DIAS – STP – 90 – 19

ETH – TH/90 – 20

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Conformally Reduced WZNW Theories and Two-Dimensional Gravity

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Abstract

The WZNW theories (for a non-compact form of the gauge groups) are reduced to a series of integrable theories that interpolate between WZNW theories and the corresponding Toda theories. They describe a set of WZNW fields in interaction with each other and with a two-dimensional gravitational field. An algorithm for constructing the general solutions, and a formula that relates the Virasoro and Kac-Moody centres of the reduced theories is given, together with a (conformally non-invariant) extension of the reduction to obtain affine Toda theories.

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In some previous reports it was shown that Toda field theories could be regarded as linearly constrained Wess-Zumino-Novikov-Witten (WZNW) theories, and that, by regarding them in this way, one could obtain a simple derivation of the general solutions of the Toda field equations [1] and a simple intuitive treatment of the associated $\mathcal{W}$-algebras [2]. In the present note we wish to show that the WZNW -Toda reduction generalizes naturally to produce a series of conformally invariant integrable theories which interpolate between the WZNW and Toda theories. These theories contain WZNW fields belonging to reducible WZNW groups, with the irreducible pieces in nearest neighbour interaction, thus providing a natural generalization of Toda theories. A remarkable feature of the theories is the emergence of a field which plays the role of the two-dimensional gravitational density $\sqrt{-g}$. Further features are the ease with which the general solutions of the field equations in these theories can be obtained from the well-known WZNW solution, and the formula for the centres of the Virasoro algebra in terms of the WZNW centre k, which is exactly the same as in the Toda case, but with the reduction parameter re-identified. A final feature is that the reduction procedure can be extended to obtain also the (non-conformally invariant) affine Toda theories.

We begin by recalling the standard WZNW theory with Lagrangian

$$S_W = \frac{k}{2}\int d^2x\, \eta^{\mu\nu}(g^{-1}\partial_\mu g)(g^{-1}\partial_\nu g) - \frac{2k}{3}\int (g^{-1}dg)^3\,, \qquad g \in G, \tag{1}$$

where G is a semi-simple Lie group and the boundary of the 3-dimensional (topological) integral is just the 2-space of the kinetic term [3]. The field equations for this Lagrangian are

$$\partial_- J(x) = \partial_+ \tilde{J}(x) = 0, \tag{2a}$$

where

$$J(x) = \partial_+ g(x) g^{-1}(x) \qquad \text{and} \qquad \tilde{J}(x) = g^{-1}(x)\partial_- g(x), \tag{2b}$$

$x^\pm$ being $\frac{1}{2}(x^0 \pm x^1)$ or $\frac{1}{2}(x^0 \pm i x^1)$, according to whether the space is Minkowskian or Euclidean. These equations show that $J(x)$ and $\tilde{J}(x)$ are functions of x^+ and x^- only, and the general solution for $g(x)$ is $g(x) = g(x^+)\,\tilde{g}(x^-)$, where g and $\tilde{g}$ are arbitrary elements of G. Because the current components $J^a = \mathrm{tr}\,(\sigma^a J)$, and similiarly for $\tilde{J}$, where σ^a are the generators of G, are the Noether currents for the invariance of S_W with respect to $g \to hg,\, gh$, where $h \in G$, h constant, they satisfy Kac-Moody (KM) algebras of the form

$$[J^a(x^+), J^b(y^+)] = \big(f^{ab}_{\ \ c}\, J^c(x^+) + k g^{ab}/\partial_{x^+}\big)\delta(x^+ - y^+). \tag{3}$$

Furthermore, because the action (1) is conformally invariant the energy-momentum density is traceless and the $T_{++} \equiv T$ and $T_{--} \equiv \tilde{T}$ components are function of x^+ and x^- only, and satisfy Virasoro algebras of the form

$$[T(x^+), T(y^+)] = \big(2T(x^+) + T'(x^+)\partial_{x^+} + \frac{c}{12}\partial^3_{x^+}\big)\,\delta(x^+ - y^+), \tag{4}$$

where c is a central term (that in general depends on the KM centre k). If one chooses for $T(x)$ the standard Sommerfield-Sugawara generators, which are normal-ordered bilinears in the currents in highest weight representations, one has

$$[T(x^+), J^a(y^+)] = J^a(x^+)\,\delta'(x^+ - y^+), \tag{5}$$

which shows that the currents are tensors (primary fields) of conformal weight unity with respect to the conformal group generated by these $T(x^+)$ [4].

