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Polyakov-Loops and Fermionic Zero Modes in QCD_2 on the Torus

U. G. Mitreuter[1], **J. M. Pawlowski**[2] **and A. Wipf**[3]
Theor.–Phys. Institut, Universität Jena
Fröbelstieg 1
D–07743 Jena
Germany

Abstract

A direct derivation of the free energy and expectation values of Polyakov-loops in QCD_2 via path integral methods is given. The chosen gauge fixing has no Gribov-copies and has a natural extension to four dimensions. The Fadeev-Popov determinant and the integration over the space component of the gauge field cancel exactly. It only remains an integration over the zero components of the gauge field in the Cartan sub-algebra. This way the Polyakov-loop operators become Vertex-operators in a simple quantum mechanical model. The number of fermionic zero modes is related to the winding-numbers of A_0 in this gauge.

[1]e–mail: Mitreuter@tpi.uni-jena.de
[2]e–mail: Pawlowski@tpi.uni-jena.de
[3]e–mail: Wipf@tpi.uni-jena.de

1 Introduction

A long standing and yet unsolved problem is proving quark confinement in QCD. An important first step in this direction would be to show confinement of static quarks. In this way the problem reduces to understanding the behaviour of electric flux strings in pure $SU(N)$ gauge theories (without dynamical quarks). The relevant observables are products of Wilson-loop operators [1]. At finite temperature the operators related to Polyakov-loops can be used to discriminate between the confining and deconfining phases [2].
A rigorous construction and investigation of gauge theories in (3+1) dimensions is beyond present days knowledge. (1+1) dimensional models are much simpler and can be used as a testing ground to get more insight into gauge theories on a sound mathematical basis. In particular, the unique and ambiguity-free gauge fixing described below can be extend to QCD_4 [3].
Pure Yang-Mills theories in (1+1) dimensions are prototypes of (almost) topological field theories without propagating degrees of freedom. Nevertheless they have interesting features, particularly in the large N limit or/and on multiple connected space-times [4]. The partition functions depend on g^2V, where g is the coupling constant and V the volume of space-time, as well as invariants of the gauge group and topological invariants of space-time. Polyakov-loops can be computed in both the strong and weak coupling phase and the two phases are related by duality. It has been shown, that the strong coupling expansion can be rewritten as a lower dimensional string theory [7]. When defined on Riemann surfaces with non-zero genus they have degrees of freedom related to the gauge group holonomy on the homology cycles of the surface. On cylindrical space-time they can be solved explicitly and posess quantum mechanical degrees of freedom corresponding to the eigenvalues of the Wilson loop operator which is winding about the compact space direction [8]. Such models are also connected with one dimensional integrable quantum systems [9].
The free energy $e^{-\beta F} = \mathrm{Tr} e^{-\beta H}$ at finite temperature $T = 1/\beta$ is given by a path integral over gauge fields on some manifolds $S^1 \times M$ with Euclidean time x^0 identified with $x^0 + \beta$. (For discussion of the path integral formulation of finite temperature gauge theory see [10].) Relevant gauge invariant order parameters are Polyakov-loop operators

$$P(x^1) = \mathrm{Tr}\,\Gamma(\mathcal{P}\,(\beta, x^1)), \quad \text{where} \quad \mathcal{P}(x^0, x^1) = \mathcal{P}\,\exp\Big(i\int\limits_0^{x^0} A_0(\tau, x^1)\mathrm{d}\tau\Big). \tag{1.1}$$

Here Γ is the representation of the gauge group which acts on the fermionic fields. For example, the two-point function

$$e^{-\beta F(x^1, y^1)} = \langle P(x^1) P^\dagger(y^1)\rangle_\beta \tag{1.2}$$

yields the free energy $F(x^1, y^1)$ in the presence of a heavy quark (in the fundamental representation) at x^1 and a heavy antiquark at y^1. In the confining phase $F(x^1, y^1)$ increases for large separations of the quark-antiquark pair and thus $\langle P(x^1)P^\dagger(y^1)\rangle \to 0$. In the deconfining phase the free energy reaches a constant value for large separations and thus $\langle P(x^1)P^\dagger(y^1)\rangle \to const \neq 0$. Inferring clustering we see that $\langle P\rangle_\beta$ vanishes in the confining

phase but not in the deconfining one. In other words, it is an order parameter for confinement. If we include massless dynamical fermions, the generating functional gains as a factor the determinant of the Dirac operator. As a consequence gauge field configurations which support fermionic zero modes do not contribute to the partition function or to expectation values of Polyakov-loops. Therefore, the question of the number of zero modes for a given gauge field configuration is an important first step from pure to full QCD.
In this paper we examine Yang-Mills theories on two dimensional tori. They correspond to finite temperature gluodynamics on a spatial circle. As shown by Grignani, Semenoff and Sodano in an interesting paper [11][4], correlators of Polyakov-loop operators can be computed as correlators in particular one dimensional models. For the case of pure gauge theories (and Polyakov-loop operators in an arbitrary representation) one can explicitly solve these quantum mechanical models. In contrast to other approaches we directly calculate, after an appropriate gauge fixing, the partition function and the correlation function $\langle P(x^1)P^\dagger(y^1)\rangle$ for arbitrary semi-simple gauge groups. This will be a starting point for further investigations concerning QCD_4 [3].
In this paper we quantise Lie-algebra valued gauge fields (non-compact QCD_2),whereas in [4, 5] the group valued fields are quantised (compact QCD_2). As Hetrick has shown [6], the non-compact theory has additional spectral values connected with states, which lie on the boundary of a Weyl chamber. (Since these states lie in more than one chamber, they must be added with an appropriate weight.) In the compact version these states are projected to zero-dimensional characters and are missing in the spectrum. If these states are added to the partition function calculated in [4, 5], the results for the partition function Z and expectation values of products of Polyakov loop operators agree. Due to this difference between compact and noncompact QCD_2, the numerical value of the string tension for the static quark potential is different, but the physics is qualitatively the same.
In addition we go beyond these results in that our approach leads to a simple relation between the winding numbers and the number of fermionic zero modes.
In the first section we discuss the gauge fixing and topological questions connected with the definition of gauge theories on T^2 (the corresponding results in 4 dimensions are briefly sketched). In the following two sections we calculate the partition function and the free energy of a static quark-antiquark pair. In the last section we derive a formula relating the number of fermionic zero modes to the winding numbers of the gauge-fixed A_μ. In the discussion we compare our results with those of [4] and [5]. The appendices contain our Lie-algebra conventions and a proof concerning antiholomorphic transition functions on the torus.

