Chapter 6

Functional Schrödinger Equation for Fermions in External Gauge Fields

In some applications the language of wave functionals and the functional Schrödinger equation has provided valuable insights (See, e. g., [1] and [2] for a review). One big advantage of the Schrödinger picture is that the intuitive picture of evolving wave functions, so successful in quantum mechanics, can be extended to problems in field theory. It is of course still an open problem whether the existence of the Schrödinger picture can be proved rigorously. At least in the case of renormalizable scalar field theories it has been demonstrated that a functional Schrödinger equation with respect to a global time parameter exists at each order of perturbation theory [3]. For arbitrary local time variations an explicit calculation has verified the validity of the Schrödinger equation up to two loops [4].

An important field of application is quantum gravity. Since quantum general relativity is non-renormalizable at the perturbative level, one has to develop non-perturbative methods, provided the theory is viable at all. There have been remarkable developments in canonical quantum gravity in recent years which have so far culminated in the discovery, by using the functional Schrödinger picture, of exact formal solutions to all constraint equations [5]. The use of wave functionals has also been useful in performing semi classical approximations, for example in the derivation of formal correction terms to the Schrödinger equation from quantum gravity [6]. It may thus turn out to be very useful for later applications to explore the potentialities of the functional Schrödinger picture in ordinary field theory.

In this chapter couple fermionic matter to gauge fields. Apart from the last paragraph we limit ourselves to the case where the gauge field can be treated semi classically, i.e. we discuss the functional Schrödinger equation for the fermionic wave functional in a prescribed external gauge field. Most of our work deals with QED but we also give some results for the non-Abelian case.

We start by giving a brief review of the functional Schrödinger equation for fermions following, with elaborations, the work of Floreanini and Jackiw [8]. Gaussian states are used as generalized vacuum states, but contrary to the bosonic case one has to fix a filling prescription for the Dirac sea to select a particular vacuum. Section 6.1.3 is concerned with the time-dependent Schrödinger equation. We give its formal solution for arbitrary external fields in terms of solutions of the (first-quantized) Dirac equation.

We then proceed to calculate the exact ground state for arbitrary external fields in two dimensional QED in both the massless and the massive case (section 6.2). We give explicit expressions for the expectation values of the Hamiltonian, the electric charge, and the axial charge with respect to this ground state. Regularization is performed through gauge-invariant point splitting. All results are given for the case of finite as well as infinite space intervals. The finite case allows a careful discussion of the dependence of the Casimir energy on the chosen boundary conditions.

The extension to non-Abelian fields in two dimensions is straightforward and is worked out in section 6.3. We give the exact ground state as well as the expectations values for the Hamiltonian, the electric and axial charges.

In section 6.4 then proceed to discuss applications of the time - dependent Schrödinger equation. The particle creation rate for constant external electric fields is calculated in this framework and the classical result found by Schwinger is recovered (section 6.4.1). In the massless case in two dimensions we calculate the anomalous particle production rate for arbitrary external fields. Its interpretation in the functional language is very transparent – the anomalous production rate is basically due to the dependence of the filling prescription on the external field (section 6.4.2).

Finally we go beyond the external field approximation and discuss briefly some subtleties connected with the interpretation of Gauss law. We show that, except for the case when anomalies violating gauge invariance are present, the interpretation of the Gauss constraint as a generator of gauge transformation can be rescued even if it does no longer annihilate gauge invariant states. We also present a brief outlook on possible future work.

6.1 Functional Schrödinger equation for fermions

6.1.1 Commutation relations and inner product

In this section we give a brief review of the canonical formalism for QED and the functional Schrödinger picture. Unless otherwise stated, the dimension D of spacetime is left arbitrary. The Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}(D_\mu\gamma^\mu - m)\psi, \tag{6.1}$$

where

$$D_\mu = \partial_\mu + ieA_\mu$$

is the covariant derivative associated with the electromagnetic potential A_μ. The canonical momenta read

$$\pi_0 = 0, \quad \pi_i = F_{i0} \equiv E^i, \quad \pi_\psi = i\psi^\dagger \tag{6.2}$$

so that the total Hamiltonian is given by

$$\begin{aligned} H &= \int dx \left(\frac{1}{2}\mathbf{E}^2 + \frac{1}{4}F^{ij}F_{ij}\right) + \int dxdy\psi^\dagger(x)h(x,y)\psi(y) \\ &\quad + \int dxA^0(e\psi^\dagger\psi - \nabla\mathbf{E}), \end{aligned} \tag{6.3}$$

where

$$h(x,y) = -i\gamma^0\gamma^i\frac{\partial}{\partial x^i}\delta(x-y) + \gamma^0(m + e\gamma^i A_i)\delta(x-y) \tag{6.4}$$

plays the role of a *first quantized* Dirac Hamiltonian in an external electromagnetic field. We will denote with $h_{(0)}$ the first quantized Hamiltonian without external field. We note that x and y is a shorthand notation for a vector in $(D-1)$ dimensional space, and the metric convention for spacetime is diag$(1, -1-1, ...)$. Variation of (6.3) with respect to A^0 yields the *Gauss constraint*

$$\nabla\mathbf{E} = e\psi^\dagger\psi. \tag{6.5}$$

In the following we use the gauge condition $A^0 = 0$. The commutation relations read

$$[A_i(x), E^j(y)] = i\delta_i^j\delta(x-y) \tag{6.6}$$

for the electromagnetic field, and

$$\{\psi_\alpha(x), \psi^\dagger_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y) \tag{6.7}$$

for the fermion fields. All other commutators (anticommutators) vanish.

In the functional Schrödinger picture we represent these commutation relations by acting with the field operators on physical states $\Psi[u, u^\dagger, \mathbf{A}]$ according to

$$E^j \to \frac{1}{i}\frac{\delta}{\delta A_j}, \quad \psi_\alpha \to \frac{1}{\sqrt{2}}\left(u_\alpha + \frac{\delta}{\delta u_\alpha^\dagger}\right), \quad \psi_\alpha^\dagger \to \frac{1}{\sqrt{2}}\left(u_\alpha^\dagger + \frac{\delta}{\delta u_\alpha}\right), \tag{6.8}$$

and $\mathbf{A}$ is represented by multiplication. Note that u_α and $u_\alpha^\dagger$ are Grassmann variables, and Ψ is *not* an eigenstate of either ψ or $\psi^\dagger$. An alternative representation has been used, for example, in [9], where ψ is represented, as in the bosonic case, by multiplication with u, and $\psi^\dagger$ is represented by $\delta/\delta u$. Since, however, the Hermitian conjugate of u in that representation is not given by $u^\dagger$, but by $\delta/\delta u$, we find it easier for our discussion to resort to the representation (6.8).

The Grassmann character of the fermion fields requires a careful treatment of the inner product [8]. If one defines the inner product by the functional integration (we do in the following not explicitly write out the electromagnetic field and the spinor indices)

$$\langle\Psi_1|\Psi_2\rangle \equiv \int \mathcal{D}u^\dagger \mathcal{D}u \Psi_1^* \Psi_2 = \langle\Psi_2|\Psi_1\rangle^*, \tag{6.9}$$

the dual Ψ^* of a state Ψ is not given by ordinary complex conjugation, but by the expression

$$\Psi^*[u, u^\dagger] = \int \mathcal{D}\bar{u}^\dagger \mathcal{D}\bar{u} \exp\left(\bar{u}u^\dagger + \bar{u}^\dagger u\right) \bar{\Psi}[\bar{u}, \bar{u}^\dagger]. \tag{6.10}$$

Here, $\bar{\Psi}$ is the hermitian conjugate of Ψ. We have used a compact notation, i. e., $\bar{u}u \equiv \int dx \bar{u}_\alpha(x) u_\alpha(x)$, etc. Note the analogy to the Bargmann representation for the harmonic oscillator in quantum mechanics.

A special role is played by Gaussian states,

$$\Psi = \exp\left(u^\dagger \Omega u\right), \tag{6.11}$$

since this generalizes the notion of a Fock vacuum; Ω is sometimes called the "covariance." If we apply the above rules to such a state we find

$$\bar{\Psi}[\bar{u}, \bar{u}^\dagger] = \exp\left(\bar{u}^\dagger \Omega^\dagger \bar{u}\right), \tag{6.12}$$

and for the dual, applying the familiar rules of Grassmann integration,

$$\begin{aligned} \Psi^*[u, u^\dagger] &= \int \mathcal{D}\bar{u}^\dagger \mathcal{D}\bar{u} \exp\left(\bar{u}u^\dagger + \bar{u}^\dagger u + \bar{u}^\dagger \Omega^\dagger \bar{u}\right) \\ &= \det(-\Omega^\dagger) \exp\left(u^\dagger (\Omega^\dagger)^{-1} u\right). \end{aligned} \tag{6.13}$$

One then finds for $\langle\Psi|\Psi\rangle$ the expression

$$\begin{aligned}\langle\Psi|\Psi\rangle &= \det(-\Omega^\dagger)\int \mathcal{D}u^\dagger\mathcal{D}u\exp\left(u^\dagger\left[(\Omega^\dagger)^{-1}+\Omega\right]u\right)\\ &= \det(1+\Omega^\dagger\Omega).\end{aligned}\tag{6.14}$$

An important difference to the bosonic case is the fact that the state $\Psi[u,u^\dagger]$ is *not* an overlap with fields states, $\Psi[u,u^\dagger]\neq\langle u,u^\dagger|\Psi\rangle$, since the inner product is an ordinary number, whereas Ψ can be expanded in terms of Grassmann variables.

