rangle = J^{a_1}_{-1} J^{a_2}_{-1} \ldots J^{a_N}_{-1} \mid \alpha\rangle \qquad a_1, a_2, .., a_N = 1, 2, \ldots, D. \tag{3.20}$$

Using (3.8) we can compute the actual L_0 eigenvalue of the grade-N states

$$L_0 \mid \Psi_N\rangle = \Big(\frac{CQ}{2(Q-2K)} + N\Big) \mid \Psi_N\rangle. \tag{3.21}$$

4. Investigation of unitarity.

After this review of the highest weight representation of KM-Virasoro algebras, we turn to the question of unitarity (positivity of the Fock space norms). The highest weight representations of non-compact KM algebras just discussed are never unitary because the norms of the states $J^a_{-1} \mid \alpha\rangle$

$$\mid J^a_{-1} \mid \alpha\rangle|^2 = \langle \alpha \mid J^a_1 J^a_{-1} \mid \alpha\rangle = \langle \alpha \mid [J^a_1, J^a_{-1}] \mid \alpha\rangle = -Kg^{aa}\langle \alpha \mid \alpha\rangle \tag{4.1}$$

are indefinite (since g^{ab} is). This observation in itself does not answer the question whether string propagation on non-compact group manifolds can be physical. The Hilbert space corresponding to the highest weight representations is the analogue of the Fock-space spanned by the Lorentzian oscillators in the case of the free bosonic string, which is also indefinite. The real question is whether the physical subspace defined by the Virasoro constraints (1.9) is free of negative-norm, ghost states. If we allow for an arbitrary intercept α_0, the constraints for the left-moving part of the string are

$$L_n \mid \Phi\rangle = 0 \qquad n > 0 \tag{4.2a}$$

$$L_0 \mid \Phi\rangle = \alpha_0 \mid \Phi\rangle. \tag{4.2b}$$

These constraints, which express reparametrization invariance of the physical states, are known to lead to a positive definite physical Hilbert space in the case of the free bosonic string [6]. In the case of string propagation on $M_4 \times G_c$, the Hilbert space is also restricted by these constraints, but once again their role is to eliminate the time-like oscillations from the Minkowski-space part, M_4, of the theory, thus essentially the same as for the free string. What we wish to investigate here is whether all negative-norm states of non-compact KM algebras are eliminated by the constraints (4.2). We shall see that the imposition of the constraints (4.2) is not sufficient to eliminate all negative norm states of highest weight KM representations. We shall show this by explicitly constructing 'physical' states that satisfy the Virasoro constraints (4.2) but still have negative norms. (To be precise one should not call such states physical, but for simplicity we continue to do so.)

We begin by demonstrating that as one might expect from its being the only simple non-compact group with a single time-like direction $SU(1,1)$ is the only viable candidate. At first sight this may seem very restrictive since for $SU(1,1)$ $D=3$ and c_v may be less than 26, however one can always form a direct product of of $SU(1,1)$ with some other compact group to make $D \geq 4$ (and if necessary $c_v = 26$). The idea is to note that any other non-compact simple group contains both an $SU(1,1)$ and an $SU(2)$ subgroup and that these two subgroups are mutually exclusive. In this section we consider simple non-compact groups only, since the additional compact factors, which may be necessary to account for the difference between 26 and the Virasoro charge c_v of the non-compact group, are not playing any role in the problem of positivity.

We label the vacuum states by $|jh\rangle$, where $j(j+1)$ and h are the eigenvalues of the Casimir-operator and J^3 respectively, of either $SU(2)$ or $SU(1,1)$. (Here we suppress additional quantum numbers that may be necessary to characterize them.) We now take the most general grade one state in the J^3 basis:

$$|\Phi_1\rangle = x\, J^+_{-1}\,|jh{-}1\rangle + y\, J^-_{-1}\,|jh{+}1\rangle + z\, J^3_{-1}\,|jh\rangle \tag{4.3}$$

where x, y and z are numerical coefficients. The grade-one state $|\Phi_1\rangle$ satisfies (4.2a) if