Our reduction will require us to set some of the current-components equal to (non-zero) constants, but as they are vectors with respect to T this cannot be done without violating the conformal invariance generated by $T(x^+)$ and $\tilde{T}(x^-)$ (just as, in QFT, the vacuum expectation value of a tensor of non-zero rank cannot be set equal to a non-zero constant without violating Lorentz invariance). Hence our procedure will be to modify the energy-momentum tensor densities so that at least some of the currents become conformal scalars. The modification is as follows: First we choose as gauge-group G the (maximally non-compact) real Lie group which is generated by the real linear span of the Cartan basis $\{H_i, E_\alpha\}$ of the Lie algebra. Letting $\vec{\alpha}_j$ denote the l (=rank) simple roots of G and $\vec{m}_i$ their duals, i.e. the l fundamental coweights, satisfying

$$(\vec{m}_i, \vec{\alpha}_j) = \delta_{ij}. \tag{6}$$

Then choose a vector $\vec{\delta}$ in root space which is a sum of any subset of the $\vec{m}_i$, i.e. choose

$$\vec{\delta} = \sum_a \vec{m}_a, \qquad \{\vec{m}_a\} \subset \{\vec{m}_i\}, \tag{7}$$

(e.g. $\vec{\delta} = (1,1,0,1,0),\ (1,0,0,1,1)$ etc. in a coweight bases for $l=5$) and use $\vec{\delta}$ to define a privileged element $H = \delta^i H_i$ of the Cartan algebra. Because $(\vec{\delta}, \vec{\alpha}_j)$ is zero or unity, the element H has the property that for the simple root-vectors

$$[H, E_{\alpha_a}] = E_{\alpha_a} \qquad \text{and} \qquad [H, E_{\alpha_j}] = 0, \qquad \alpha_j \neq \alpha_a, \tag{8}$$

and since all the roots E_α of G are obtained by commutation from the E_{α_j} and $E_{-\alpha_j}$, we see that H provides a natural integer grading of the Lie algebra,

$$[H, E^h_\alpha] = h_\alpha \, E^h_\alpha, \tag{9}$$

where $h_\alpha = (\vec{\delta}, \vec{\alpha}) \in Z$ and $h_\alpha \geq 0$ for positive roots and $h_\alpha \leq 0$ for negative roots. In particular the little algebra of H (which includes at least the Cartan subalgebra of G) has zero grade. We shall denote by B the little group of G generated by this little algebra. The set of all possible little groups B is characterized by the fact that their compact forms B_c are just the little groups in the adjoint representation of the compact form G_c of G. (This can be seen by noting that in the compact form of G every element of the Lie algebra can be conjugated into the Cartan, where its little group is determined by the number of zeros in its coweight basis). Letting $\{J^i(x^+), J^\alpha(x^+)\}$ denote the KM-current components in the Cartan basis, the required modification of the energy momentum tensor may be written as

$$T(x^+) \to L(x^+) = T(x^+) - J'_H(x^+), \qquad \text{where} \qquad J_H(x^+) = \mathrm{tr}\big(J(x^+)H\big), \tag{10}$$

that is where $J_H(x^+)$ is the current component in the direction H (and a similar modification for $\tilde{T}$ but with plus $\tilde{J}'_H(x^-)$). The subtraction of J'_H from T has no effect on the currents belonging to the little group B of G (except for J_H itself) since J_H commutes with all these currents, but it has a two-fold effect on the remaining currents. First, the transformation law for J_H itself becomes

$$\{L(x^+), J_H(y^+)\} = J_H(x^+)\, \delta' - k \,\mathrm{tr}\, H^2 \, \delta'', \tag{11}$$

where k is the KM centre. This equation shows that J_H no longer transforms linearly, and hence not as a conformal tenor but transforms as a spin-one connection. The second effect of the modification (10) is to change the transformation law (5) for the currents J^α to

$$\{L(x^+), J^\alpha(y^+)\} = (1 + h_\alpha) J^\alpha(x^+)\, \delta' + h_\alpha (J^\alpha)'(x^+)\, \delta \tag{12}$$

which shows that they transform as tensors of conformal spin $(1{+}h_\alpha, 0)$ instead of $(1, 0)$. In particular the current components J^α_{-1} (we use the same notation as earlier, i.e. the subscript denotes the grade of the root α) transform as conformal scalars. Similarly for the $\tilde{J}^\alpha_1$.