6.1.2 Solution of the stationary Schrödinger equation

Here we look for the *ground state* of the Dirac Hamiltonian in an external electromagnetic field, i. e., we solve the stationary Schrödinger equation

$$\left(\int dxdy\psi^\dagger(x)h(x,y)\psi(y)\right)\Psi\equiv H_\psi\Psi=E_0\Psi. \tag{6.15}$$

If ψ_n are the eigenmodes of the first quantized Hamiltonian h,

$$h\psi_n=E_n\psi_n, \tag{6.16}$$

we can expand the field operators ψ and $\psi^\dagger$ as

$$\psi=\sum_n a_n\psi_n \quad , \quad \psi^\dagger=\sum_n a_n^\dagger\psi_n^\dagger,$$

where a_n $(a_n^\dagger)$ is the usual annihilation (creation) operator. Then,

$$H_\psi=\sum_n E_n a_n^\dagger a_n.$$

We can also expand u and $u^\dagger$ in terms of these eigenmodes

$$u(x)=\sum_n u_n\psi_n(x),\quad u^\dagger(x)=\sum_n u_n^\dagger\psi_n^\dagger(x).$$

Note that

$$\frac{\delta}{\delta u(x)}=\sum_n\psi_n^\dagger(x)\frac{\delta}{\delta u_n}$$

to guarantee that $\delta u(x)/\delta u(y)=\delta(x-y)$. Inserting these expansions into the expression for H_ψ, we find

$$H_\psi=\frac{1}{2}\sum_n E_n\left(u_n^\dagger+\frac{\delta}{\delta u_n}\right)\left(u_n+\frac{\delta}{\delta u_n^\dagger}\right). \tag{6.17}$$

We want to apply this Hamiltonian on the Gaussian state (6.11). To that purpose we note that

$$u^\dagger\Omega u=\sum_{n,m}u_n^\dagger\Omega_{nm}u_m \tag{6.18}$$

with

$$\Omega(x,y) = \sum_{n,m} \Omega_{nm} \psi_n(x) \psi_m^\dagger(y). \tag{6.19}$$

We then find

$$\begin{aligned} H_\psi \Psi &= \frac{1}{2} \mathrm{Tr} h (1+\Omega) \Psi \\ &+ \frac{1}{2} \sum_{k,l,n} u_n^\dagger (\delta_{nk} - \Omega_{nk}) E_k (\delta_{kl} + \Omega_{kl}) u_l \Psi. \end{aligned} \tag{6.20}$$

Upon comparison with (6.15) we see that the ground state energy is given by

$$E_0 = \frac{1}{2} \mathrm{Tr} h (1+\Omega) = \frac{1}{2} \sum_n E_n (1 + \Omega_{nn}), \tag{6.21}$$

and that, since the second term in (6.20) must vanish, the elements of Ω_{nn} are given by

$$\Omega_{nm} = \pm \delta_{nm}. \tag{6.22}$$

There still remains some arbitrariness how one distributes the numbers 1 and -1 among the elements of Ω. This arbitrariness can be removed by the use of the annihilation operators introduced above. We have

$$\begin{aligned} a_n^\dagger a_n \Psi &= \left(u_n^\dagger + \frac{\delta}{\delta u_n} \right) \left(u_n + \frac{\delta}{\delta u_n^\dagger} \right) \Psi \\ &= \frac{1}{2} (1 + \Omega_{nn}) \Psi. \end{aligned} \tag{6.23}$$

We demand that the ground state Ψ be annihilated by a_n for *positive* energies E_n, i. e.,

$$a_n^\dagger a_n \Psi = \begin{cases} 0 & \text{if } \Omega_{nn} = -1 \leftrightarrow E_n > 0 \\ \Psi & \text{if } \Omega_{nn} = +1 \leftrightarrow E_n < 0 \end{cases} \tag{6.24}$$

This selects a specific ground state and is equivalent to say, in a more heuristic language, that a specific prescription for the filling of the Dirac sea has been chosen. From (6.19) we thus find for the covariance

$$\Omega(x,y) = \sum_{E_n<0} \psi_n(x) \psi_n^\dagger(y) - \sum_{E_n>0} \psi_n(x) \psi_n^\dagger(y). \tag{6.25}$$

It is very convenient, and we will make extensive use of it later on, to express this relation in terms of *projectors*,

$$\Omega \equiv P_- - P_+, \quad \text{where} \quad P_\pm \equiv \frac{1 \mp \Omega}{2} \tag{6.26}$$

project on positive and negative energies, respectively:

$$P_+ P_- = P_- P_+ = 0,\ P_+^2 = P_+,\ P_-^2 = P_-,\ P_+ + P_- = 1. \tag{6.27}$$

We also note the operator expression for Ω, which follows from the vanishing of the second term in (6.20), reads:

$$\frac{1}{4}(1-\Omega)h(1+\Omega) = 0 = P_+ h P_- . \tag{6.28}$$

In case that the external electromagnetic field vanishes we can give easily an explicit expression for Ω. In momentum space, the solution corresponding to the filling prescription (6.24) reads

$$\Omega_{(0)}(p,p') = -\frac{h_{(0)}}{\sqrt{p^2+m^2}}\delta(p-p'), \tag{6.29}$$

where $h_{(0)}$ is the **A**-independent part of (6.4). This can most easily be seen by calculating the vacuum energy E_0. From (6.21) we have, since $h_{(0)}$ has vanishing trace,

$$\begin{aligned} E_0 &= \frac{1}{2}\mathrm{Tr} h_{(0)}\Omega_{(0)} = \frac{1}{2}\sum_n E_n \Omega_{nn} = -\frac{1}{2}\sum_n |E_n| \\ &= -\frac{1}{2}\mathrm{Tr}\sqrt{p^2+m^2} = -\frac{1}{2}\frac{V}{(2\pi)^3}\int d^3p\sqrt{p^2+m^2}. \end{aligned} \tag{6.30}$$

Use has been made here of the fact that the square of h is given by $h_{(0)}^2 = p^2+m^2$. For later use we give the explicit result for two and four spacetime dimensions. In two dimensions we have,

$$\Omega_{(0)} = -\frac{1}{\sqrt{p^2+m^2}}\begin{pmatrix} -p & m \\ m & p \end{pmatrix}, \quad \Omega_{(0)} = \frac{1}{\sqrt{p^2+m^2}}\begin{pmatrix} -m & p \\ p & m \end{pmatrix} \tag{6.31}$$

in the chiral and Dirac representations, respectively. In the four dimensional case we have, in the Dirac representation,

$$\Omega_{(0)} = -\frac{1}{\sqrt{p^2+m^2}}\begin{pmatrix} m & \sigma\cdot p \\ \sigma\cdot p & -m \end{pmatrix}, \tag{6.32}$$

where σ are the Pauli matrices.
We conclude this part with a discussion of the two-point function $\langle\psi_\alpha(x)\psi_\beta^\dagger(y)\rangle$, where the expectation value is computed with respect to the above ground state. For this we need the two-point function of $uu^\dagger$ which we now calculate, using (6.11) and (6.13),

$$\begin{aligned} \frac{\langle u_\alpha(x)u_\beta^\dagger(y)\rangle}{\langle\Psi|\Psi\rangle} &= \frac{\det(-\Omega^\dagger)}{\langle\Psi|\Psi\rangle}\int \mathcal{D}u^\dagger\mathcal{D}u\, u_\alpha(x)u_\beta^\dagger(y) \\ &\cdot\exp\left(u^\dagger[(\Omega^\dagger)^{-1}+\Omega]u\right) \end{aligned}$$

$$= \frac{\det(-\Omega^\dagger)}{\langle\Psi|\Psi\rangle} \frac{\delta^2}{\delta\eta_\alpha(x)\delta\eta^\dagger_\beta(y)} \int \mathcal{D}u^\dagger\mathcal{D}u$$
$$\cdot \exp\left(u^\dagger[(\Omega^\dagger)^{-1} + \Omega]u + \eta u + \eta^\dagger u^\dagger\right)|_{\eta=\eta^\dagger=0}$$
$$= \frac{\det(1+\Omega^\dagger\Omega)}{\langle\Psi|\Psi\rangle} \frac{\delta^2}{\delta\eta_\alpha(x)\delta\eta^\dagger_\beta(y)} \exp\left(\eta[(\Omega^\dagger)^{-1} + \Omega]^{-1}\eta^\dagger\right)|_{\eta=\eta^\dagger=0}$$
$$= -[(\Omega^\dagger)^{-1} + \Omega]^{-1}_{\alpha\beta}(x,y),$$

where (6.14) has been used. In the present case, where $\Omega = \Omega^\dagger$ and $\Omega^2 = 1$, this reads

$$\frac{\langle u_\alpha(x)u^\dagger_\beta(y)\rangle}{\langle\Psi|\Psi\rangle} = -\frac{1}{2}\Omega_{\alpha\beta}(x,y). \tag{6.33}$$

If we apply $\psi_\alpha(x)\psi^\dagger_\beta(y)$ on the ground state, we find

$$\psi_\alpha(x)\psi^\dagger_\beta(y)\Psi = \frac{1}{2}(\delta_{\alpha\beta}\delta(x-y) - \Omega_{\alpha\beta}(x,y))\Psi$$
$$+\frac{1}{2}(u_\alpha(x) + \Omega_{\alpha\delta}(x,w)u_\delta(w))(u^\dagger_\beta(y) - u^\dagger_\gamma(z)\Omega_{\gamma\beta}(z,y))\Psi,$$

where a summation (integration) over repeated indices (variables) is understood.