$$z = -\frac{1}{h}\Big[d_\epsilon(j,h-1)x + d_\epsilon(j,h)y\Big] \tag{4.4}$$

where

$$d_\epsilon(j,h) = \sqrt{\epsilon[j(j+1) - h(h+1)]} \tag{4.5}$$

is a normalization coefficient and $\epsilon = +1$ and -1 for the case of $SU(2)$ and $SU(1,1)$ respectively. Using (4.4) and the KM algebra, the norm of the physical state is

$$\begin{aligned}\langle\Phi_1 \mid \Phi_1\rangle = &\Big[2\epsilon(1-K-h) - (\frac{K}{h^2} + \frac{2}{h})\, d^2_\epsilon(j,h-1)\Big]x^2 \\ &+ \Big[2\epsilon(1-K+h) - (\frac{K}{h^2} - \frac{2}{h})\, d^2_\epsilon(j,h)\Big]y^2 \\ &- 2\frac{K}{h^2}\, d_\epsilon(j,h)d_\epsilon(j,h-1)xy.\end{aligned} \tag{4.6}$$

To ensure that all physical states (parametrized by x and y) have positive norms it is necessary that the trace of the matrix defining the quadratic form (4.6) be positive

$$-2\epsilon K\Big(1 + \frac{j(j+1)}{h^2}\Big) \geq 0 \tag{4.7}$$

where of course j and h are different for $SU(2)$ and $SU(1,1)$. For $SU(2)$ $\epsilon = +1$ and $j(j+1)/h^2$ is positive so (4.7) implies $K < 0$. For $SU(1,1)$ $\epsilon = -1$, and $j(j+1)/h^2$ is indefinite, but h^2 can be arbitrarily large so (4.7) implies $K > 0$. Thus $SU(2)$ and $SU(1,1)$ are mutually exclusive as claimed. So we are left with the $SU(1,1)$ group only and with the condition $K > 0$.

Let us now turn to the L_0-constraint (4.2b). From (3.21), one concludes that the physical states at grade N must correspond to vacuum representations with Casimir

$$C_N = 2\frac{2K-Q}{Q}(N-\alpha_0) \tag{4.8a}$$

which reduces for both $SU(2)$ and $SU(1,1)$ to

$$C_N = 2(K-1)(N-\alpha_0). \tag{4.8b}$$

We see that the Hilbert space decomposes into infinitely many subspaces, each one characterized by a highest weight state. According to (4.8) the physical states at each grade are built on their own distinct vacuum. From this point of view (4.8) is the analogue of the mass-shell condition for the free bosonic string,

$$p^2 = m^2 = \frac{1}{\alpha'}(N-\alpha_0), \tag{4.9}$$

p^2 playing the role of the Casimir-operator and $K-1$ that of the string tension. This shows that there is a natural division of the range of K into $0 < K < 1$ and $K > 1$.

The construction of the unitary representations of $SU(1,1)$ is summarized in Appendix A. Equation (4.8) determines the Casimir eigenvalue of the $SU(1,1)$ vacuum-representation, corresponding to grade-N physical states. Note that for large N the class of representations is determined by the magnitude of K. For $K < 1$, the asymptotic representations are the principal ones, whereas for $K > 1$ they are of the discrete, or highest weight type. We will consider these two cases separately.

Before going into details, let us define the notion of extremal representations, which will turn out to be useful in both cases. Let us start from a given irreducible vacuum representation characterized by the parameter j. By acting with the generators J^a_{-1} on the vacuum states we can generate three irreducible $SU(1,1)$ representations corresponding to j-values $j+1, j$ and $j-1$ at grade 1. This follows from the lemma of Appendix B with $V^a = J^a_{-1}$. Using this lemma m times, it is easy to see that at grade m the 3^m independent states (3.20) belong to 3^m irreducible $SU(1,1)$ representations. Their j-values range from $j - m$ to $j + m$. We will call the representations corresponding to $j \pm m$ the extremal representations:

$$\mathbf{C} \mid E^{\pm}_m\rangle = (j \pm m)(j \pm m + 1) \mid E^{\pm}_m\rangle. \tag{4.10}$$