Since the currents J^{α}_{-1} and $\tilde{J}^{\alpha}_{1}$ are conformal scalars they can be made constant without violating conformal invariance, and our main constraints will in fact be

$$J^{\alpha}_{-1}(x^{+}) = \mu^{\alpha} \qquad \text{and} \qquad \tilde{J}^{\alpha}_{1}(x^{-}) = \nu^{\alpha}, \tag{13}$$

where $\mu^{\alpha}, \nu^{\alpha}$ are non-zero constants. However, since all the negative and positive currents J^{α}_{-p} and $\tilde{J}^{\alpha}_{p}$, $p{=}1,2,..$ can be generated by commutation from J^{α}_{-1} and $\tilde{J}^{\alpha}_{1}$ (and the currents of the little group B) respectively, we cannot impose (13) consistently unless we also impose the constraints

$$J^{\alpha}_{-p}(x^{+}) = 0 \qquad \text{and} \qquad \tilde{J}^{\alpha}_{p}(x^{-}) = 0, \quad p = 2,3,... \ . \tag{14}$$

Eqns. (14,15) represent our full set of constraints on the KM currents of the WZNW theory and may be summarized as

$$J_{neg}(x^{+}) = M \qquad \text{and} \qquad \tilde{J}_{pos}(x^{-}) = \tilde{M}, \tag{15}$$

where M and $\tilde{M}$ are constant matrices of grade -1 and 1 respectively, that is $[H,M]{=}{-M}$ and $[H,\tilde{M}]{=}\tilde{M}$. They are first-class constraints, and from (15) are seen to be just special solutions of (some of) the WZNW field equations. To obtain a more intuitive picture of their meaning let us consider for example the case $G = SL(9,R)$ and $\vec{\delta} = (0,0,0,1,0,1,0,0)$, so that the little group B is $S\big(GL(4,R) \times GL(2,R) \times GL(3,R)\big)$. Then the constrained currents are as shown in Fig. 1. Finally we note that, like all first class constraints, the constraints (15) generate a system of gauge transformations, and that these are just the KM transformations corresponding to the $(\mathrm{dim}G{-}\mathrm{dim}B)/2$ -dimensional nilpotent subgroups of G generated by the root-vectors E_{α} and $E_{-\alpha}$ $\big((\vec{\delta},\vec{\alpha}) > 0\big)$, respectively. The gauge freedom can be used to gauge $(\mathrm{dim}G{-}\mathrm{dim}B)/2$ of the remaining currents to zero. This leaves only $\mathrm{dim}B$ 'true' currents, and the gauge can be chosen so that these are the currents $j_b = \partial_{+}b\,b^{-1}$ and $\tilde{j}_b = b^{-1}\partial_{-}b$, $b \in B$, of the little group B.

We now wish to show that the constraints (15) reduce the WZNW theory for G to a theory which contains two-dimensional gravity and a set of WZNW fields belonging to the subgroup B interacting with each other and the gravitational field. To show this we first note that the WZNW group G admits a local Gauss decomposition

$$G = ABC, \quad ; \quad g = abc \qquad g \in G, a \in A, \text{etc.}, \tag{16}$$

where B is the little group and A and C are the nilpotent subgroups discussed above. (Although the decomposition is only local, the whole group can be covered by a finite number of patches with the decomposition (16) multiplied by a constant matrix in each patch). We then show that the partial constraints (15) for the full KM currents J and $\tilde{J}$ are equivalent to the full constraints

$$j_c(x) = b^{-1}(x) M\, b(x) \qquad \text{and} \qquad \tilde{j}_a(x) = b(x) \tilde{M}\, b^{-1}(x) \tag{17}$$

for the partial currents j_c and $\tilde{j}_a$ belonging to the subgroups C and A respectively. Note that contrary to the full currents J and and $\tilde{J}$ the partial currents are not chiral since the group elements a, b and c in the Gauss decomposition (16) are not chiral. To establish the equivalence of (15) and (17) let us consider J and j_c for example. In an obvious notation we have, for $g = abc$

$$\begin{aligned} J = \partial_+ g g^{-1} &= (abc_+ + ab_+c + a_+bc)c^{-1}b^{-1}a^{-1} \\ &= abj_c b^{-1}a^{-1} + aj_b a^{-1} + j_a. \end{aligned} \tag{18}$$