Using the result (6.33) we find eventually for the desired two-point function the expression

$$\frac{\langle\psi_\alpha(x)\psi^\dagger_\beta(y)\rangle}{\langle\Psi|\Psi\rangle} = \frac{1}{2}(\delta_{\alpha\beta}\delta(x-y) - \Omega_{\alpha\beta}(x,y)),$$

or, in operator notation and with respect to a normalized state,

$$\boxed{\langle\psi(x)\psi^\dagger(y)\rangle = \frac{1}{2}(1 - \Omega(x,y)) = P_+(x,y).} \tag{6.34}$$

Thus, if one knows the covariance, one can calculate all two-point functions, and vice versa. We finally note that excited states can be easily generated by applying the above creation operator $a^\dagger_n$ on the ground state, leading to a Gaussian times some polynomial.

6.1.3 Solution of the time-dependent Schrödinger equation

In this subsection we discuss the solution of the functional Schrödinger equation for fermions in an external electromagnetic field,

$$\left(\int dxdy\psi^\dagger(x)h(x,y)\psi(y)\right)\Psi \equiv H_\psi\Psi = i\dot{\Psi}, \tag{6.35}$$

where, again, h is given explicitly by (6.4). Equation (6.35) follows from a semiclassical expansion of the full functional Schrödinger equation [7]. We make again a Gaussian ansatz,

$$\Psi = N(t)\exp\left(u^\dagger \Omega(t) u\right), \tag{6.36}$$

where Ω and N now depend on time. The state (6.36) may be thought as an evolving vacuum state. Inserting this ansatz into (6.35) we find two equations for N and Ω which read, in operator notation,

$$i\frac{d\ln N}{dt} = \frac{1}{2}\mathrm{Tr}h\Omega \tag{6.37}$$

$$i\dot{\Omega} = \frac{1}{2}(1-\Omega)h(1+\Omega). \tag{6.38}$$

An important special case is given if Ω can be written in terms of the projectors (6.26). As in the case of the stationary equation this is equivalent to $\Omega^2 = 1$.

One physical application we have in mind is to choose the free solution in, say, the asymptotic past and study its evolution under the influence of an external electromagnetic field according to (6.35). It is important to note that (6.38) preserves the property $\Omega^2 = 1$. Thus, $\Omega(t)$ can always be written as in (6.26) provided $\Omega^2(t_0) = 1$ for some "initial time" t_0. This can easily be seen: One first verifies that the inverse of Ω, Ω^{-1}, obeys the same differential equation as (6.38). From the uniqueness of the solution we thus have $\Omega(t_0) = \Omega^{-1}(t_0) \Rightarrow \Omega(t) = \Omega^{-1}(t) \Leftrightarrow \Omega^2(t) = 1$.

Eq. (6.38) is solved by

$$\Omega(t) = \left(Q(t) - C\right)\left(Q(t) + C\right)^{-1}, \tag{6.39}$$

where C is a time-independent operator, and the operator $Q(t)$ satisfies

$$i\dot{Q} = hQ. \tag{6.40}$$

One may wish, for example, to choose for Ω the "free solution" (6.25) in the asymptotic past, i.e., one demands that Ω approaches $\Omega_0 = P_- - P_+$ for $t \to -\infty$. This would correspond to the choice

$$C = P_+, \quad \text{and} \quad Q(t) \stackrel{t\to-\infty}{\longrightarrow} P_-.$$

The time evolution according to (6.35) will then in general induce a time dependence of Ω which may deviate, at late times, from the asymptotic "free" solution. This can then be interpreted as particle creation and will be explicitly discussed below.

The significance of the result (6.39,6.40) consists in the reduction of the solution of the full functional equation (6.35) to the solution of a "first quantized" problem – Eq. (6.40) is nothing but the Dirac equation with an external electromagnetic field.

After the solution for Ω has been found, the prefactor N can be immediately determined from (6.37) to read

$$N(t) = N_0 \exp\left(-\frac{i}{2}\int^t \mathrm{Tr}(h\Omega)ds\right).$$

The time-independent factor N_0 can be fixed if Ψ is normalized, i. e. $\langle\Psi|\Psi\rangle = 1$, and one finds, using (6.14),

$$N(t) = \det{}^{-1/2}(1+\Omega^\dagger\Omega)\exp\left(-\frac{i}{2}\int^t \mathrm{Re}\ \mathrm{Tr}(h\Omega)ds\right). \tag{6.41}$$

We now address the question of particle creation. We first note that the absolute square of the matrix element of two Gaussians, Ψ_1 and Ψ_2, with corresponding covariances Ω_1 and Ω_2, is given by the expression

$$|\langle\Psi_1|\Psi_2\rangle|^2 = \det\frac{(1+\Omega_1^\dagger\Omega_2)(1+\Omega_2^\dagger\Omega_1)}{(1+\Omega_1^\dagger\Omega_1)(1+\Omega_2^\dagger\Omega_2)}. \tag{6.42}$$

In the following we will take for Ψ_1 the time-evolved in-vacuum and for Ψ_2 the vacuum state at late times. The corresponding covariances will be called $\Omega(t)$ and Ω_0, respectively. As discussed above, we demand $\Omega(t)$ to approach the "free covariance" Ω_0 at $t\to-\infty$. Since $\Omega_0 = \Omega^\dagger$ and $\Omega_0^2 = 1$, the desired transition element (6.42) reads

$$|\langle\Psi_1|\Psi_2\rangle|^2 = \det\frac{(1+\Omega_0\Omega(t))(1+\Omega^\dagger(t)\Omega_0)}{2(1+\Omega^\dagger(t)\Omega(t))}. \tag{6.43}$$

To get the desired expression (6.39) for Ω, which for the present case reads

$$\Omega(t) = (Q(t)-P_+)\,(Q(t)+P_+)^{-1}, \tag{6.44}$$

it is first necessary to solve (6.40) for $Q(t)$. This is most conveniently done by the ansatz

$$Q(t) = \sum_n \chi_n(t)\chi_n^\dagger,$$

where χ_n (without argument) denotes a negative frequency eigenfunction of the Dirac Hamiltonian h, and $\chi_n(t)$ denotes the solution of (6.40) which approaches χ_n in the asymptotic limit $t\to-\infty$. Therefore,

$$Q(t) \stackrel{t\to-\infty}{\longrightarrow} \sum_n \chi_n\chi_n^\dagger \equiv P_-,$$

as required. It will prove to be convenient if one expands $\chi_n(t)$ as follows,

$$\chi_n(t) = \alpha_{nm}(t)\chi_m + \beta_{nm}(t)\psi_m, \tag{6.45}$$

where ψ_m is a positive frequency eigenfunction of h, and α, β are the time-dependent Bogolubov coefficients associated with this expansion. Since h is hermitian, the norm $(\chi_n(t), \chi_m(t))$ is conserved, and we choose it to be equal to one. The Bogolubov coefficients are then normalized according to

$$|\alpha|^2 + |\beta|^2 = 1. \tag{6.46}$$

Note that this is different from the bosonic case where the analogous expression contains a minus sign.

The operator $Q(t) + P_+$ in (6.44) is then given by the expression

$$Q(t) + P_+ = \sum_{n,m} \left(\alpha_{nm}\chi_m\chi_n^\dagger + \beta_{nm}\psi_m\chi_n^\dagger\right) + \sum_n \psi_n\psi_n^\dagger,$$

from where its inverse is found to read

$$(Q(t) + P_+)^{-1} = \sum_n \psi_n\psi_n^\dagger - \sum_{n,s,t} \psi_n\alpha_{st}^{-1}\beta_{tn}\chi_n^\dagger + \sum_{n,s} \chi_n\alpha_{sn}^{-1}\chi_s^\dagger.$$

One can then write down the desired expression for $\Omega(t)$,

$$\begin{aligned} \Omega(t) &= \sum_n(\chi_n\chi_n^\dagger - \psi_n\psi_n^\dagger) + 2\sum_{n,s,t}\psi_n\alpha_{st}^{-1}\beta_{tn}\chi_s^\dagger \\ &= \Omega_0 + 2\sum_{n,s,t}\psi_n\alpha_{st}^{-1}\beta_{tn}\chi_s^\dagger \equiv \Omega_0 + 2B, \end{aligned} \tag{6.47}$$

where we have introduced an operator B, which in the position representation is given by

$$B(x,y) = \sum_{n,s,t} \psi_n(x)\alpha_{st}^{-1}\beta_{tn}\chi_s^\dagger(y).$$

It maps negative energy eigenfunctions into positive ones, and it annihilates positive energy eigenfunctions. Conversely, its adjoint

$$B^\dagger(x,y) = \sum_{n,s,t} \chi_s(x)\bar{\alpha}_{st}^{-1}\bar{\beta}_{tn}\psi_n^\dagger(y)$$

maps positive energy eigenfunctions into negative ones and annihilates negative energy eigenfunctions. Note that B and $B^\dagger$ are nilpotent operators.