2 Gauge Fixing

We view the torus T^d as R^d modulo a d-dimensional lattice, whose points are denoted by $a, b, \dots$ with coordinates $a_\mu = n_\mu L_\mu$, $n_\mu \in Z$ (no sum). Matter fields and gauge potentials

[4] after we completed this work, Grignani et. al revised their paper, see [12]

on R^d can be put on the torus if they are (anti)periodic up to gauge transformations [13]

$$\begin{aligned} \psi(x+a) &= (-1)^{n_0}\Gamma(U_a^{-1}(x))\psi(x), \\ A(x+a) &= U_a^{-1}(x)A(x)U_a(x) + iU_a^{-1}(x)\mathrm{d}U_a(x), \end{aligned} \tag{2.1}$$

where the factor $(-1)^{n_0}$ enforces the finite temperature boundary conditions for fermions ($L_0 = \beta = 1/T$). Since $\psi((x+a)+b) = \psi((x+b)+a)$, the transition functions U_a must obey the cocycle conditions [13]

$$U_a(x)U_b(x+a) = U_b(x)U_a(x+b)Z_{ab}, \qquad Z_{ab} = Z_{ba}^{-1},$$

where the twists Z_{ab} are in the kernel of Γ, i.e. $\Gamma(Z_{ab}) = 1\!\!1$. This kernel is a subgroup of the center Z of G. For fermions in the fundamental representation no twists are allowed whereas for fermions in the adjoined representation the twists can be any element of the center of G.

Performing a (not necessarily periodic) gauge transformation with $V(x)$, the new transition functions for the transformed fields are

$$\tilde{U}_a(x) = V^{-1}(x)U_a(x)V(x+a) \tag{2.2}$$

The $\tilde{U}_a$ fulfill the cocycle condition with the same Z_{ab} as the U_a. Thus the twists are gauge invariant. Note that the Polyakov-loop operators (1.1) transform as

$$P(\vec{x}) \longrightarrow \tilde{P}(\vec{x}) = \mathrm{Tr}\{V(0,\vec{x})V^{-1}(\beta,\vec{x})\,\mathcal{P}(\beta,\vec{x})\}, \quad \text{where} \quad x = (x^0,\vec{x}) \tag{2.3}$$

and are only invariant if $V(x)$ is periodic in time. A twisted G-bundle over T^d is uniquely characterized by the transition functions modulo gauge transformation (2.2).

In the following we shall consider 2-dimensional gauge theories. In 2 dimensions and for simply connected gauge groups G the G-bundles over T^2 are trivial [5] (all Chern classes are zero [16]). Thus, in the untwisted case ($Z_{ab} = 1\!\!1$) the transition functions can be chosen to be the identity. With twists this is not true, but writing a twist as $Z_{01} = Z = \exp(-2\pi i T)$, we can always choose the transition functions as

$$U_\beta = 1\!\!1 \quad \text{and} \quad U_L = e^{-2\pi i T x^0/\beta}, \tag{2.4}$$

where U_β relates the fields at x^0 and $x^0+\beta$ and U_L those at x^1 and x^1+L.

In explicit calculations we must fix the gauge. The field-dependent gauge transformation which transform an A_μ into the gauge fixed form may be non-periodic and thus lead to nontrivial transition functions.

After these general remarks we now discuss the explicit gauge fixing. Since Polyakov-loops only depend on A_0, we shall choose a gauge for which A_0 is as simple as possible. Actually,

[5] This is not true, for example, for $U(1)$ or $SO(3)$-bundles over T^2 [15].

with our fixing A_0 will decouple in the path integral and the expectation values of products of Polyakov-loops can easily be calculated[6]. Below we prove that there is a (non-periodic) gauge transformation which transforms any A_μ into

$$A_0 = A_0^c = 2\pi H \frac{x^1}{V} + A_0^{per}(x^1) \quad , \quad H \in \mathcal{L}_\Gamma \quad \text{and} \quad \int\limits_0^\beta A_1^c(x^0, x^1) dx^0 = C, \tag{2.5}$$

where $V = \beta L$ is the volume and we have introduced the following discrete lattice in the Cartan-subalgebra $\mathcal{H}$:

$$\mathcal{L}_\Gamma \equiv \Big\{ H \in \mathcal{H} | \Gamma(\exp(2\pi i H)) = \mathbb{1} \Big\}. \tag{2.6}$$

In particular, for the fundamental and adjoint representations $\Gamma = f$ and $\Gamma = adj$ we have

$$\mathcal{L}_f = \Big\{ H \in \mathcal{H} | \exp(2\pi i H) = \mathbb{1} \Big\} \quad , \quad \mathcal{L}_{adj} = \Big\{ H \in \mathcal{H} | \exp(2\pi i H) \in Z \Big\}. \tag{2.7}$$

In (2.5) A^c is that part of A which lies in the Cartan subalgebra, A_0^{per} is periodic in x^1 and A_1 periodic in x^0. Below the constant C and $\int dx^1 A_0^{per}$ are further restricted such that the gauge fixing (2.5) becomes unique.
Note that the gauged fixed fields are not periodic in x^1. Indeed, the transition functions for the gauged fixed configurations are

$$\tilde{U}_\beta(x) = \mathbb{1} \qquad \text{and} \qquad \tilde{U}_L(x^0) = \exp\left\{ -2\pi i H \frac{x^0}{\beta} \right\} \quad \text{with} \quad H \in \mathcal{L}_\Gamma. \tag{2.8}$$

Hence, the periodicity property of A_1 is given by

$$A_1(x^0, x^1 + L) = e^{2\pi i \frac{x^0}{\beta} H} A_1(x^0, x^1) e^{-2\pi i \frac{x^0}{\beta} H}. \tag{2.9}$$

Only $A_1^c \in \mathcal{H}$ is periodic in x^1.
To prove, that (2.5) can be achieved we perform a gauge transformation with

$$V(x^0, x^1) = \mathcal{P}(x^0, x^1) \mathcal{P}^{-x^0/\beta}(\beta, x^1) W(x^1), \tag{2.10}$$

where $\mathcal{P}(x^0, x^1)$ has been defined in (1.1) and W diagonalizes $\mathcal{P}(\beta, x^1)$, i.e.

$$\mathcal{P}(\beta, x^1) = W(x^1) \exp\{2\pi i H(x^1)\}\, W^{-1}(x^1). \tag{2.11}$$

This representation allows one to take powers of $\mathcal{P}(\beta, x^1)$ and (2.10) becomes

$$V(x^0, x^1) = \mathcal{P}(x^0, x^1)\, W(x^1)\, \exp\Big(-2\pi i \frac{x^0}{\beta} H(x^1) \Big). \tag{2.12}$$

[6]similar fixings have been studied in [14]. After we discovered the gauge fixing used in this work J. Fuchs pointed out to us that E. Langmann et.al. found a very similar fixing.