Since the last two terms in (18) are non-negative by the definition of A, B and C, we then have

$$J_{neg} = (abj_c b^{-1}a^{-1})_{neg}, \tag{19}$$

and thus the condition for J in (15) may be written as

$$(abj_c b^{-1}a^{-1})_{neg} = M. \tag{20}$$

Since M is already negative, (20) can be written as

$$(abj_c b^{-1}a^{-1} - M) = Q \qquad \text{where} \quad Q \geq 0, \tag{21}$$

or, by conjugating with a, as

$$bj_c b^{-1} - M = (a^{-1}Ma - M) + a^{-1}Qa. \tag{22}$$

But since $(a-1)$ and $(a^{-1}-1)$ are strictly positive and M has grade minus one, both expressions on the right of (22) are non-negative. On the other hand, the expression on the left of (22) is strictly negative by definition. Hence each side of (22) must be zero separately. Thus, the condition (15) for $J(x^+)$ implies (17) for $j_c(x)$. Conversely, it is easy to check from (18) that (17) for $j_c(x)$ implies (15) for $J(x^+)$. This result, together with the corresponging result for $\tilde{J}$ and $\tilde{j}_a$, establishes the required equivalence of (15) and (17).

Let us now decompose the WZNW field equations with respect to the Gauss decomposition $g = abc$. After some straightforward algebra one obtains

$$a^{-1}\,\partial_- J\, a = \partial_- j_b - [b j_c b^{-1}, \tilde{j}_a] + \partial_-(b j_c b^{-1}) + b\,\{\partial_+(b^{-1}\tilde{j}_a b)\}\, b^{-1} = 0, \tag{23}$$

and similiarly for $\tilde{J}$. If we now impose the constraints in the form (17) the last two terms in (23) vanish because $M, \tilde{M}$ are constant, and we obtain

$$\partial_- j_b = [M, b\tilde{M}b^{-1}] \qquad \text{and} \qquad \partial_+ \tilde{j}_b = [b^{-1}Mb, \tilde{M}], \tag{24}$$

where one set of field equations follow from the other by conjugation with b. The eqs. (24) are the required field equations for the fields b of the little group B. Note that they do not involve the fields a, c of the subgroups A and C of G, and thus are self-contained. They can be derived from the effective action

$$S_{eff}[b] = S_W[b] - \int d^2x\, \mathrm{tr}\,(Mb\tilde{M}b^{-1}). \tag{25}$$

This action shows that the b-fields are just a set of WZNW fields belonging to the (reducible) WZNW group B, with a (non-derivative) coupling between the nearest-neighbour irreducible blocks, which are linked by the non-zero grade $\pm$ constant matrices $M, \tilde{M}$ (see Fig. 1). In fact the action (25) is the natural generalization of the Toda action for abelian fields to the case of non-abelian WZNW fields, and in the special case when B is abelian (i.e. is the Cartan subgroup of G) all simple roots have weight one and we have

$$\mathrm{tr}\,(Mb\tilde{M}b^{-1}) = \sum_{i,j} \mu^i \nu^j\, \mathrm{tr}(E_{-\alpha_i} e^{\phi^p H_p} E_{\alpha_j} e^{-\phi^q H_q}) = \sum_{\text{simple } \alpha} \frac{2(\mu,\nu)}{\vec{\alpha}^2}\, e^{(\vec{\alpha},\vec{\phi})}, \tag{26}$$

so (25) reduces to the usual Toda action [5]. Thus in general the constraints reduce the standard WZNW theory for the irreducible group G to an interacting WZNW theory for the reducible little group B.

We next wish to show that the Lagrangian (25) contains also a two-dimensional gravitational field, and that, with respect to this field it is not only conformally, but general coordinate invariant. To show this we note that since, by definition, the group $GL(1)$ generated by the privileged element H is in the centre of the little group B, the little group B may be written (locally) as a direct product $B = GL(1) \times \hat{B}$. Hence if we write the b-fields in the form $b(x) = \hat{b}(x)\, \exp(\phi(x)H)$ the action (25) may be written as

$$S_{eff}(\hat{b}, h) = S_W(\hat{b}) + \frac{\mathrm{tr}\, H^2}{2} \int d^2x\, \partial_+\phi\partial_-\phi - \int d^2x\, e^{\phi(x)}\mathrm{tr}(M\hat{b}\tilde{M}\hat{b}^{-1}), \tag{27}$$

where the factor $\exp(\phi(x))$ appears in the last integral because the $M, \tilde{M}$ have H-grade ± 1. We then recall that the current components of j_b orthogonal to H have conformal weights one and that $\partial_\pm \phi$ transforms as a spin one connection. It follows that $\hat{b}$ is a conformal scalar and $\exp(\phi)$ has conformal weights $(1,1)$. On account of this it is permissible, indeed quite natural, to introduce a curved 2-manifold with metric tensor $g_{\mu\nu}$ defined as