One then finds for the various terms in the transition element (6.43) the expressions

$$\Omega^\dagger(t)\Omega_0 = 1 - 2B^\dagger, \quad , \quad \Omega_0\Omega(t) = 1 - 2B,$$
$$\Omega^\dagger(t)\Omega(t) = 1 - 2B - 2B^\dagger + 4B^\dagger B, \tag{6.48}$$

and one has

$$|\langle\Psi_1|\Psi_2\rangle|^2 = \det \frac{(1-B)(1-B^\dagger)}{(1-B-B^\dagger+2B^\dagger B)}. \tag{6.49}$$

Written in the basis $(\psi, \chi)^T$, the various operators in (6.48) are given by the matrix expressions

$$B = \begin{pmatrix} 0 & \alpha^{-1}\beta \\ 0 & 0 \end{pmatrix} \quad , \quad B^\dagger = \begin{pmatrix} 0 & 0 \\ (\alpha^{-1}\beta)^\dagger & 0 \end{pmatrix}, \tag{6.50}$$

One immediately verifies that $\det(1-B) = \det(1-B^\dagger) = 1$. Therefore, using (2.63),

$$\begin{aligned} &|\langle\Psi_1|\Psi_2\rangle|^2 = \det{}^{-1}(1 - B - B^\dagger - 2B^\dagger B) \\ &= \det{}^{-1}(1 + \alpha^{-1}\beta\beta^\dagger\alpha^{-1\dagger}) = \det{}^{-1}(1 + \beta^\dagger(1-\beta\beta^\dagger)^{-1}\beta) \\ &= \det{}^{-1}\beta^{-1}(1-\beta\beta^\dagger)^{-1}\beta = \det(1-\beta\beta^\dagger). \end{aligned} \tag{6.51}$$

The interpretation of this result is obvious. The determinant is less than one for non-vanishing Bogolubov coefficient β, which signals particle creation. Note that the analogous expression in the bosonic case reads [7] $\det^{-1}(1 + \beta\beta^\dagger)$, which is only equal to (6.51) for small β. We will apply the above result to the calculation of particle creation in an external electric field in section 6.4.

6.2 Ground state for QED_2

6.2.1 The massless case

Calculation of the covariance: In the following we shall give explicit results for the ground state of QED_2 in arbitrary external electromagnetic fields by applying the method developed in the last section. Two-dimensional massless QED is also known as the *Schwinger model* [10]. It has been explicitly solved and found to be equivalent to the theory of a free *massive* scalar field (see [11] for some literature on the Schwinger model). In this paper we also address some issues for the Schwinger model on a finite space [12, 42]. The Hamiltonian formalism for the Schwinger model has been discussed in [13] and [14].

It is convenient to discuss the massless and the massive case separately since it is adequate to use the chiral representation for the Gamma matrices in the massless case and the Dirac representation in the massive case. For $m = 0$ we thus use

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \gamma^0\gamma^1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (6.52)$$

The first-quantized Hamiltonian h (6.4) is then given explicitly by (with $A^1 \equiv A$)

$$h(x,y) = \begin{pmatrix} i\frac{\partial}{\partial x} - eA(x) & 0 \\ 0 & -i\frac{\partial}{\partial x} + eA(x) \end{pmatrix} \delta(x-y). \qquad (6.53)$$

To find the ground state of the stationary Schrödinger equation we have to solve the "first-quantized" problem (6.16), i. e., to find the spectrum of (6.53),

$$h\psi_n = E_n\psi_n. \qquad (6.54)$$

We quantize the fields in a finite interval, $x \in [0, L]$, and impose the boundary condition

$$\psi_n(x+L) = e^{2\pi i(\alpha+\beta\gamma_5)}\psi_n(x), \qquad (6.55)$$

where α and β are the vectorial and chiral twists, respectively. Writing

$$\psi_n = \begin{pmatrix} \varphi_n \\ \chi_n \end{pmatrix}, \qquad (6.56)$$

the diagonality of h yields two decoupled equations for φ_n and χ_n, corresponding to a decomposition into right- and left handed fermions. One finds from (6.54) and (6.55) for the right handed part

$$\begin{aligned} \varphi_n(x) &= \frac{1}{\sqrt{L}} \exp\left[-i\left(E_n^R x + e\int_0^x A\right)\right], \\ E_n^R &= \frac{2\pi}{L}(n-\alpha-\beta) - \frac{e}{L}\int_0^L A \equiv \frac{2\pi}{L}(n-\phi), \end{aligned} \qquad (6.57)$$

and for the left handed part

$$\begin{aligned} \chi_n(x) &= \frac{1}{\sqrt{L}} \exp\left[i\left(E_n^L x - e\int_0^x A\right)\right], \\ E_n^L &= -\frac{2\pi}{L}(n-\alpha+\beta) + \frac{e}{L}\int_0^L A \equiv -\frac{2\pi}{L}(n-\tilde{\phi}). \end{aligned} \qquad (6.58)$$

Here we have introduced

$$\phi = \alpha + \beta + \frac{e}{2\pi}\int_0^L A \quad \text{and} \quad \tilde{\phi} = \alpha - \beta + \frac{e}{2\pi}\int_0^L A. \qquad (6.59)$$

The covariance (6.25) also splits into a right- and left handed part

$$\Omega(x,y)=\begin{pmatrix}\Omega_+(x,y) & 0\\ 0 & \Omega_-(x,y)\end{pmatrix}, \tag{6.60}$$

where

$$\begin{aligned}\Omega_+(x,y) &= \sum_{E_n^R<0}\varphi_n(x)\varphi_n^\dagger(y)-\sum_{E_n^R>0}\varphi_n(x)\varphi_n^\dagger(y),\\ \Omega_-(x,y) &= \sum_{E_n^L<0}\chi_n(x)\chi_n^\dagger(y)-\sum_{E_n^L>0}\chi_n(x)\chi_n^\dagger(y).\end{aligned} \tag{6.61}$$

From (6.57) and (6.58) one recognizes that $E_n^R>0$ for $n>\phi$ and $E_n^L>0$ for $n<\tilde{\phi}$. Inserting all this into (6.61) one finds

$$\begin{aligned}\Omega_+(x,y) &= \frac{1}{L}\sum_{E_n^R<0}\exp\left(iE_n^R(y-x)+ie\int_x^y A\right)\\ &\quad -\frac{1}{L}\sum_{E_n^R>0}\exp\left(iE_n^R(y-x)+ie\int_x^y A\right)\\ &= \frac{1}{L}\exp\left(ie\int_x^y A+i\frac{2\pi\phi}{L}(x-y)\right)\times\\ &\quad \left(\sum_{n<\phi}\exp\left[-\frac{2\pi in}{L}(x-y)\right]-\sum_{n>\phi}\exp\left[-\frac{2\pi in}{L}(x-y)\right]\right)\\ &= \frac{i}{L}\exp\left(ie\int_x^y A+\frac{2\pi i}{L}(\phi-[\phi]-\frac{1}{2})(x-y)\right)\times\\ &\quad \frac{1}{\sin\frac{\pi}{L}(x-y)},\end{aligned} \tag{6.62}$$

where $[\phi]$ denotes the biggest integer smaller or equal than ϕ.

The left handed part, $\Omega_-(x,y)$, is calculated in the same way, and found to read

$$\Omega_-(x,y)=-\frac{i}{L}\exp\left(ie\int_x^y A+\frac{2\pi i}{L}(\tilde{\phi}-[\tilde{\phi}]-\frac{1}{2})(x-y)\right)\frac{1}{\sin\frac{\pi}{L}(x-y)}. \tag{6.63}$$

In the limit $L\to\infty$ the covariance is given by the expression

$$\Omega(x,y)=\frac{i}{\pi}\exp\left(ie\int_x^y A\right)\mathcal{P}\left(\frac{1}{x-y}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{6.64}$$

where $\mathcal{P}$ denotes the principal value. This result is in accordance with [8]. We make a final remark on the existence of *large* gauge transformations, i. e. gauge transformations which cannot be obtained from the identity in a continuous way. As can be seen from the expressions for the energy, (6.57) and (6.58), such gauge transformations change the fluxes ϕ and $\tilde{\phi}$ by an integer. Since the eigenfunctions in (6.57) and (6.58) remain unchanged, and the covariance contains only the fractional part of the flux (see (6.62) and (6.63)), the wave functional (6.11) remains invariant.

Charges and energy: Now we shall calculate the expectation values of the charge, chiral charge, and energy with respect to the ground state derived above.

The components of the electric current are given by

$$\begin{aligned} j^0 &= \psi^\dagger\psi = \varphi^\dagger\varphi + \chi^\dagger\chi \equiv j_+ + j_-, \\ j^1 &= \psi^\dagger\gamma^0\gamma^1\psi = -\varphi^\dagger\varphi + \chi^\dagger\chi \equiv -j_+ + j_-. \end{aligned} \tag{6.65}$$