Now it is easy to see that the gauge transformed A_0 reads

$$\tilde{A}_0 = \frac{2\pi}{\beta} H(x^1) \tag{2.13}$$

and hence depends only on x^1 and lies in the Cartan subalgebra.
By construction the gauge transformations (2.10) are periodic in time so that the Polyakov-loops are unchanged, as required, and the transition function in the time direction, $\tilde{U}_\beta$, remains the identity, see (2.8). To find $\tilde{U}_L$ we use

$$\mathcal{P}(x^0, x^1 + L) = \exp(2\pi i T \frac{x^0}{\beta}) \mathcal{P}(x^0, x^1) \tag{2.14}$$

from which follows, that

$$\exp\{2\pi i H(x^1 + L)\} \;=\; \exp\{2\pi i (T + H(x^1)\} \quad \text{and} \quad W(x^1 + L) = W(x^1) \tag{2.15}$$

or equivalently that

$$H(x^1 + L) = H(x^1) + H \quad \text{with} \quad H \in \mathcal{L}_\Gamma. \tag{2.16}$$

With (2.13) and the consistency condition we end up with the form (2.5) for A_0. The new transition function $\tilde{U}_L$ is easily calculated from (2.2), with U_L from (2.4), V from (2.12) and $V(x^1 + L)$ from (2.14,2.15). The result is the transition function $\tilde{U}_L$ given in (2.8).
We have not yet fixed the gauge freedom completely. Indeed, the residual gauge transformations are

$$V(x) = w \cdot \exp\Big\{2\pi i (H_{per}(x^1) + H_0 \frac{x^0}{\beta} + H_1 \frac{x^1}{L})\Big\}, \tag{2.17}$$

where all H's are in the Cartan subalgebra and in addition H_{per} is periodic in x^1, $H_i \in \mathcal{L}_f$ and w is an element of normalizer($\mathcal{H}$)/centralizer($\mathcal{H}$) $\cong$ Weyl-group [17]. More explicitly, w_α acts on a generator H_β in $\mathcal{H}$ as

$$w_\alpha^{-1} H_\beta w_\alpha = H_{\sigma_\alpha \beta}, \tag{2.18}$$

where σ_α is the Weyl reflection related to the root α. The H_{per} part in (2.17) is fixed by imposing the second condition in (2.5). The H_i-parts are fixed if we further impose

$$\frac{\beta}{L} \int_0^L A_0^{per} dx^1 \equiv \frac{\beta}{L} \tilde{C} \in 2\pi \mathcal{H}/\mathcal{L}_\Gamma \qquad \text{and} \qquad \frac{L}{\beta} \int_0^\beta A_1^c dx^0 \equiv \frac{L}{\beta} C \in 2\pi \mathcal{H}/\mathcal{L}_\Gamma. \tag{2.19}$$

It remains to fix the Weyl-transformations w in (2.17). This can be done by imposing the condition, that $\tilde{C}$ is in the first Weyl-chamber. However, the Weyl group is a finite group and permutes transitively and freely the Weyl chambers, so the integration over the $\tilde{C}$, subject to (2.19), is a multiple of the integration over the first Weyl chamber. If we consider normalised observables, this overcounting cancels with that in the normalisation. For later purposes it is important to note that the transition functions for the gauge fixed configurations possess abelian winding numbers. This can be seen as follows: Since $\Gamma(\tilde{U}_L(x^0)) = \Gamma(\tilde{U}_L(x^0+\beta))$ the map

$$x^0 \longrightarrow \Gamma(U_L(x^0)), \tag{2.20}$$

is a map from S^1 to $S^1 \times \ldots \times S^1$ (r=rank(G) factors) and thus allows for r integer winding numbers.

3 The functional integral

In the following we decompose the Lie algebra valued gauge potential (for conventions see the appendix) as follows

$$\begin{aligned} A_0(x) &= \sum_{\alpha\in\Delta} p^\alpha(x) H_\alpha + \sum_{\varphi\in\Phi^+} a^\varphi(x) E_\varphi + \sum_{\varphi\in\Phi^+} \bar{a}^\varphi(x) E_{-\varphi} \\ A_1(x) &= \sum_{\alpha\in\Delta} q^\alpha(x) H_\alpha + \sum_{\varphi\in\Phi^+} b^\varphi(x) E_\varphi + \sum_{\varphi\in\Phi^+} \bar{b}^\varphi(x) E_{-\varphi}, \end{aligned}$$

where Δ, Φ^+ and Φ^- denote the simple, positive and negative roots, respectively and H_α is the generator in the Cartan subalgebra $\mathcal{H}$ belonging to the simple root α.
The gauge fixing conditions (2.5) read

$$\mathrm{Tr}\, E_\varphi A_0 = \mathrm{Tr}\, E_{-\varphi} A_0 = \mathrm{Tr}\, H_\alpha (\partial_0 A_0 - \frac{1}{\beta}\int dx^0 \partial_1 A_1) = 0. \tag{3.1}$$

For the gauge fixed configurations (see (2.5)) we find for the field strength

$$F_{01} = \sum_\alpha (\dot{q} - p')^\alpha H_\alpha - i \sum_\varphi [M_\varphi b^\varphi E_\varphi - (M_\varphi b^\varphi)^* E_{-\varphi}], \qquad M_\varphi = (i\partial_0 + \sum_\alpha K_{\varphi\alpha} p^\alpha) \tag{3.2}$$

is hermitean. Correspondingly the gauge fixed action reads

$$S = \frac{1}{8g^2}\int \mathrm{d}x^0 \mathrm{d}x^1\ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} = \frac{1}{4g^2}\int \Big\{ (\dot{q} - p', C\,(\dot{q} - p')) + \sum_\varphi \frac{4}{\varphi^2}(b^\varphi, M_\varphi^2 b^\varphi) \Big\}, \tag{3.3}$$

where $C = (C_{\alpha\beta})$ is the symmetric Coxeter matrix and the r-component real vector fields p and q have entries p^α and q^α, respectively. The last scalar product containing the operators M_φ is a complex one, $(b, c) = \bar{b}\cdot c$. To calculate the Fadeev-Popov determinant we observe that the gauge variation of the gauge fixings $\mathrm{Tr}(E_{\pm\varphi}A_0) = 0$ are

$$\delta_\theta \mathrm{Tr} E_{-\varphi} A_0 = \frac{2i}{\varphi^2} M_\varphi \theta^\varphi \quad \text{and} \quad \delta_\theta \mathrm{Tr} E_\varphi A_0 = -\frac{2i}{\varphi^2}(M_\varphi \theta^\varphi)^* \quad \text{(no sum)}. \tag{3.4}$$