$$g_{\mu\nu}(x) = e^{\phi(x)}\, \eta_{\mu\nu}, \tag{28}$$

where $\eta_{\mu\nu}$ is the 2-dimensional Minkowskian (or Euclidean) metric. Then the action (27) may be written in the form

$$S_{eff}(\hat{b}, g_{\mu\nu}) = \int d^2x\, \sqrt{-g} \Big\{ \frac{1}{2} \mathrm{tr}\, H^2 \; R\, \nabla^{-2} R - \mathrm{tr}\big(M \hat{b} \tilde{M} \hat{b}^{-1}\big) \Big\} + S_W(\hat{b}, g_{\mu\nu},) \tag{29}$$

where of course $S_W(\hat{b}, g_{\mu\nu})$ means that in the kinetic term of $S_W(\hat{b})$, the Minkowskian $\eta^{\mu\nu}\partial_\mu\,.\,\partial_\nu\,.$ is replaced by $\sqrt{-g} g^{\mu\nu} \partial_\mu\,.\partial_\nu\,.\,.$ If it is understood that the conformal scalarity of $\hat{b}$ is extended to scalarity with respect to general coordinate transformations then the general coordinate invariance of S_{eff} in the form (29) is manifest. One sees, therefore, that the reduction provides us not only with a conformally invariant self-interacting WZNW theory, but with a unified theory of WZNW theory and (2-dimensional) gravity.

Note that, in the form (29), the energy momentum tensor $T_{\mu\nu}$ defined as $1/\sqrt{-g} \cdot \delta S / \delta g^{\mu\nu}$ is automatically traceless due to the field equation for $\exp(\phi) = \sqrt{-g}$,

$$\frac{\delta S}{\delta \sqrt{-g}} = \frac{\delta S}{\delta g^{\mu\nu}} \frac{\delta g^{\mu\nu}}{\delta \sqrt{-g}} = -g^{\mu\nu}\, T_{\mu\nu} = 0, \tag{30}$$

and it is easy to verify that the Virasoro densities $L = T - \partial_+ J_H$ and $\tilde{L} = \tilde{T} + \partial_- \tilde{J}_H$ introduced earlier in order to reduce the WZNW theory coincide exactly with the T_{++} and T_{--} components of this energy-momentum tensor $T_{\mu\nu}$. Note also that once $\exp(\phi(x))$ is identified as $\sqrt{-g}$ it defines a covariant derivative (e.g. on vectors $\nabla_\pm = \partial_\pm + \Gamma_\pm$) with Christoffel symbols $\Gamma_\pm = \partial_\pm \phi$, and that the existence of this covariant derivative explains why the H-components of the currents loose their tensorial (primary field) character under reduction. The point is that the currents $\nabla_+ b\, b^{-1}$ and $b^{-1} \nabla_- b$ formed with $\nabla_\pm$ are tensors, whereas the actual currents $\partial_+ b\, b^{-1}$ and $b^{-1} \partial_- b$ are not, but since

$$j_b = \partial_+ b\, b^{-1} = \nabla_+ b\, b^{-1} - H \Gamma_+, \tag{31}$$

and similiarly for $\tilde{j}_b$, the non-covariant pieces occur only for the H-components of the currents.

We now turn to the general solution of the field equation (24). Since the constraints satisfy the field equations (indeed are special solutions to some of them) the general solution must be of the usual WZNW form [3]

$$g(x) = g(x^+)\tilde{g}(x^-). \tag{32}$$

The only new feature is that $g(x^+)$ and $\tilde{g}(x^-)$ are no longer completely free but are subject to the constraints on the currents. To see the effect of the constraints, we ignore them for the moment and make a Gauss decompostion of all the g's in (32) to obtain

$$a(x)b(x)c(x) = a(x^+)b(x^+)c(x^+) \cdot \tilde{a}(x^-)\tilde{b}(x^-)\tilde{c}(x^-). \tag{33}$$