The total charge and chiral charge are

$$Q = \int dx j_+ + \int dx j_- \equiv Q_+ + Q_-, \quad \text{and} \quad Q_5 = Q_+ - Q_-, \tag{6.66}$$

respectively. These expressions contain products of the field operators and thus require a regularization prescription. The procedure employed here is to first perform a point splitting and then to subtract the expectation value for vanishing external field. After the point splitting is removed, one is left with a finite result. The crucial point to note is that the point splitting has to be done in a gauge invariant way. We thus define the following "point splitted" quantities

$$\rho_+(x,y) = \varphi^\dagger(x)e^{ie\int_x^y A}\varphi(y) \quad \text{and} \quad \rho_-(x,y) = \chi^\dagger(x)e^{ie\int_x^y A}\chi(y)$$

and they are explicitly gauge invariant. Applying ρ_+ on the vacuum state (6.11) we find

$$\begin{aligned} \rho_+\Psi &= \frac{1}{2}\exp\left(ie\int_x^y A\right)\left(u_1^\dagger(x) + \frac{\delta}{\delta u_1(x)}\right)\left(u_1(y) + \frac{\delta}{\delta u_1^\dagger(y)}\right)\Psi \\ &= \frac{1}{2}\exp\left(ie\int_x^y A\right)(\delta(x-y) + \Omega_+(y,x))\Psi \\ &\quad + \frac{1}{2}\exp\left(ie\int_x^y A\right)(u_1^\dagger(x) - \Omega_+(z,x)u_1^\dagger(z)) \times \\ &\quad (u_1(y) + \Omega_+(y,z)u(z))\Psi, \end{aligned} \tag{6.67}$$

where, again, an integration over repeated variables is understood. If we set $x = y$ and integrate over x, the last term on the right-hand side of (6.67) vanishes since $(1-\Omega_+)(1+\Omega_+) = 0$ according to (6.28). Subtracting the expression for vanishing A field, the first term after the second equation sign on the right-hand side of (6.67) reads

$$\begin{aligned}&\frac{1}{2}\exp\left(ie\int_x^y A\right)\Omega_+(y,x)-\frac{1}{2}\Omega_{(0)}(y,x) \\ &=\frac{i}{2\pi}\exp\left(\frac{2\pi i}{L}(\phi-[\phi]-\frac{1}{2})(y-x)\right)\frac{1}{y-x}-\ \phi\leftrightarrow\phi_0+\mathcal{O}(x-y),\end{aligned} \tag{6.68}$$

where we have expanded the sine in the expression (6.62) for the covariance and kept only the term proportional to $(x-y)^{-1}$. We have also introduced

$$\phi_0=\alpha+\beta \quad \text{so that} \quad \phi=\phi_0+\frac{e}{2\pi}\int_0^L A\equiv\phi_0+\varphi$$

(compare (6.59)). Expanding also the exponential in (6.68) we note that the terms which become singular in the limit $x\to y$ drop out. We can thus remove the point splitting and perform the x integration to find

$$\langle Q_+\rangle=[\phi]-\phi-([\phi_0]-\phi_0). \tag{6.69}$$

The left handed sector is calculated analogously, with the result

$$\langle Q_-\rangle=[\tilde{\phi}]-\tilde{\phi}-([\tilde{\phi_0}]-\tilde{\phi_0}), \tag{6.70}$$

where

$$\tilde{\phi_0}=\alpha-\beta \tag{6.71}$$

so that

$$\tilde{\phi}=\tilde{\phi_0}+\frac{e}{2\pi}\int_0^L A\equiv\tilde{\phi_0}+\varphi \tag{6.72}$$

(compare (6.59)).

The results for the expectation values of the total charge and chiral charge are then given by

$$\begin{aligned}\langle Q\rangle &= \langle Q_+\rangle+\langle Q_-\rangle \\ &= [\alpha+\beta+\varphi]-[\alpha+\beta]-[\alpha-\beta+\varphi]+[\alpha-\beta]\end{aligned} \tag{6.73}$$

and

$$\begin{aligned}\langle Q_5\rangle &= \langle Q_+\rangle-\langle Q_-\rangle \\ &= [\alpha+\beta+\varphi]-[\alpha+\beta]+[\alpha-\beta+\varphi]-[\alpha-\beta]-2\varphi.\end{aligned} \tag{6.74}$$

Note that $< Q >= 0$ for vanishing chiral twist, $\beta=0$ (see (6.55)), and that $< Q_5 >= 2([\varphi]-\varphi)$ for $\alpha=\beta=0$. The above expectation values have been calculated, using zeta regularization, by [14] for the special case $\alpha=1/2$ and $\beta=0$. Their result is in agreement with ours.

We now proceed to calculate the expectation value of the Hamiltonian H_ψ (6.17). We first operate with H_ψ on the ground state wave functional to find the expression (6.20). We then use the explicit solution (6.22) for the covariance to recognize that only the first term in (6.20) contributes to the expectation value $< H_\psi >$:

$$\langle H_\psi \rangle = \frac{1}{2} \sum_n E_n (1 + \Omega_{nn}). \tag{6.75}$$

We regularize again by point splitting. We thus introduce a "point splitted" expectation value which for the contribution from the right handed sector reads

$$\langle \Psi | H_\psi^\dagger (x,y) | \Psi \rangle = \frac{1}{2} \exp \left(-ie \int_x^y A \right) h_x \sum_n (1 + \Omega_{nn}) \varphi_n(x) \varphi_n^\dagger(y). \tag{6.76}$$

Note that this expression is explicitly gauge-invariant and reduces to (6.75) after setting $x = y$ and integrating over x (the action of the first-quantized Hamiltonian $h_x \equiv i\partial/\partial x - eA(x)$ just produces the energy E_n when acting on the ψ_n). The completeness of the φ_n, as well as (6.19), enables one to write (6.76) as

$$\langle \Psi | H_\psi^\dagger (x,y) | \Psi \rangle = \frac{1}{2} \exp \left(-ie \int_x^y A \right) h_x (\delta(x-y) + \Omega_+(x,y)). \tag{6.77}$$

Using the explicit expression (6.62) for $\Omega_+(x,y)$ one finds, up to order $x - y$,

$$\begin{aligned} &\exp \left(-ie \int_x^y A \right) h_x \Omega_+(x,y) = \\ &\left(-\frac{2i}{L(x-y)} (\phi - [\phi] - \frac{1}{2}) + \frac{1}{\pi (x-y)^2} - \frac{\pi}{6L^2} \right) \times \\ &\exp \left(\frac{2\pi i}{L} (\phi - [\phi] - \frac{1}{2})(x-y) \right) + \mathcal{O}(x-y). \end{aligned}$$

Expanding also the exponential, this reads

$$\exp \left(-ie \int_x^y A \right) h_x \Omega_+(x,y) = \frac{1}{\pi (x-y)^2} - \frac{\pi}{6L^2} + \frac{2\pi}{L^2} (\phi - [\phi] - \frac{1}{2})^2 + \mathcal{O}(x-y),$$

so that we find

$$\begin{aligned} \langle \Psi | H_\psi^\dagger (x,y) | \Psi \rangle &= \frac{1}{2} \exp \left(-ie \int_x^y A \right) (i \frac{\partial}{\partial x} - eA) \delta(x-y) \\ &+ \frac{1}{2\pi (x-y)^2} - \frac{\pi}{12L^2} + \frac{\pi}{L^2} (\phi - [\phi] - \frac{1}{2})^2 + \mathcal{O}(x-y). \end{aligned}$$

Since

$$\exp \left(-ie \int_x^y A \right) i \frac{\partial}{\partial x} \delta(x-y) = i\delta'(x-y) + eA(x) \delta(x-y),$$

we have

$$\langle\Psi|H_\psi^\dagger(x,y)|\Psi\rangle = \frac{i}{2}\delta'(x-y)+\frac{1}{2\pi(x-y)^2}-\frac{\pi}{12L^2}+\frac{\pi}{L^2}(\phi-[\phi]-\frac{1}{2})^2+\mathcal{O}(x-y). \tag{6.78}$$

From this expression one has to subtract the expectation value for vanishing external field. To retain finite-size effects we subtract the "free" value for $L \to \infty$. This removes the divergent terms in (6.78). Setting $x = y$ and integrating over x, one finds the result

$$\langle H_\psi^\dagger\rangle = \frac{\pi}{L}\left(\phi-[\phi]-\frac{1}{2}\right)^2 - \frac{\pi}{12L}. \tag{6.79}$$

This vanishes in the limit $L \to \infty$. The expression for finite L is nothing but the *Casimir energy* which is also present for vanishing external field:

$$\langle H_\psi^\dagger\rangle = \frac{\pi}{L}\left(\phi_0-[\phi_0]-\frac{1}{2}\right)^2 - \frac{\pi}{12L}.$$

Note that the resulting force between the boundaries at $x = 0$ and $x = L$ can be attractive or repulsive, depending on the chosen boundary conditions. For the conditions chosen in [14] the expectation value is given by $-\pi/12L$ and thus leads to an attractive force.

The expectation value of the Hamiltonian in the left handed sector is calculated in the same way by making use of (6.63) and using $-h_x = -i\partial/\partial x + eA(x)$. Instead of (6.79) one finds

$$\langle H_\psi^-\rangle = \frac{\pi}{L}\left(\tilde{\phi}-[\tilde{\phi}]-\frac{1}{2}\right)^2 - \frac{\pi}{12L}. \tag{6.80}$$

The total Casimir energy is the sum of the expressions (6.79) and (6.80).