Vanishing variations imply vanishing θ^φ and then the variation of the remaining gauge fixings simplifies to

$$\delta_\theta \mathrm{Tr} H_\alpha \Big(\partial_0 A_0 + \frac{1}{V}\int dx^0 \partial_1 A_1\Big) = -C_{\alpha\beta}\Big(\partial_0^2 + \frac{1}{\beta}\int dx^0 \partial_1^2)\theta^\beta \tag{3.5}$$

Now we see, that for simply laced groups (for which the length of all roots can be taken to be 2) the field-dependent Fadeev-Popov determinant coming from the θ^φ cancel exactly against the functional integral over the non-Cartan fields b^φ. Thus we obtain the following partition function

$$\begin{aligned} Z &= N \textstyle\int \mathcal{D}q(x)\mathcal{D}p(x^1)\, \delta(\mathcal{F}(A))\, \det(C)\, \det(-\partial_0^2 - \frac{1}{L}\int \mathrm{d}x^0\ \partial_1^2)^r \\ &\cdot \quad \exp\Big\{\tfrac{1}{4g^2}\textstyle\int (\dot q - p', C(\dot q - p'))\Big\} \end{aligned} \tag{3.6}$$

with a normalization factor N. Here $\delta(\mathcal{F}(A))$ indicates the implementation of the zero mode fixings (2.19). Since A_1^c is periodic in x^0 and A_0 depends only on x^1, the integration over q^α decouples completely and we end up with

$$Z = N' \int \mathcal{D}p(x^1) \exp\Big\{-\frac{\beta}{4g^2}\int (p', C\, p')\, \mathrm{d}x^1\Big\}. \tag{3.7}$$

For simplicity we restrict ourselves to $\Gamma = f$ (fermions in the fundamental representation), $G = SU(N)$ with rank $r = N-1$ and choose the basis

$$E_i = E_{\alpha_i} = E_{i,i+1} \quad \text{and} \quad H_i = [E_{\alpha_i}, E_{-\alpha_i}] \quad \text{for simple} \quad \alpha_i, \tag{3.8}$$

where $E_{i,j}$ is the $N\times N$-matrix whose only non-zero entry is a 1 in the i'th row and j'th column. The step-operators belonging to the non-simple roots are obtained by commutation of the E_i. The center consists of the N'th roots of unity and is generated by the

$$T_\tau = \frac{\tau}{N}\,\mathrm{diag}(1,1,\dots,1-N), \qquad \tau = 0,1,\dots N-1. \tag{3.9}$$

Then $C = K$ has 2's on the diagonal and -1 on the two off-diagonals above and below the diagonal. We can decompose the $p = (p^1,\dots,p^r)$, $p^i = p^{\alpha_i}$ as follows

$$p(x) = \frac{1}{\beta}[\tilde p(x) + h] + \frac{2\pi n}{V}x \qquad (x = x^1), \tag{3.10}$$

where we have separated the constant part $h = (h^1,\dots,h^r)$ of the periodic piece, so that the $\tilde p^i$ are periodic in x and integrate to zero. Since $\Gamma(\exp[2\pi\vec n\cdot H]) = 1\!\!1$, the $\vec n$ lie in Z^r for matter in the fundamental representation and $n^i \in \tau/N + Z$ for matter in the adjoint representation. In the explicit calculations below we assume that there are no twists. We shall give the corresponding results for the twisted case at the end of the next section.

Inserting the decomposition (3.10) we find for the partition function

$$Z \sim \int \mathcal{D}\tilde{p}\, d^r h \, \exp\Big\{\frac{1}{4g^2\beta}\int(\tilde{p}, K\partial_1^2\tilde{p})dx\Big\} \cdot \sum_{\vec{n}\in Z^r} \exp\Big\{-\frac{\pi^2}{g^2V}(\vec{n}, K\vec{n})\Big\}. \tag{3.11}$$

Due to the fixing of the time dependent residual gauge freedom (2.17) the h^i-integrations are restricted to the interval $[-\pi, \pi]$.
Using zeta-function regularisation the Gaussian integration over $\tilde{p}^i$ yields

$$\left(\det{}'\left[\frac{-K\partial^2}{2g^2\beta}\right]\right)^{-1/2} = \det{}^{1/2}\frac{K}{2\beta L^2 g^2}.$$

After a Poisson-resummation in (3.11) we end up with

$$Z \sim \sum_{\vec{m}\in Z^r} \exp\{-g^2 V(\vec{m}, K^{-1}\vec{m})\} \tag{3.12}$$

with the inverse of the Cartan matrix

$$(K^{-1})_{ij} = \frac{1}{N}(N-j)i, \quad \text{for} \quad i \le j, \quad (K^{-1})_{ij} = (K^{-1})_{ji}. \tag{3.13}$$

4 Calculation of Polyakov-loops

For the gauge fixed configurations and $\Gamma = f$ the Polyakov-loops (1.1) simplify to

$$P(x) = \mathrm{Tr}\exp\{i\beta\, p(x)\cdot H\} = \sum_{k=1}^{N} \exp\Big\{i\beta[p^k(x) - p^{k-1}(x)]\Big\}, \tag{4.1}$$

where $p^0 \equiv p^N \equiv 0$. We get for the expectation value of the product of two Polyakov-loops

$$\langle P(x)P^\dagger(y)\rangle = \frac{1}{Z}\int \mathcal{D}\tilde{p}\, d^r h \sum_{\vec{m}} \exp\left\{-\frac{\pi^2}{g^2V}(\vec{m}, K\vec{m}) + \frac{1}{4g^2\beta}\int(\tilde{p}, K\partial^2\tilde{p})\right\} P(x)P^\dagger(y). \tag{4.2}$$