From (33) one sees that the B-component $b(x)$ of $g(x)$ is the the B-component in the Gauss-decomposition of $b(x^+)c(x^+) \cdot \tilde{a}(x^-)\tilde{b}(x^-)$. Furthermore, if

$$c(x^+) \cdot \tilde{a}(x^-) = \alpha(x)\beta(x)\gamma(x) \tag{34}$$

is the Gauss decomposition of $c\tilde{a}$ alone, then the required Gauss-decomposition of $bc\tilde{a}\tilde{b}$ is

$$[b(x^+)\alpha(x)b^{-1}(x^+)]\,[b(x^+)\beta(x)\tilde{b}(x^-)]\,[\tilde{b}^{-1}(x^-)\gamma(x)\tilde{b}(x^-)], \tag{35}$$

and so the required $b(x)$ component of $g(x)$ is

$$b(x) = b(x^+)\beta(x)\tilde{b}(x^-). \tag{36}$$

Let us now impose the constraints, which from (17) are just

$$\partial_+ c\, c^{-1} = b^{-1}Mb \qquad \text{and} \qquad \tilde{a}^{-1}\partial_- \tilde{a} = \tilde{b}\tilde{M}\tilde{b}^{-1}, \tag{37}$$

and thus determine $c(x^+)$ and $\tilde{a}(x^-)$ in terms of $b(x^+)$ and $\tilde{b}(x^-)$ respectively (up to constant matrices which can be absorbed in $b, \tilde{b}$). Since the matrix $\beta(x)$ is determined uniquely by $c(x^+)$ and $\tilde{a}(x^-)$ from (34) we then see that the role of the constraints is to determine the matrix $\beta(x)$ in terms of the matrices $b(x^+)$ and $\tilde{b}(x^-)$. Thus, the general solution of the field equation for the reduced system is (36), where $b(x^+)$ and $\tilde{b}(x^-)$ are arbitrary and $\beta(x)$ is determined in terms of $b(x^+)$ and $\tilde{b}(x^-)$ by (34) and (37). Eq. (36) should be compared to the non-interacting WZNW solution for $b(x)$ which is (36) with $\beta{=}1$.

From the above discussion we see that the algorithm for constructing the general solution is to take arbitrary matrices $b(x^+)$, $\tilde{b}(x^-)$, solve (37) for $c(x^+)$ and $\tilde{a}(x^-)$, and determine $\beta(x)$ (algebraically) from (34). It might be thought that this procedure only shifts the problem to solving another set of differential equations, namely (37), but because of the nilpotency of the groups A and C, these can be solved by successive integration of already known quantities. In fact, if $c(x^+)$ is decomposed into its H-components $c_h(x^+)$ the solution is given by the finite series

$$c(x^+) = 1 + \sum_h c_h(x^+) \quad \text{where} \quad c_{h+1} = \int_0^{x^+} dy\, b^{-1}(y) M\, b(y) c_h(y) \tag{38}$$

and similiarly for $\tilde{a}(x^-)$. To illustrate the algorithm in more detail let us consider the 'Liouville' analogue in which the little group B has only two irreducible blocks, for definiteness $G = SL(n,R)$ and $B = S\big(GL(p,R)\otimes GL(q,R)\big)$ for $p{+}q{=}n$. Then

$$H = \begin{pmatrix} \frac{p}{n}1_q & 0 \\ 0 & -\frac{q}{n}1_p \end{pmatrix} \quad b(x) = \begin{pmatrix} b_1(x) & 0 \\ 0 & b_2(x) \end{pmatrix} \quad M = \begin{pmatrix} 0 & 0 \\ m & 0 \end{pmatrix} \quad \tilde{M} = \begin{pmatrix} 0 & \tilde{m} \\ 0 & 0 \end{pmatrix}, \tag{39}$$

and the Gauss decomposition (34) is

$$\begin{aligned} c(x^+)\tilde{a}(x^-) &= \begin{pmatrix} 1 & 0 \\ l(x^+) & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{r}(x^-) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & u(x) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\Delta^\dagger)^{-1} & 0 \\ 0 & \Delta(x) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v(x) & 1 \end{pmatrix}, \end{aligned} \tag{40}$$

where $\Delta(x) = 1 + l(x^+)\tilde{r}(x^-)$ and $(u,v) = (\tilde{r}\Delta^{-1}, \Delta^{-1}l)$. Thus $\beta =$diag$((\Delta^\dagger)^{-1}, \Delta)$ and the solutions for the blocks in $b(x)$ are

$$b_1(x) = b_1(x^+)\frac{1}{\Delta^\dagger(x))}\,\tilde{b}_1(x^-) \quad \text{and} \quad b_2(x) = b_2(x^+)\Delta(x)\,\tilde{b}_2(x^-). \tag{41}$$

It remains therefore only to compute $l(x^+)$ and $r(x^-)$ from the finite series (38). It is easy to see that the series terminates after one step and thus

$$l(x^+) = \int_0^{x^+} dy\, b_2^{-1}(y)\, m\, b_1(y) \qquad \text{and} \qquad r(x^-) = \int_0^{x^-} dy\, \tilde{b}_1(y)\, \tilde{m}\, \tilde{b}_2^{-1}(y). \tag{42}$$