6.2.2 The massive case

Calculation of the covariance: In the massive case we use the Dirac representation for the Gamma matrices, i. e.,

$$\gamma^0 = \sigma_3 \quad , \quad \gamma^1 = -i\sigma_2 \quad , \quad \gamma^0\gamma^1 = -\sigma_1 \tag{6.81}$$

The first-quantized Hamiltonian is then given by the expression

$$h(x,y) = \begin{pmatrix} m & i\frac{\partial}{\partial x} - eA(x) \\ i\frac{\partial}{\partial x} - eA(x) & -m \end{pmatrix} \delta(x-y). \tag{6.82}$$

We are again looking for the eigenfunctions of h,

$$h\psi_n = E_n\psi_n. \tag{6.83}$$

If we make the ansatz

$$\psi_n = \frac{1}{\sqrt{L}} \exp\left(-ie\int_0^x A - i\lambda_n x\right) c_n, \tag{6.84}$$

Eq. (6.83) yields an algebraic equation for c_n,

$$\begin{pmatrix} m - E_n & \lambda_n \\ \lambda_n & -m - E_n \end{pmatrix} \begin{pmatrix} c_{n,1} \\ c_{n,2} \end{pmatrix} = 0. \tag{6.85}$$

The boundary condition

$$\psi_n(x+L) = e^{2\pi i\alpha}\psi_n(x) \tag{6.86}$$

yields a quantization condition for the λ_n,

$$\lambda_n = \frac{2\pi}{L}\left(n - \alpha - \frac{e}{2\pi}\int_0^L A\right) \equiv \frac{2\pi}{L}(n-\phi), \tag{6.87}$$

where $n \in Z$. From (6.85) one then finds the values for the energy,

$$E_n = \pm\sqrt{m^2+\lambda_n^2} = \pm\sqrt{m^2 + \frac{4\pi^2}{L^2}(n-\phi)^2} \equiv \pm\omega_n. \tag{6.88}$$

We already note at this point that the massless limit of (6.88) yields $E_n = \pm\frac{2\pi}{L}|n-\phi|$ instead of $E_n = \pm\frac{2\pi}{L}(n-\phi)$ which was found by starting from $m = 0$ ab initio. This will be relevant for the discussion of anomalies in chapter 5.

The normalized eigenfunctions ψ_n read

$$\psi_{n,+} = \frac{1}{\sqrt{2\omega_n(\omega_n+m)L}} \begin{pmatrix} \omega_n + m \\ \lambda_n \end{pmatrix} \exp\left(-i\lambda_n x - ie\int_0^x A\right) \tag{6.89}$$

for $E_n = \omega_n$, and

$$\psi_{n,-} = \frac{1}{\sqrt{2\omega_n(\omega_n+m)L}} \begin{pmatrix} -\lambda_n \\ \omega_n + m \end{pmatrix} \exp\left(-i\lambda_n x - ie\int_0^x A\right) \tag{6.90}$$

for $E_n = -\omega_n$.

We now use again (6.25) and the filling prescription (6.24) to calculate the covariance,

$$\Omega(x,y) = \sum_n \psi_{n,-}(x)\psi_{n,-}^\dagger(y) - \sum_n \psi_{n,+}(x)\psi_{n,+}^\dagger(y) \equiv P_- - P_+. \tag{6.91}$$

Noting that $\lambda_n = (\omega_n+m)(\omega_n-m)$, we find

$$P_+(x,y) = \frac{1}{2L}e^{-ie\int_y^x A}\sum_n \frac{e^{-i\lambda_n(x-y)}}{\omega_n}\begin{pmatrix} \omega_n + m & \lambda_n \\ \lambda_n & \omega_n - m \end{pmatrix} \tag{6.92}$$

and

$$P_-(x,y) = \frac{1}{2L} e^{-ie\int_y^x A} \sum_n \frac{e^{-i\lambda_n(x-y)}}{\omega_n} \begin{pmatrix} \omega_n - m & -\lambda_n \\ -\lambda_n & \omega_n + m \end{pmatrix}. \qquad (6.93)$$

To evaluate the various sums in these expressions we make use of Poisson's summation formula:

$$2\pi \sum_{n=-\infty}^{\infty} f(2\pi n) = \sum_{n=-\infty}^{\infty} F(n) \quad \text{where} \quad F(u) = \int_{-\infty}^{\infty} dz f(z) e^{izu}. \qquad (6.94)$$

We then have

$$\sum_n \frac{\lambda_n}{\omega_n} e^{-2\pi i n(x-y)/L} \equiv \sum_n f(2\pi n) \quad \text{and} \quad \sum_n \frac{1}{\omega_n} e^{-2\pi i n(x-y)/L} \equiv \sum_n \tilde{f}(2\pi n),$$

where

$$f(z) = \frac{z - 2\pi\phi}{\sqrt{(z-2\pi\phi)^2 + m^2L^2}} e^{-i(x-y)z/L}$$

and

$$\tilde{f}(z) = \frac{L}{\sqrt{(z-2\pi\phi)^2 + m^2L^2}} e^{-i(x-y)z/L}.$$

From (6.94) we then find

$$\begin{aligned} F(u) &= e^{2\pi i\phi(u-(x-y)/L)} \int_{-\infty}^{\infty} dp \frac{p e^{ip(u-(x-y)/L)}}{\sqrt{p^2 + m^2L^2}} \\ &= 2ie^{2\pi i\phi(u-(x-y)/L} mLK_1(mLu - m(x-y)) \end{aligned} \qquad (6.95)$$

and

$$\begin{aligned} \tilde{F}(u) &= Le^{2\pi i\phi(u-(x-y)/L)} \int_{-\infty}^{\infty} dp \frac{e^{ip(u-(x-y)/L)}}{\sqrt{p^2 + m^2L^2}} \\ &= 2Le^{2\pi i\phi(u-(x-y)/L} K_0(mLu - m(x-y)). \end{aligned} \qquad (6.96)$$

Here K_0 and K_1 denote Bessel functions and use has been made of [15] to evaluate the integrals. From (6.94) we then find for the sums

$$\sum_n \frac{\lambda_n}{\omega_n} e^{-2\pi i n(x-y)/L} = -\frac{imL}{\pi} e^{-2\pi i\phi(x-y)/L} \sum_n e^{-2\pi i n\phi} K_1(mLn + m(x-y))$$

and

$$\sum_n \frac{1}{\omega_n} e^{-2\pi i n(x-y)/L} = \frac{L}{\pi} e^{-2\pi i\phi(x-y)/L} \sum_n e^{-2\pi i n\phi} K_0(mLn + m(x-y)).$$

In the expressions below we will for simplicity not explicitly write out the argument $mLn + m(x-y)$ of the Bessel functions K_0 and K_1. For the remaining sum we have

$$\sum_n e^{-\frac{2\pi i n}{L}(x-y)} = L \sum_n \delta(x-y-nL) = L\delta(x-y)$$

since $|x-y| < L$. Inserting all these results into the expressions (6.92) and (6.93) we find

$$P_+ = \frac{1}{2}\delta(x-y)\mathbf{I} + \frac{m}{2\pi} e^{-ie\int_y^x A} \sum_n \begin{pmatrix} K_0 & -iK_1 \\ -iK_1 & -K_0 \end{pmatrix} e^{-2\pi i n\phi} \tag{6.97}$$

and

$$P_- = \frac{1}{2}\delta(x-y)\mathbf{I} - \frac{m}{2\pi} e^{-ie\int_y^x A} \sum_n \begin{pmatrix} K_0 & -iK_1 \\ -iK_1 & -K_0 \end{pmatrix} e^{-2\pi i n\phi}. \tag{6.98}$$

We verify that $P_+ + P_- = \mathbf{I}$. Our final result for the covariance (6.26) is then given by the expression

$$\Omega(x,y) = P_- - P_+ = \frac{m}{\pi} e^{-ie\int_y^x A} \sum_n \begin{pmatrix} -K_0 & iK_1 \\ iK_1 & K_0 \end{pmatrix} e^{-2\pi i n\phi}. \tag{6.99}$$

In the limit $L \to \infty$ we find

$$\Omega(x,y) = \frac{m}{\pi} e^{-ie\int_y^x A} \begin{pmatrix} -K_0(m(x-y)) & iK_1(m(x-y)) \\ iK_1(m(x-y)) & K_0(m(x-y)) \end{pmatrix}. \tag{6.100}$$

Using the asymptotic expressions for the Bessel functions one verifies that Ω approaches the result (6.64) in the limit of vanishing mass. Furthermore, in the opposite limit of large mass (or large $|x-y|$) the covariance reads

$$\Omega(x,y) = \sqrt{\frac{m}{2\pi|x-y|}} e^{-ie\int_y^x A} \exp\left(-m|x-y|\right) \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}. \tag{6.101}$$

In summary, we have found the exact ground state for arbitrary external fields in the massive Schwinger model. The excited states, Ψ_n, can then be constructed in the usual way through the application of the creation operator. A general state, Ψ, of the fully quantized theory can then be expanded into these energy eigenstates according to

$$\Psi[A, u, u^\dagger] = \sum_n \varphi_n[A] \Psi_n[A, u, u^\dagger],$$

where the functionals $\varphi_n[A]$ can be determined from the full functional Schrödinger equation which contains the kinetic term $-\delta^2/2\delta A^2$ in the Hamiltonian.

Charges and energy: We define again a "point splitted" charge operator

$$\rho(x,y) = \psi^\dagger(x) \exp\left(ie\int_x^y A\right)\psi(y)$$

and find for its action on the vacuum state an expression analogous to (6.67) (there is now no distinction between a left and a right handed sector):

$$\begin{aligned}\rho(x,y)\Psi &= \frac{1}{2}\exp\left(ie\int_x^y A\right)(2\delta(x-y)+\sum_{\alpha=1}^{2}\Omega_{\alpha\alpha}(y,x))\Psi \\ &+\frac{1}{2}\exp\left(ie\int_x^y A\right)(u_\alpha^\dagger(x)-\Omega_{\beta\alpha}(z,x)u_\beta^\dagger(z))\times \\ &(u_\alpha(y)+\Omega_{\alpha\gamma}(y,z)u_\gamma(z))\Psi,\end{aligned} \tag{6.102}$$

where a summation (integration) over repeated indices (variables) is understood. Like in the massless case, the second term on the right-hand side vanishes after setting $x=y$ and integrating over x. The first term is again regularized by subtracting its value for vanishing external field. This yields for the vacuum expectation value of the total charge

$$\langle Q\rangle = \frac{1}{2}\sum_{\alpha=1}^{2}\int_0^L dx \lim_{x\to y}\left(\Omega_{\alpha\alpha}(y,x)-\Omega_{\alpha\alpha}^{(0)}(y,x)\right)=0, \tag{6.103}$$

since the covariance (6.99) is traceless with respect to the spinor indices. The result (6.103) has of course been expected since the total charge should annihilate the vacuum state (see also the discussion at the end of this chapter). This is true in any number of dimensions.