It is not difficult to see that the representations $\mid E^{\pm}_N\rangle$ (built on the corresponding vacuum states labelled by j_N) are physical for any grade N. To prove this, we note that since the Virasoro constraints are SU(1,1)-invariant, the states $L_n \mid E^{\pm}_N\rangle$ have the same j-values as the extremal states themselves, namely $j_N \pm N$. On the other hand, the grade of these states is $m = N - n$ and hence their j-values should be in the range $j_N \pm (N - n)$. This excludes $j_n \pm N$, therefore the states must vanish:

$$L_n \mid E^{\pm}_N\rangle = 0. \tag{4.11}$$

Let us now consider the case $K < 1$. From (4.8) we find that for large enough N, the j-value of the vacuum is

$$j_N = -\frac{1}{2} + i\omega_N, \quad \text{where} \quad \omega_N = \sqrt{2(1-K)(N-\alpha_0) - \frac{1}{4}}. \tag{4.12}$$

Therefore the vacuum carries a principal representation and consequently the eigenvalue of the Casimir on the N^{th} extremal state have nonvanishing imaginary part, $2iN\omega_N$. It follows then that the representation is non-unitary. This result is also consistent with one's naive expectations since for negative string tension all but the first few physical states are tachyonic.

Finally we turn to the most promising case of $K > 1$, which is the analogue of the free string with a positive string tension. In this case the vacuum states carry a highest weight representation with highest weight state $\mid j_N j_N\rangle$ satisfying

$$J^+_0 \mid j_N\, j_N\rangle = 0 \qquad \text{and} \qquad J^3_0 \mid j_N\, j_N\rangle = j_N \mid j_N\, j_N\rangle, \tag{4.13}$$

where

$$j_N = -\frac{1}{2} - \sqrt{2(K-1)(N-\alpha_0) + \frac{1}{4}}. \tag{4.14}$$

Here we can construct the extremal representation explicitly. We define

$$\mid E^+_N\rangle = (J^+_{-1})^N \mid j_N\, j_N\rangle. \tag{4.15}$$

Using the KM algebra we can show that

$$J_0^+ \mid E_N^+\rangle = 0 \qquad \text{and} \qquad J_0^3 \mid E_N^+\rangle = (j_N + N) \mid E_N^+\rangle \tag{4.16}$$

so the extremal states also form a highest weight representation with highest weight state

$$\mid E_N^+\rangle = \mid j_N{+}N \; j_N{+}N\rangle. \tag{4.17}$$

But since from (4.14) asymptotically

$$j_N \sim -\sqrt{N}, \tag{4.18}$$

for large N the highest weight $h_N = j_N{+}N$ becomes positive. Thus $\mid E_N^+\rangle$ is simultaneously a highest weight state and an eigenstate of J_0^3 with positive eigenvalue and such a state is incompatible with unitarity. This can be seen either from Appendix A or can be verified directly by noting that

$$\frac{\langle E_N^+ | J^+ J^- \mid E_N^+\rangle}{\langle E_N^+ \mid E_N^+\rangle} = -2h_N < 0. \tag{4.19}$$

For large K the magnetic quantum number h_N first becomes positive for $N \sim 2K$ so negative-norm states first occur at grades of order of $2K$. In the limit $f^{ab}{}_c \to 0$ $(Q \to 0)$ the KM algebra (3.1) reduces to the algebra of the free bosonic oscillators. Indeed in this limit $K \to +\infty$ (relative to Q) and the ghost states disappear from the physical spectrum.

Finally we note that despite of the noncompactness of the group the Fock space basis is well defined (that is the basis states have finite norms relative to the vacuum states). This is because of the structure of the KM algebra and the fact that the J_n^a annihilate the vacuum for $n > 0$. This is in contrast with the embedding of a noncompact Lie Algebra into a larger one (e.g. $SO(2,1)$ into $SO(3,1)$) in which case the relative normalizations are often not finite (e.g. the decomposition of a unitary irreducible representation of $SO(3,1)$ with respect $SO(2,1)$ is a direct integral). So in the present KM case the real question was therefore not whether the norms of the states (3.20) are finite but whether they are positive.