After integration of the h^i only the diagonal elements in the double sum (coming from the 2 Polyakov-loop operators) contribute and

$$\begin{aligned}\langle PP^\dagger\rangle = \frac{1}{Z}\sum_{k=1}^{N}\sum_{\vec{m}} &\exp\left\{-\frac{\pi^2}{g^2V}(\vec{m}, K\vec{m}) + 2\pi i m^k\xi - 2\pi i m^{k-1}\xi\right\} \\ &\int \mathcal{D}\tilde{p}\exp\left\{\frac{1}{4g^2\beta}\int(\tilde{p}, K\partial^2\tilde{p})\mathrm{d}x + i[\tilde{p}^k(x) - \tilde{p}^k(y)] - i[\tilde{p}^{k-1}(x) - \tilde{p}^{k-1}(y)]\right\},\end{aligned}$$

where we have introduced $\xi = (x-y)/L$ (recall that $x \equiv x^1$). To calculate the functional integral over the $\tilde{p}^i$ we need the zero mode truncated Greens function of $-1/2g^2\beta \cdot K\partial^2$ which is

$$G(x,y) = K^{-1}\Delta(x,y), \quad \text{where} \quad \Delta(x,y) = g^2V\Big(\xi^2 - |\xi| + \frac{1}{6}\Big) \ \text{ for } \ \xi \in [-1, 1]. \tag{4.3}$$

Now we perform a Poisson resummation of the $N-1$ sums and calculate the Gaussian functional integral over the periodic $\tilde{p}^i$. We emphasize that there are no zero mode problems, as it must be for a complete gauge fixing. The result is

$$\begin{aligned}\langle P(x)^\dagger P(y)\rangle \;=\; & \frac{1}{Z}\sum_{k,\vec{m}} \exp\left\{-g^2 V \left(m_i - \xi[\delta_{ik} - \delta_{i(k-1)}]\right) K^{-1}_{ij} \left(m_j - \xi[\delta_{jk} - \delta_{j(k-1)}]\right)\right\} \\ & \exp\left\{\frac{N-1}{N}[\Delta(x,y) - \Delta(0,0)]\right\}\end{aligned} \tag{4.4}$$

where $K^{-1}_{0i} = K^{-1}_{Ni} = 0$. Thus the expectation value of the product of two Polyakov loops reads

$$\langle P(x)P^\dagger(y)\rangle = \frac{1}{Z}\sum_{k,\vec{m}} \exp\left\{-g^2 V \left(\vec{m}, K^{-1}\vec{m} - 2\xi m_i \left[K^{-1}_{ik} - K^{-1}_{i(k-1)}\right] + |\xi|\frac{N-1}{N}\right)\right\} \tag{4.5}$$

for $\xi \in [-1,1]$. For $SU(2)$ this simplifies to

$$\langle P(x)P^\dagger(y)\rangle = \frac{2}{Z}\sum_{m\in Z} \exp\left\{-\frac{g^2 V}{2}\left[m^2 - 2\xi m + |\xi|\right]\right\}. \tag{4.6}$$

The free energy for the static quark-antiquark pair in the fundamental representation is gotten from (1.2). Z is given by $\langle P(x)P^\dagger(x)\rangle = 1$. For large separations of the pair we find for the free energy for $SU(N)$

$$\lim_{L\to\infty} F(x,y) = g^2\frac{N-1}{N}|x-y|. \tag{4.7}$$

We conclude this section with the analogous results for the free energy of a static quark-antiquark pair in the adjoint representation, for which $adj[\exp(2\pi i H)] = \mathbb{1}$. In this case the gauge fixed A_0 has the decomposition

$$A_0 = \sum p^k H_k \quad \text{with} \quad p^k(x) = \frac{1}{\beta}[\tilde{p}^k + h^k] + \frac{2\pi}{V}(k\frac{\tau}{N} + m^k)x, \qquad \tau = 0,\dots,N-1 \tag{4.8}$$

and the Polyakov-loop is

$$P(x) \;=\; \mathrm{Tr}\,\Gamma_{adj}\left(\exp\left(i\int_0^\beta A_0(\tau,x)\mathrm{d}\tau\right)\right) \;=\; \mathrm{Tr}\exp\left(i\int_0^\beta A_0^k(\tau,x)\Gamma^*_{adj}(H_k)\mathrm{d}\tau\right).$$

where Γ^* is the Lia algebra representation induced by Γ. Now one proceeds as in the untwisted case. One obtains with $\tilde{m} = \sum_l m_l l$ and $\xi \in [-1,1]$

$$\begin{aligned}\langle P(x)P^\dagger(y)\rangle \;=\; & \frac{1}{Z}\left[r^2 + 2N\sum_{p=1}^{r}\sum_{j=1}^{p}\sum_{\vec{m}} \exp\left\{-g^2 V\left(\vec{m}K^{-1}\vec{m} - 2\xi\sum_{i=j}^{p} m_i\right)\right\}\right. \\ & \left.\exp\left\{-g^2 V|\xi|2(p-j+1)\right\}\sum_{n=-\infty}^{\infty}\delta_{nN,\tilde{m}}\right].\end{aligned} \tag{4.9}$$

For $SU(2)$ (4.9) simplifies to

$$\langle P(x)P^\dagger(y)\rangle = 1 + \frac{4}{Z}\sum_m \exp\left\{-2g^2V\left(m^2 - 2m\xi + |\xi|\right)\right\}. \tag{4.10}$$

For large separations of the pair we get for the free energy in the twisted case for $SU(N)$

$$\lim_{L\to\infty} F(x,y) \sim -\frac{1}{\beta}\frac{2N}{r^2}\sum_{p=1}^{r}\sum_{j=1}^{p}\exp\left\{-g^2\beta|x-y|2(p-j+1)\right\}. \tag{4.11}$$

For $\frac{1}{g^2\beta} \ll |x-y| \ll L$ the free energy becomes zero and due to the cluster decomposition theorem the expectation value of one Polyakov-loop operator is one in agreement with [5].

5 Zero Modes of the Dirac-operator

In this section we characterize and count the number of zero modes (in the fundamental representation) of $\not{D}$ for gauge theories on T^2. We will show that the number of fermionic zero modes for gauge fixed configurations A_μ with transition functions (2.8) is just

$$n_0 = \mathrm{Tr}|H|. \tag{5.1}$$

The analogous result on S^2 has been derived in [18]. To prove (5.1) we introduce the complexification G^c of G and assign to each gauge fixed A (in the gauge (2.5)) the set of G^c-valued prepotentials

$$\mathcal{G}_A \quad = \quad \{g(z,\bar z) \in G^c |\ A_z = ig^{-1}\partial_z g\} \tag{5.2}$$

with $A_z := \frac{1}{2}(A_0 - iA_1)$ and $z = x^0 + ix^1$. Since the G-bundles over T^2 are trivial each gauge fixed A is a gauge transform of a periodic potential A^p,

$$A_z = V_A^{-1}A_z^pV_A + iV_A^{-1}\partial_z V_A \tag{5.3}$$

and the prepotentials belonging to A are

$$\mathcal{G}_A = \Big\{g(z,\bar z) = h(\bar z)\cdot g_A(z,\bar z)V_A |\ g_A(z,\bar z) = \mathcal{P}\exp\{i\int_z^0 \mathrm{d}uA_z^p(u,\bar z)\}\Big\}. \tag{5.4}$$