For the chiral charge we give first a general expression which is valid in any even dimension. We define the "point splitted" chiral charge

$$\begin{aligned}\rho_5(x,y) &= \bar\psi(x)\gamma^5\gamma^0\exp\left(ie\int_x^y A\right)\psi(y) \\ &= -\psi^\dagger(x)\gamma^5\exp\left(ie\int_x^y A\right)\psi(y).\end{aligned} \tag{6.104}$$

Operating with this on the vacuum state yields (compare (6.67))

$$\begin{aligned}\rho_5(x,y)\Psi &= -\frac{1}{2}\exp\left(ie\int_x^y A\right)\mathrm{Tr}\gamma^5(\delta(x-y)+\Omega(y,x))\Psi \\ &-\frac{1}{2}\exp\left(ie\int_x^y A\right)\int dvdwu^\dagger(v)\left(\delta(v-x)-\Omega(v,x)\right)\gamma^5 \\ &\cdot\left(\delta(y-w)+\Omega(y,w)\right)u(z)\Psi.\end{aligned} \tag{6.105}$$

The second term can be written, after setting $x=y$, integrating over x, and performing the expectation value, as

$$-2\mathrm{Tr}P_+\gamma^5P_- = 0$$

since $P_+P_- = 0$ (compare (6.79)), and use has been made of (6.33). We are thus left with

$$\langle\Psi|Q_5|\Psi\rangle = -\frac{1}{2}\mathrm{Tr}\int_0^L dx \lim_{x\to y}\gamma^5\Omega(y,x)\exp\left(ie\int_x^y A\right), \tag{6.106}$$

from where the result for $A = 0$ has to be subtracted. Using the explicit results in two dimensions we find

$$-\frac{1}{2}\ \mathrm{Tr}\ \gamma^5\Omega(y,x)\exp\left(ie\int_x^y A\right) = \frac{im}{\pi}\sum_n K_1(mLn + m(y-x))e^{-2\pi in\phi}.$$

Subtracting from this the expression with $A = 0$ we get

$$\frac{im}{\pi}\sum_n \left(K_1(m(y-x)+nmL)e^{-2\pi in\phi} - K_1(m(y-x)+nmL)e^{-2\pi in\alpha}\right)$$

so that we have

$$\begin{aligned}\langle Q_5\rangle &= \frac{im}{\pi}\int_0^L dx \sum_{n\neq 0} K_1(nmL)\left(e^{-2\pi in\phi} - e^{-2\pi in\alpha}\right)\\ &= \frac{2mL}{\pi}\sum_{n>0} K_1(nmL)(\sin(2\pi n\phi) - \sin(2\pi n\alpha)).\end{aligned} \tag{6.107}$$

In the limit $m \to 0$ we obtain

$$\begin{aligned}\lim_{m\to 0}\langle Q_5\rangle &= \frac{2}{\pi}\sum_{n>0}\frac{1}{n}(\sin(2\pi n\phi) - \sin(2\pi n\alpha))\\ &= 2([\phi] - \phi + \frac{1}{2}) - 2([\alpha] - \alpha + \frac{1}{2})\\ &= 2([\alpha+\varphi] + [\alpha] - \varphi).\end{aligned} \tag{6.108}$$

This is equal to our earlier result (6.74) when evaluated for $\beta = 0$ (φ was defined in (6.72)). Recalling the asymptotic formula for K_1 in the limit of large arguments one finds that

$$\langle Q_5\rangle \stackrel{L\to\infty}{\sim} \sqrt{\frac{2mL}{\pi}}(\sin(2\pi n\phi) - \sin(2\pi n\alpha))e^{-mL} \stackrel{L\to\infty}{\to} 0. \tag{6.109}$$

We finally calculate the vacuum expectation value of the Hamiltonian H_ψ (6.17) in the massive case. We start from the expectation value (6.77) for the point splitted Hamiltonian but insert in that expression

$$h_x = -i\gamma^0\gamma^1(\partial_x - iA) + m\gamma^0 \tag{6.110}$$

as well as the full covariance Ω instead of Ω_+. With our result (6.99) for the covariance we then find

$$\exp\left(ie\int_y^x A\right) h_x\Omega(x,y) = -\frac{im^2}{\pi}\gamma^0\gamma^1\sum_n \begin{pmatrix} -K_0' & iK_1' \\ iK_1' & K_0' \end{pmatrix} e^{-2\pi in\phi}$$
$$+\frac{m^2}{\pi}\gamma^0 \exp\left(-ie\int_y^x A\right)\sum_n \begin{pmatrix} -K_0 & iK_1 \\ iK_1 & K_0 \end{pmatrix} e^{-2\pi in\phi}.$$

Thus,

$$\langle\Psi|H_\psi(x,y)|\Psi\rangle = \frac{1}{2}\mathrm{Tr}\exp\left(-ie\int_y^x A\right)(h_x\Omega(x,y) + h_x\delta(x-y))$$
$$= \mathrm{Tr}\left\{\frac{m^2}{2\pi}\sum_n \begin{pmatrix} -K_0 - K_1' & iK_1 + iK_0' \\ -iK_1 - iK_0' & -K_1' - K_0 \end{pmatrix} e^{-2\pi in\phi}\right. \tag{6.111}$$
$$\left.+\frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i\delta'(x-y) + \frac{m}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\delta(x-y)\right\}.$$

From this we subtract the expectation value for $L \to \infty$ and vanishing A. Using the relations

$$K_1'(\xi) + K_0(\xi) = -\frac{K_1(\xi)}{\xi}, \quad K_1 = -K_0',$$

this yields

$$\begin{aligned}
\langle H_\psi\rangle &= \int_0^L dx \lim_{x\to y}\left(\langle\Psi|H_\psi(x,y)|\Psi\rangle - \langle\Psi_0|H_\psi(x,y)|\Psi_0\rangle\right) \\
&= \frac{m^2}{\pi}\int_0^L dx \lim_{x\to y}\left(\sum_n \frac{e^{-2\pi in\phi}}{m(x-y)}K_1(m(x-y))\right. \\
&\quad \left.-\frac{1}{m(x-y)+nmL}K_1(m(x-y)+nmL)\right) \\
&= \frac{m}{\pi}\sum_{n\neq 0}\frac{1}{n}K_1(nmL)e^{-2\pi in\phi} \\
&= \frac{2m}{\pi}\sum_{n>0}\frac{1}{n}K_1(nmL)\cos(2\pi n\phi). \qquad (6.112)
\end{aligned}$$

This vanishes in the limit $L \to \infty$ but remains finite for finite L even for vanishing electromagnetic field where we have

$$\langle H_\psi\rangle_{A=0} = \frac{2m}{\pi}\sum_{n>0}\frac{1}{n}K_1(nmL)\cos(2\pi n\alpha). \tag{6.113}$$

In the limit of vanishing mass we obtain from (6.112) the result of section 6.2.1. The expectation value of the Hamiltonian vanishes for $L \to \infty$ as can be easily seen from (6.112).

6.3 Non-Abelian gauge fields

Calculation of the covariance: We consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i D_\mu \gamma^\mu - m)\psi, \tag{6.114}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \quad , \quad D_\mu = \partial_\mu + i A_\mu. \tag{6.115}$$

We have introduced here the (hermitian) matrix-valued vector field, $A_\mu(x)$, which is defined by

$$A_\mu = A_\mu^i T_i \quad , \quad (T_i, T_j) = \delta_{ij}.$$

The gauge coupling constant has been set equal to one. In two dimensions the discussion is greatly simplified since the gauge $A^0 = 0$ removes the commutator in (6.115). This enables us to proceed analogously to the Abelian case. Denoting $A_1 \equiv A$, the total Hamiltonian density reads explicitly

$$\mathcal{H} = \frac{1}{2}\pi_A^2 - i\psi^\dagger \gamma^0 \gamma^1 (\partial_x + iA)\psi + m\psi^\dagger \gamma^0 \psi \equiv \frac{1}{2}\pi_A^2 + \psi^\dagger h \psi. \tag{6.116}$$

The first-quantized Hamiltonian reads

$$h = g h_{(0)} g^{-1}, \quad \text{where} \quad h_{(0)} = -i\gamma^0\gamma^1\partial_x + m\gamma^0 \tag{6.117}$$

and

$$g(x) = \mathcal{P}\exp\left(-i\int_0^x A\right), \tag{6.118}$$

where $\mathcal{P}$ denotes path-ordering (we will suppress this letter in the following). It follows immediately that if $\psi^{(0)}$ is an eigenfunction of $h_{(0)}$ with eigenvalue E then $\psi = g\psi^{(0)}$ is an eigenfunction of h with the same eigenvalue E. From (6.89) and (6.90) we see that the free eigenfunctions are given by

$$\psi_{n,+}^{(0)} = c f_n, \quad \psi_{n,-}^{(0)} = c g_n, \tag{6.119}$$

where c is a constant vector in the representation space of the above generators, and

$$f_n(x) = \frac{1}{\sqrt{2\omega_n(\omega_n + m)L}} \begin{pmatrix} \omega_n + m \\ \lambda_n \end{pmatrix} \exp(-i\lambda_n x)$$

and

$$g_n(x) = \frac{1}{\sqrt{2\omega_n(\omega_n + m)L}} \begin{pmatrix} -\lambda_n \\ \omega_n + m \end{pmatrix} \exp(-i\lambda_n x).$$