5. Summary and Remarks

In this paper we have considered the problem of string propagation on non-compact group manifolds. We have shown that the negative-norm, ghost states are not all eliminated from the Hilbert space by the Virasoro constraints. This can be interpreted as if truly consistent string propagation (positivity) rules out certain manifolds, hence restricting the possible vacua of the string. In our example the time direction is not decoupled from the non-trivial (curved) part of the manifold as happens for the usually considered $M_4 \times B$ case where M_4 is a four-dimensional Minkowski space and B is a compact internal space. It is clearly a very important question, whether for other, more physical background space-time manifolds the string propagation is truly consistent.

We would also like to make a few comments on the recent work of Polyakov et al. In ref. [7] it has been discovered that the induced string action for two-dimensional gravity can be solved exactly in the light-cone gauge. The key observation was that there is an $SL(2,R)$ current algebra satisfied by the components of the world-sheet metric. It is natural to ask what is the relation between the traditional approach adopted in our paper and that of ref. [7]. First, the stress-energy tensor of ref. [7] is not just the Sommerfield-Sugawara one but contains an extra term as well:

$$T(z) = T^{SS}(z) + \partial_z I^3(z). \tag{5.1}$$

I^-, I^+ and I^3 satisfy the commutation relations (3.14), illustrating the fact that the Lie-algebras SL(2,R) and SU(1,1) are isomorphic. However, all the generators are anti-hermitian in contrast to our case (3.13). More importantly, in addition to requiring the full energy-momentum tensor (5.1) to vanish, there is an extra constraint coming from the residual reparametrization symmetry of the light-cone gauge:

$$I_n^- \mid \Phi\rangle = 0 \qquad n \geq 0. \tag{5.2}$$

This latter constraint breaks the SL(2,R) symmetry and is extremely restrictive. Though the problem of unitarity is not investigated in ref. [7], it is likely that the physical subspace is positive. To see how strong the constraint (5.2) is let us consider a vacuum representation of the highest weight type. We write a general state $\mid \Phi\rangle$

in the form

$$| \Phi\rangle = \sum_{r=0}^{\infty} f_r \mid j, j-r\rangle \tag{5.3}$$

where the amplitudes f_r will be determined by (5.2). Before imposing the constraint (5.2) on the state (5.3) we have to change the generators from the $SL(2,R)$ basis of ref. [7] to the $SU(1,1)$ one used in this paper:

$$\begin{aligned} I^+ &= -\frac{i}{2}(2J^3 - J^+ - J^-) \\ I^- &= -\frac{i}{2}(2J^3 + J^+ + J^-) \\ I^3 &= \frac{1}{2}(J^+ - J^-). \end{aligned} \tag{5.4}$$

Using (5.4) we find the following recurrence relation among the amplitudes f_r:

$$f_{r+1} = -\frac{1}{\sqrt{(r+1)(r-2j)}}\left[2(j-r)f_r + \sqrt{r(r-2j-1)}f_{r-1}\right] \tag{5.5}$$

showing that apart from normalization only a single physical state is left in the vacuum sector.

Finally one can consider the general problem of obtaining the unitarity of non-compact KM algebras by imposing additional constraints. E.g. (5.2) breaks even the $SL(2,R)$ invariance. However one might be interested in group invariant constraints. A natural set of such constraints would be the analogue of the Virasoro operators for higher order Casimirs. However higher order Casimirs do not occur for $SU(1,1)$ and such operators alone would probably not produce unitarity for higher rank groups because the number of time-like directions is always greater than the number of Casimirs.

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Appendix A

In this appendix we summarize the classification of unitary irreducible representations of $\widetilde{SU(1,1)}$ [8], where $\widetilde{SU(1,1)}$ is the universal covering group of of $SU(1,1)$. We use the conventions introduced in section 3. These representations can be grouped into three classes according to the spectra of $\mathbf{C}$ and J^3. In all cases one can choose an orthonormal basis $|\, j\, h\rangle$ such that

$$\begin{aligned} \mathbf{C}\,|\, j\, h\rangle &= j(j+1)\,|\, j\, h\rangle & J^3 &= h\,|\, j\, h\rangle \\ J^+\,|\, j\, h\rangle &= d(j,h)\,|\, j\, h+1\rangle & J^-\,|\, j\, h\rangle &= d(j,h-1)\,|\, j\, h-1\rangle. \end{aligned} \qquad (A1)$$

Note in particular that the eigenvalues of J^3 are real and integer spaced. The normalization constants

$$d^2(j,h) = h(h+1) - j(j+1) \qquad (A2)$$

must be non negative for the representation to be unitary.