Accordingly, (41) and (42), where $\Delta(x) = 1 + l(x^+) r(x^-)$ is the general solution for the two-block case. Note that the solution (41) for $b_1(x)$ generalizes the general solution

$$b_1^2(x) = \partial_+ l(x^+)\,[1 + l(x^+)\tilde{r}(x^-)]^{-2}\,\partial_- \tilde{r}(x^-) \tag{43}$$

of the Liouville equation, and reduces to it for $G = SL(2,R)$. Note also that when $p{=}q$ the two-block system admits the reflexion symmetry

$$b_1 \longleftrightarrow (b_2^\dagger)^{-1} \quad , \quad \tilde{M} \longleftrightarrow (M^\dagger)^{-1} \tag{44}$$

and that if one identifies b_1 with $(b_2^t)^{-1}$ one obtains a simple WZNW field in self-interaction (or, more precisely a WZNW field $\tilde{b}$ in interaction with itself and with the gravitational Liouville field $\exp(\phi)$).

As a first step toward quantization of the reduced theories we compute their Virasoro centres in terms of the KM centres k, assuming only that quantization requires the use of highest-weight (Fock-space) representations. There are two main contributions. First, there is the direct contribution of the modified energy-momentum tensor $L(x^+)$, namely,

$$c_L = \frac{\mathsf{dim}G}{1 + g/k} - 12k \operatorname{tr} H^2, \tag{45}$$

where g is the dual Coxeter number of the group G, the first term is the well known centre of the Sommerfield-Sugawara energy-momentum tensor [4], and the second term comes from the $J'_H(x^+)$-modification. Second there is the contribution from the BRST ghost-pairs due to the constraints (15) or (17) and the corresponding gauge-fixing constraints. As is well known the ghost-contribution takes the form

$$c_{gh} = -2\sum_{j>0}[1 + 6j(j-1)], \tag{46}$$

where j is the conformal weight of the ghosts [6]. But since these are just the weights of the corresponding constrained currents, (46) may be written as

$$\begin{aligned} c_{gh} &= -(\mathsf{dim}G - \mathsf{dim}B) - 6\operatorname{tr}(\mathsf{adj}\,H)^2 + 12\sum_{\alpha>0}(\vec{\alpha}, \vec{\delta}) \\ &= \mathsf{dim}B - \mathsf{dim}G - 12\operatorname{tr}H^2 + 24(\vec{\rho}, \vec{\delta}), \end{aligned} \tag{47}$$

where $\vec{\rho}$ is half the sum of the positive roots. Adding (47) and (45), and using the Freudenthal-deVries formula $g\,\text{dim}G = 12\vec{\rho}^{\,2}$, we obtain finally

$$\begin{aligned} c = c_L + c_{gh} &= \text{dim}B - 12\frac{\vec{\rho}^{\,2}}{k+g} - 12(g+k)\text{tr}H^2 + 24(\vec{\rho},\vec{\delta}) \\ &= \text{dim}B - 12\Big(\frac{\vec{\rho}}{\sqrt{k+g}} - \sqrt{k+g}\,\vec{\delta}\Big)^2. \end{aligned} \tag{48}$$

This generalizes the formula obtained previously for the Toda theory [2], and reduces to it for B=Cartan ($\text{dim}B = l$) and $\vec{\delta}$=sum of all the fundamental coweights=half the sum of the positive coroots. Note that for the simply-laced Toda case the formula reduces to $c = l - 12\vec{\rho}^{\,2}(\beta - 1/\beta)^2$, where $\beta^2 = k+g$, which is reminiscent of the general formulae for decenerate conformal field theories. But the formula (48) was obtained only under the assumption that the representations were highest weight, and, until some other conditions such as unitarity or rationality are added, the value of $(k+g)$ is unknown. It could even be negative, in which case the two negative signs in (48) would be replaced by positive one, a result that has been obtained using a completely different quantization procedure in ref. [7].

Finally, we consider the possibility of generalizing the reduction process even further. For this we note that the reduction of the WZNW field equations in (23) is actually valid for the decomposition $g = abc$ of $g \in G$ into any three a, b and c and for any choice of constant matrices $M\,\tilde{M}$, not merely when B is a little group, A, C are nilpotent, and M, $\tilde{M}$ have grade ± 1. The only difference is that, in the general case, the reduction is not necessarily conformally invariant, there is no analogue of the gravitational field $\exp(\phi(x))$, and the constraints

$$j_c = b^{-1}Mb \quad \text{and} \quad \tilde{j}_a = b\tilde{M}b^{-1}, \tag{49}$$

are not, in general, expressible as linear constraints for the full WZNW currents $J(x^+)$ and $\tilde{J}(x^-)$.