¿From the periodicity of A^p and the known transition functions (2.8) of A one can read off the nonperiodicity of the V:

$$V_A(x^0+n\beta, x^1+mL) = V_A(x^0,x^1)e^{-2\pi imHx^0/\beta} \tag{5.5}$$

Now we classify the non-periodicity of the prepotentials g in (5.2,5.4). Since A^p is periodic it follows that

$$g(z+n\beta+imL,\bar z+n\beta-imL) \;=\; h_{nm}(\bar z)g(z,\bar z)e^{-2\pi im\frac{x^0}{\beta}H} \tag{5.6}$$

The antiholomorphic h_{nm} are transition functions of homomorphic vector bundles over the 2-dimensional torus[7] and must obey the cocycle conditions

$$h_{nm}(\bar z+p\beta-iqL)h_{pq}(\bar z)=h_{pq}(\bar z+n\beta-imL)h_{nm}(\bar z). \tag{5.7}$$

To continue, we note that if $g\in\mathcal{G}_A$ has transition functions $h_{nm}(\bar z)$, then $h(\bar z)g\in\mathcal{G}_A$ has transition functions

$$\tilde h_{nm}(\bar z) \;=\; h(\bar z+n\beta-imL)h_{nm}(\bar z)h^{-1}(\bar z). \tag{5.8}$$

Using this gauge freedom we can always find a representative in $\mathcal{G}_A$ such that $h_{n0}(\bar z)=\mathbb{1}$. To see that we write the h_{n0} as

$$h_{n0}=\mathcal{P}\exp\Big\{i\int_{\bar z}^{\bar z+n\beta}\mathrm{d}\bar u\;a(\bar u)\Big\}. \tag{5.9}$$

Then

$$h(\bar z)=\mathcal{P}\exp\Big\{i\int_{\bar z}^{0}\mathrm{d}\bar u\;a(\bar u)\Big\} \tag{5.10}$$

transforms the h_{n0} into the identity, as required.
It follows from the cocycle conditions (5.7) that the remaining nontrivial transition functions must be periodic in time,

$$h_{0m}(\bar z+n\beta)=h_{0m}(\bar z). \tag{5.11}$$

In the appendix B we shall prove, that the h_{0m} can be written as

$$h_{0m}(\bar z)=V_L^{m^2}\cdot e^{2\pi im\frac{\bar z}{\beta}H_A}\cdot\mathcal{P}\exp\Big\{i\int_{\bar z}^{\bar z-imL}\mathrm{d}\bar u\;b_p(\bar u)\Big\}, \tag{5.12}$$

where b_p is periodic in x^0, H_A lies in the Cartan subalgebra and is quantized, $\exp(2\pi iH_A)=\mathbb{1}$, and

$$V_L \;=\; e^{\pi\frac{L}{\beta}H_A}\quad\text{and}\quad[H_A,b_p(\bar u)] \;=\; 0. \tag{5.13}$$

[7] e.g. $g=g_AV_A$ has transition function $h_{nm}(\bar z)=\mathcal{P}\exp\big\{-i\int_0^1 A^p_{\bar z}\big(-\tau(n\beta+imL),\bar z\big)\cdot(n\beta+imL)\mathrm{d}\tau\big\}$.

Now we make a further gauge transformation with

$$h(\bar{z}) = \mathcal{P}\exp\{i\int_{\bar{z}}^{0} \mathrm{d}\bar{u}\; b_p(\bar{u})\}. \tag{5.14}$$

The new transition functions h_{nm} read

$$h_{nm}(\bar{z}) \;=\; V_\beta^n \cdot V_L^{m^2} \cdot e^{2\pi i m\frac{\bar{z}}{\beta}H_A} \qquad \text{with} \qquad V_\beta \;=\; \mathcal{P}\exp\Big\{i\int_\beta^0 b_p(\bar{u})d\bar{u}\Big\}. \tag{5.15}$$

Setting $V_\beta = e^{v_\beta}$ we can factorize g as

$$g(z,\bar{z}) = e^{v_\beta x^0/\beta} e^{\pi H_A (x^1)^2/V} \tilde{g}(z,\bar{z}).$$

The non-periodicity of $\tilde{g}$ is simply

$$\tilde{g}(z+n\beta+imL, \bar{z}+n\beta-imL) \;=\; e^{2\pi i m\frac{x^0}{\beta}H_A}\cdot \tilde{g}\cdot e^{-2\pi i m\frac{x^0}{\beta}H}.$$

In terms of $\tilde{g}$ the gauge fixed potential reads

$$A_z = \tilde{g}^{-1}\Big(i\partial_z + A_I\Big)\tilde{g}, \qquad \text{where} \quad A_I = \frac{\pi x^1}{V}H_A + \frac{i}{2\beta}v_\beta \tag{5.16}$$

is an abelian instanton potential, $[A_I, v_\beta] = 0$.
Now it is easy to see that the Dirac-operator $\not{D}(A)$ can be related to the one in the instanton background as

$$\not{D}(A) = \tilde{G}^\dagger \not{D}(A_I)\tilde{G}, \quad \tilde{G} = \begin{pmatrix} \tilde{g}^{\dagger-1} & 0 \\ 0 & \tilde{g} \end{pmatrix}, \quad \not{D}(A_I) = 2\begin{pmatrix} 0 & \partial_z - iA_I \\ \partial_{\bar{z}} - iA_I^\dagger & 0 \end{pmatrix}. \tag{5.17}$$

It follows at once that

$$\psi_0 = \tilde{G}^{-1}\tilde{\psi}_0 \tag{5.18}$$

is a zero mode of $\not{D}(A)$ if $\tilde{\psi}_0$ is a zero mode of $\not{D}(A_I)$.
Let us calculate the left-handed ($\gamma_5 = -1$) zero modes in the instanton background A_I. Comparing A_I with the general gauge fixed form (2.5) we see that $H_A \equiv H$. The Dirac-eqation reads

$$(\partial_z - i\frac{\pi x^1}{V}H + \frac{1}{2\beta}v_\beta)\tilde{\psi}_0 = 0$$

and is solved by the spinor fields

$$\tilde{\psi}_0(x^0,x^1) = e^{-\pi(x^1)^2 H/V - i v_\beta x^1/\beta}\chi(\bar{z}). \tag{5.19}$$