(i) The *discrete series* are characterized by a highest (lowest) weight state,

$$J^+\,|\, j\, j\rangle = 0 \qquad (J^-\,|\, j\, j\rangle = 0) \qquad (A3)$$

and it follows that

$$j < 0 \quad h = j,\ j-1, \ldots \qquad (j > -1 \quad h = j,\ j+1, \ldots) \qquad (A4)$$

(ii) For the *principal representations* there is no highest weight and the spectrum of J^3 is unbounded

$$j = -\frac{1}{2} + i\kappa, \qquad h = a,\ a \pm 1, \ldots \quad 0 \leq a < 1 \qquad (A5)$$

and κ is a real parameter.

(iii) In addition there are the *exceptional series* for which $-1 < j < 0$ and h is again unbounded.

Beside the unitary representations there are representations for which the $U(1)$ subgroup generated by J^3 is still unitary, i.e. J^3 has real and integer spaced spectrum, but for which the Casimir operator has complex eigenvalues. Such representations are necessarily non-unitary since on a positive Hilbert space the formally self-adjoint Casimir would always possess a real spectrum. Indeed all eigenstates of C have zero norm. This follows at once from the equations

$$j(j+1)(\Psi,\Psi) = (\Psi,\mathbf{C}\Psi) = (\mathbf{C}\Psi,\Psi) = \overline{j(j+1)}(\Psi,\Psi)$$

which require $(\Psi,\Psi) = 0$ for a complex $j(j+1)$. Note however, that the algebraic structure (A1), (A2) remains unchanged for these non-unitary representations.

Appendix B

We wish to show that the familiar $SU(2)$ Clebsch-Gordan decomposition $\vartheta_1 \otimes \vartheta_l = \vartheta_{l-1} \oplus \vartheta_l \oplus \vartheta_{l+1}$ also holds for $SU(1,1)$. Let V^j be a vector-operator, i.e.

$$[J^i, V^j] = i\epsilon^{ij}{}_k V^k \tag{B1}$$

where the $i\epsilon^{ij}{}_k$ are the structure constants of $SU(1,1)$. Furthermore, let $|\, j\, h\rangle$ be the eigenstates of C and J^3 in a given representation, then it follows that the set of states

$$V^i\,|\, j\, h\rangle \qquad i = 1,2,3 \tag{B2}$$

decompose into irreducible representations with Casimirs parametrized by $j-1$, j and $j+1$. In other words the same branching rule holds as for $SU(2)$. To show that this is indeed the case we first observe that the states (B2) are obtained by acting with $J^{\pm}$ on the three basic states

$$V^+J^-\,|\, j\, h\rangle, \quad V^-J^+\,|\, j\, h\rangle \quad \text{and} \quad V^3\,|\, j\, h\rangle. \tag{B3}$$

They all have the same J^3 eigenvalue h. With respect to these states the Casimir has matrix elements

$$\begin{pmatrix} c+2h & 0 & 1 \\ 0 & c-2h & -1 \\ -2d^2(j,h-1) & 2d^2(j,h) & c+2 \end{pmatrix}, \quad \text{where} \quad c = j(j+1). \tag{B4}$$

One sees at once that the state $(1,1,-2h)$ is a right-eigenvector with eigenvalue $j(j+1)$. Together with the trace and determinant this eigenvalue uniquely determines the remaining eigenvalues to be $(j+1)(j+2)$ and $j(j-1)$ as stated. In particular for the principal representations $j = -\frac{1}{2} + i\kappa$, and the two representations labeled by $j-1$ and $j+1$ have complex Casimirs and therefore are non-unitary. Actually $j(j-1)$ and $(j+1)(j+2)$ are complex conjugate of each other. Note that the two zero norm states corresponding to these complex conjugate Casimirs are not necessarily orthogonal to each other.

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