For example, if we choose B as in the usual Toda theory, but choose the matrices $M, \tilde{M}$ in (49) as

$$M = \sum_{\alpha\, simple} \mu^\alpha E_{-\alpha} + \mu E_\gamma \qquad \text{and} \qquad \tilde{M} = \sum_{\alpha\, simple} \nu^\alpha E_\alpha + \nu E_{-\gamma}, \tag{50}$$

where $\vec{\gamma}$ is the highest root, then the field equations for the fields ϕ^α belonging to the diagonal subgroup B ($b{=}\exp(\phi^\alpha H_\alpha)$) are, from (24),

$$\nabla^2\vec{\Phi} = -\sum_{\text{simple}\,\alpha} \vec{\alpha}e^{(\vec{\alpha},\vec{\Phi})} + \vec{\gamma}e^{-(\vec{\gamma},\vec{\Phi})}, \qquad \text{where} \qquad \vec{\Phi} = \sum \frac{2\vec{\alpha}}{\vec{\alpha}^{\,2}}\,\phi^\alpha, \tag{51}$$

and we have set $2\mu^\alpha\nu^\alpha = \vec{\alpha}^{\,2}$ and $2\mu\nu = \vec{\gamma}^{\,2}$. These are just the (non-conformally-invariant) equations of affine Toda field theory [8]. In particular for $G = SL(2,R)$ we have

$$\nabla^2\phi = -\sinh\phi, \qquad \text{where} \qquad \phi = \Phi/\sqrt{2}, \tag{52}$$

which is just the sinh-Gordon equation.

Acknowledgements:

We are grateful to J. Balog, L. Dabrowski, L. Feher, P. Forgacs, S. Lukyanov, A. Morozow, M. Olshanetsky and L. Palla for discussions and helpful criticism during the course of this work.

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Figure 1:

$$H = \frac{1}{9}\begin{pmatrix} 8 &&&&&&&& \\ & 8 &&&&&&& \\ && 8 &&&&&& \\ &&& 8 &&&&& \\ &&&& -1 &&&& \\ &&&&& -1 &&& \\ &&&&&& -10 && \\ &&&&&&& -10 & \\ &&&&&&&& -10 \end{pmatrix}$$

$$B = \begin{pmatrix} * & * & * & * &&&&& \\ * & * & * & * &&&&& \\ * & * & * & * &&&&& \\ * & * & * & * &&&&& \\ &&&& * & * &&& \\ &&&& * & * &&& \\ &&&&&& * & * & * \\ &&&&&& * & * & * \\ &&&&&& * & * & * \end{pmatrix}$$

$$M = \begin{pmatrix} 0 &&&&&&&& \\ 0 & 0 &&&&&&& \\ 0 & 0 & 0 &&&&&& \\ 0 & 0 & 0 & 0 &&&&& \\ m_{51} & m_{52} & m_{53} & m_{54} & 0 &&&& \\ m_{61} & m_{62} & m_{63} & m_{64} & 0 & 0 &&& \\ 0 & 0 & 0 & 0 & m_{75} & m_{76} & 0 && \\ 0 & 0 & 0 & 0 & m_{85} & m_{86} & 0 & 0 & \\ 0 & 0 & 0 & 0 & m_{95} & m_{96} & 0 & 0 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} * & * & * & * &&&&& \\ * & * & * & * &&&&& \\ * & * & * & * &&&&& \\ * & * & * & * &&&&& \\ m_{51} & m_{52} & m_{53} & m_{54} & * & * &&& \\ m_{61} & m_{62} & m_{63} & m_{64} & * & * &&& \\ 0 & 0 & 0 & 0 & m_{75} & m_{76} & * & * & * \\ 0 & 0 & 0 & 0 & m_{85} & m_{86} & * & * & * \\ 0 & 0 & 0 & 0 & m_{95} & m_{96} & * & * & * \end{pmatrix}$$

Figure caption:

Fig. 1: The special element H of the Cartan subgroup, the constant Matrix M of H-grade -1, the little group B and the contrained current J for the group $SL(9,R)$ and $\vec{\delta}=(0,0,0,1,0,1,0,0)$.