The zero modes must fulfill the boundary conditions (2.1) with transition function (2.8) so that χ must be antiperiodic in time and

$$\chi(\bar{z}-iL)=e^{2\pi iH\bar{z}/\beta+L(\pi H-iv_\beta)/\beta}\chi(\bar{z}). \tag{5.20}$$

Thus χ can be expanded as

$$\chi(\bar{z})=\sum_n e^{\pi i(2n+1)\bar{z}/\beta}a_n, \tag{5.21}$$

where the Fourier-coefficients a_n transform according to the fundamental representation of G. To proceed we use the fact that H commutes with the Dirac-operator and can be diagonalized, $Ha_n = ma_n$. Since v_β commutes with H it leaves the subspace on which $H=m$ invariant. On this subspace (5.20) translates into

$$a_n=e^{\pi(m-1-2n)L/\beta}e^{-iv_\beta L/\beta}a_{n-m}. \tag{5.22}$$

Now we see that we can choose the vectors $a_1,\dots,a_m$ freely, so that there are $m\cdot$(degeneracy of m) normalizable zero modes if m is positive. Repeating the same procedure for the right-handed zero-modes, for which m must be negative, we end up with the following formula for the number of zero-modes

$$n_0=\mathrm{Tr}|H|. \tag{5.23}$$

The explicit zero modes are given by (5.18,5.19,5.21) where the a_n are determined by the recursion relations (5.22). This way one finds, that the zero modes in the instanton backgrounds are theta-functions (similar to the abelian Schwinger model [15]).

6 Discussion

In this paper we have given a simple derivation for the expectation value of Polyakov-loops in QCD_2 at finite temperature. For the simplest case, $G=SU(2)$ and matter in the fundamental representation, i.e. with untwisted gauge fields, the interaction energy between two widely separated external sources is

$$F(x,y)\quad\stackrel{L\to\infty}{\to}\quad\frac{g^2}{2}|x-y|. \tag{6.1}$$

If we twist the gauge fields and thus introduce magnetic flux quanta we get

$$F(x,y)\quad\stackrel{L\to\infty}{\to}\quad-\frac{4}{\beta}\exp\{-2g^2\beta|x-y|\} \tag{6.2}$$

A similar behaviour is found for the higher groups and maximally twisted and untwisted fields. In the untwisted case we get a confining potential, whereas in the twisted case the potential $F(x-y)$ decays exponentially to a constant, which is to be interpreted as screening of the external charges. In figure 1 we show $F(x,y)$ for abitrary $|x-y|/L \in [0,1]$ for $SU(2)$ in the untwisted case. In figure 2 we show $F(x,y)$ for $SU(2)$ in the twisted case. For large volume $F(x,y)/V$ tents to zero everywhere.

The string tension (4.7) in non-compact QCD_2 on the torus is different from the one in compact QCD_2 [4, 5, 12]. For example, for the partition function (for $SU(2)$) the two quantisations differs on the $n=0$ contribution in

$$Z = \sum_n n^{2-2g} e^{-g^2 V n^2}$$

In [19] it has been argued, that the $n=0$ term is absent for $g \neq 1$, but on the torus there is no way to decide, which of the two quantisations is the correct one. Since in our path integral quantisation we do not need to fix the Weyl symmetry, we get twice the result of non-compact QCD_2. Therefore the 'zero representation' of Hetrick [6] must be added to the partition function of compact QCD_2 with a factor $1/2$. Another argument for a factor $1/2$ is, that the corresponding state lies on the boundary of the Weyl chamber and hence belongs to two chambers simultanously. In order to avoid double counting, we need the factor $1/2$. These weights are also present in the calculation of expecation values of Polyakov loops.

To check the cluster decomposition theorem one must compute the expectation value of one Polyakov-loop operator for the twisted and untwisted case. We get $\langle P \rangle_f = 0$ and $\langle P \rangle_{adj} = 1$. This agrees with calculations of expectation values of homologically nontrivial Wilson-loops on genus one Riemann surfaces done in [5].

In a second part we derived an explicit formula relating the r winding numbers of the gauge fixed configurations to the total number of zero modes. Indeed, the number of zero modes is just the product of the winding numbers. This is a nontrivial result and we believe it is new. It goes much beyond the well known index theorem, which is trivial in two dimensions.

We would like to point out, that the gauge fixing introduced in section 2 has a natural extension to higher dimensions. For example in 4 dimensions the generalization of (2.5) reads

$$A_0 = 2\pi H_0 \frac{x^1}{L_0 L_1} + \tilde{A}_0^c, \quad A_1 = \tilde{A}_1, \quad A_2 = 2\pi H_2 \frac{x^3}{L_2 L_3} + \tilde{A}_2, \quad A_3 = \tilde{A}_3 \tag{6.3}$$

where the Cartan-pieces of the $\tilde{A}_\mu$ are constrained by

$$\begin{array}{ccc} \tilde{A}_0^c = C_0(x^1, x^2, x^3) & , & \int dx^0 \tilde{A}_1^c = C_1(x^2, x^3) \\ \int dx^0 dx^1 \tilde{A}_2^c = C_2(x^3) & , & \int dx^0 dx^1 dx^3 \tilde{A}_3^c = C_3. \end{array} \tag{6.4}$$

The constant parts of the $C_\mu \in \mathcal{H}$ are further restricted to avoid Gribov copies. Actually a slight modification of this gauge fixings can be achieved in all instanton sectors for $G = SU(N > 2)$ and in the sectors with even instanton numbers for $SU(2)$ [3].

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Appendix

A Conventions

In sections 2,3 and 4 we used the Chevalley basis [17]

$$\begin{aligned}
&[H_\alpha, H_\beta] = 0, \quad \forall\, \alpha, \beta \in \Delta \quad &,& \quad [H_\alpha, E_\beta] = K_{\beta\alpha} E_\beta \quad \forall\, \alpha, \beta \in \Delta \\
&[E_\alpha, E_{-\alpha}] = H_\alpha, \quad \forall\, \alpha \in \Delta \quad &,& \quad [E_\alpha, E_\beta] = E_{\alpha+\beta}, \quad \forall\, \alpha + \beta \in \Phi^+ \\
&[H_\alpha, E_{\beta+\gamma}] = (K_{\beta\alpha} + K_{\gamma\alpha}) E_{\beta+\gamma} \quad &,& \quad \forall\, \alpha, \beta, \gamma \in \Delta, \quad \beta + \gamma \in \Phi^+
\end{aligned} \tag{A.1}$$

where Δ is the set of simple roots and Φ^+ the set of positive roots. The Cartan matrix and the symmetric Coxeter matrix are given by

$$K_{\alpha\beta} = \frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} \quad \text{and} \quad C_{\alpha\beta} = \frac{4\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle \langle \alpha, \alpha \rangle}. \tag{A.2}$$

In the body of this paper we used $K_{\alpha\varphi}$ for simple α and positive φ. This 'extension' of the Cartan matrix is defined as for simple roots, see (A.2). For the traces we get

$$\mathrm{Tr}(H_\alpha H_\beta) = \frac{2}{\alpha^2} K_{\alpha\beta} \quad , \quad \mathrm{Tr}(E_\alpha E_\beta) = \frac{2}{\alpha^2} \delta_{\alpha,-\beta} \quad \text{and} \quad \mathrm{Tr}(H_\alpha E_\beta) = 0, \tag{A.3}$$

where Tr is the usual matrix trace multiplied by an appropriate normalisation constant which ensures $|\alpha_{long}|^2 = 2$.

B Proof of (5.12)

To prove (5.12) we rewrite $\tilde{h}_{0m}$ as a path ordered exponential. We define $b(\bar{z})$ by

$$\tilde{h}_{0m}(\bar{z}) \;\; = \;\; \mathcal{P} \exp \Big\{ i \int_{\bar{z}}^{\bar{z}-imL} \mathrm{d}\bar{u} \; b(\bar{u}) \Big\}. \tag{B.1}$$

The periodicity of $\partial_{\bar{z}} h_{0m}$ in x^0 translates into

$$b(\bar{z} + p\beta - imL) - b(\bar{z} + q\beta - imL) = h_{0m}(\bar{z}) \Big[b(\bar{z} + p\beta) - b(\bar{z} + q\beta) \Big] h_{0m}^{-1}(\bar{z}) \tag{B.2}$$

for arbitrary integers p, q and m. It follows at once, that b must be periodic in x^0, up to a linear term,

$$b(\bar{u}) = b_1 \bar{u} + b_p(\bar{u}). \tag{B.3}$$

From (B.2) it follows that h_{0m} lies in the little group of b_1,

$$h_{01}(\bar{z}) b_1 h_{01}^{-1}(\bar{z}) = b_1 \tag{B.4}$$

As a consequence b_1 commutes with b_p and

$$h_{0m} = e^{-(m^2 L^2 b_1/2 + imLb_1 \bar{z})} \mathcal{P} \exp\Big\{ i \int\limits_{\bar{z}}^{\bar{z}-imL} \mathrm{d}\bar{u}\; b_p(\bar{u}) \Big\}. \tag{B.5}$$

Since the h_{0m} are periodic in x^0 with period β, the constant b_1 has to be $2\pi H_A/V$ with $\exp(2\pi i H_A) = 1$. Thus H_A is diagonalizable and can take it to be in the Cartan subalgebra.

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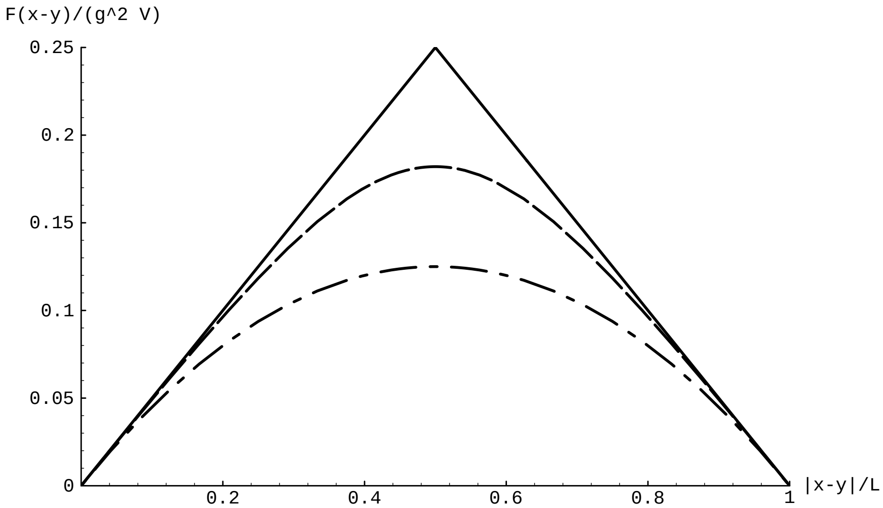

Figure 1:

$g^2V = \infty$: ———, $g^2V \sim 1$: $- - - -$, $g^2V = 0$: $- \cdot - \cdot -$

Interaction energy of two external charges for $SU(2)$, untwisted case.

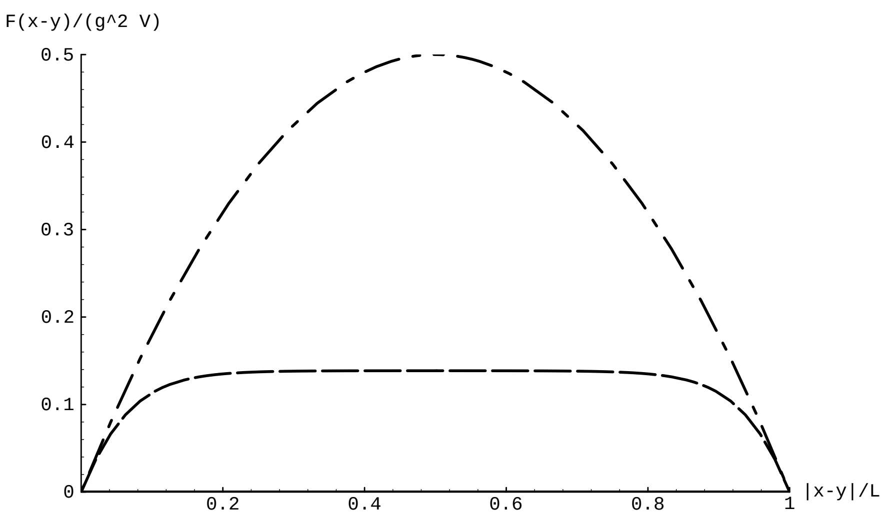

Figure 2:

$g^2V = \infty$: ———, $g^2V \sim 1$: $- - - -$, $g^2V = 0$: $- \cdot - \cdot -$

Interaction energy of two external charges for $SU(2)$, twisted case. $F/V = 0$ for $g^2V = \infty$.