We also have to implement the boundary conditions

$$\psi(L) = e^{2\pi i\alpha}\psi(0) = g(L)\psi^{(0)}(L) = g(0)e^{2\pi i\alpha}\psi^{(0)}(0). \tag{6.120}$$

We note that

$$g^{-1}(L)g(0) = \exp\left(i\int_0^L A\right) \equiv \exp(iB)\,. \tag{6.121}$$

Since B is a hermitian matrix it can be diagonalized:

$$Be_a = \mu_a e_a \quad , \quad (e_a, e_b) = \delta_{ab},$$

where a and b run form 1 to the dimension of the representation. We thus have

$$g^{-1}(L)g(0)e_a = \exp(i\mu_a)\, e_a. \tag{6.122}$$

Choosing $c = e_a$ we find from (6.120) the quantization condition

$$\lambda_{n,a} = \frac{2\pi}{L}(n - \alpha) - \frac{\mu_a}{L}, \tag{6.123}$$

and the energies are given by

$$E_{n,a} = \pm\sqrt{m^2 + \lambda_{n,a}^2} \equiv \pm\omega_{n,a} \tag{6.124}$$

in analogy to the Abelian result (6.88). From (6.119) and $\psi = g\psi^{(0)}$ the positive energy and negative energy solutions are given by

$$\psi_{n,+}^a = g(x)e_a \otimes f_n^a \quad , \quad \psi_{n,-}^a = g(x)e_a \otimes g_n^a, \tag{6.125}$$

(no summation over a). These solutions are orthonormal since

$$(\psi_{n,+}^a, \psi_{m,+}^b) = (g(x)e_a \otimes f_n^a, g(x)e_b \otimes f_m^b) = \delta_{ab}\delta_{nm}, \quad \text{etc.}$$

Under a gauge transformation mediated by $U(x)$ the following transformation laws hold:

$$\begin{aligned} \psi \to \tilde{\psi} = U(x)\psi \quad &, \quad A \to \tilde{A} = UAU^{-1} + i(\partial_x U)U^{-1}, \\ g \to \tilde{g} = U(x)g(x)U^{-1}(0) \quad &, \quad \psi^{(0)} \to \tilde{\psi}^{(0)} = U(0)\psi^{(0)}. \end{aligned} \tag{6.126}$$

Since gauge transformations should respect the boundary conditions, we must have $U(0) = U(L)$. Since the "boundary operator" $g^{-1}(L)g(0)$ transforms as

$$g^{-1}(L)g(0) \to U(0)g^{-1}(L)g(0)U^{-1}(0)$$

the quantities μ_a appearing in (6.122) are gauge invariant.

We now proceed to calculate the covariance of the ground state. For the projector on positive energies one finds, making use of the result (6.92) for the Abelian case,

$$\begin{aligned} P_+(x,y) &= \sum_{a,n} \psi^a_{n,+}(x)\psi^{a+}_{n,+}(y) \\ &= \frac{1}{2L} g(x)\Bigg[\sum_a e_a e_a^\dagger \exp\big(\frac{i}{L}(2\pi\alpha + \mu_a)(x-y)\big) \\ &\times \sum_n \frac{e^{e^{-2\pi i n(x-y)/L}}}{\omega_n} \begin{pmatrix} \omega_n + m & \lambda_n \\ \lambda_n & \omega_n - m \end{pmatrix}\Bigg] g^\dagger(y). \end{aligned} \tag{6.127}$$

Applying Poisson's summation formula (6.94) one finds, in analogy to (6.97),

$$\begin{aligned} P_+(x,y) &= \frac{1}{2}\delta(x-y)\mathbf{I} - \frac{m}{2\pi} g(x)\Bigg[\sum_{a,n} e_a \exp\big(-in\int_0^L \mu_a\big) e_a^\dagger \\ &\times \begin{pmatrix} -K_0 & iK_1 \\ iK_1 & K_0 \end{pmatrix} e^{-2\pi i n\alpha}\Bigg] g^\dagger(y). \end{aligned} \tag{6.128}$$

and $P_- = \mathbf{I} - P_+$. It is convenient to define the "diagonal matrix"

$$D = \sum_a \exp(i\mu_a)\, e_a e_a^\dagger \Rightarrow D^n = \sum_a \exp(in\mu_a)\, e_a e_a^\dagger. \tag{6.129}$$

The covariance $\Omega = P_- - P_+$ can thus be written as

$$\Omega(x,y) = \frac{m}{\pi} g(x)\left[\sum_n D^{-n} e^{-2\pi i n\alpha} \begin{pmatrix} -K_0 & iK_1 \\ iK_1 & K_0 \end{pmatrix}\right] g^\dagger(y). \tag{6.130}$$

In the Abelian case we have

$$\mu = \int_0^L A \tag{6.131}$$

so that the result (6.130) equals our earlier result (6.99).

Charges and energy: The point splitted version of the non-Abelian current operator reads, in any number of dimensions,

$$j_i^\mu(x,y) = \psi^\dagger(x)\exp\left(i\int_x^y A\right) T_i\gamma^0\gamma^\mu\psi(y). \tag{6.132}$$

Its action on the vacuum state Ψ can be found in the same way as for the Abelian case (6.102). The result is

$$
\begin{aligned}
j_i^\mu(x,y)\Psi &= \frac{1}{2}\mathrm{Tr}\exp\left(i\int_x^y A\right)T_i\gamma^0\gamma^\mu(\delta(x-y)+\Omega(y,x))\Psi \\
&+\frac{1}{2}\int dvdwu^\dagger(v)(\delta(v-x)-\Omega(v,x))\exp\left(i\int_x^y A\right)\times \\
&T_i\gamma^0\gamma^\mu(\delta(y-w)+\Omega(y,w))u(w)\Psi.
\end{aligned} \tag{6.133}
$$

Taking the expectation value of the second term in (6.133) with respect to Ψ, one gets, making use of (6.33) and (6.27)

$$
\begin{aligned}
&\frac{1}{4}\mathrm{Tr}\int dvdw\ (\delta(v-x)-\Omega(v,x))\exp\left(i\int_x^y A\right)\times \\
&T_i\gamma^0\gamma^\mu(\delta(y-w)+\Omega(y,w))\Omega(w,v) \\
=\ &\frac{1}{4}\mathrm{Tr}\int dv(\delta(y-v)+\Omega(y,v))(\delta(v-x)-\Omega(v,x))\times \\
&\exp\left(i\int_x^y A\right)T_i\gamma^0\gamma^\mu=0.
\end{aligned}
$$

The expectation value of the point splitted current with respect to Ψ is thus given by

$$
\langle\Psi|j_i^\mu(x,y)|\Psi\rangle=\frac{1}{2}\mathrm{Tr}\exp\left(i\int_x^y A\right)T_i\gamma^0\gamma^\mu(\delta(x-y)+\Omega(y,x)). \tag{6.134}
$$

For the axial current

$$
j_{5i}^\mu(x,y)=\psi^\dagger(x)\exp\left(i\int_x^y A\right)T_i\gamma^0\gamma^5\gamma^\mu\psi(y) \tag{6.135}
$$

the analogous result is (compare also (6.105))

$$
\langle\Psi|j_{5i}^\mu(x,y)|\Psi\rangle=-\frac{1}{2}\mathrm{Tr}\exp\left(i\int_x^y A\right)T_i\gamma^5\gamma^0\gamma^\mu(\delta(x-y)+\Omega(y,x)). \tag{6.136}
$$

Like in the Abelian case (see (6.103)) one finds from (6.134) that $<Q>=0$, where Q is the total charge (the first term in (6.134) vanishes after the subtraction of the "free" expectation value, the second term vanishes since Ω is traceless in spinor space - see (6.130)).

In the following we explicitly evaluate the vacuum expectation value of the chiral charge in two spacetime dimensions. From (6.136) we have

$$
\langle\rho_i^5(x,y)\rangle=-\frac{1}{2}\mathrm{Tr}\exp\left(i\int_x^y A\right)T_i\gamma^0\gamma^1(\Omega(y,x)-\Omega_{(0)}(y,x)). \tag{6.137}
$$

The trace in (6.137) consists actually of two traces: a trace Tr_S in spinor space and a trace Tr_C in the representation space of the Lie group. We evaluate the spinor trace by making use of (6.81) and (6.130):

$$
-\frac{1}{2}\mathrm{Tr}_S\gamma^0\gamma^1\Omega(y,x)=\frac{im}{\pi}g(y)\sum_n e^{-2\pi i\alpha n}D^{-n}K_1(m(x-y)+mnL)g^\dagger(x).
$$

Eq. (6.137) then becomes

$$\begin{aligned} \langle \rho_i^5(x,y) \rangle &= \frac{im}{\pi} \sum_n e^{-2\pi i \alpha n} \Big(\mathrm{Tr}_C e^{i \int_x^y A} T_i g(y) D^{-n} g^\dagger(x) \\ &\quad - \mathrm{Tr}_C T_i \Big) K_1(m(x-y) + mnL). \end{aligned} \tag{6.138}$$

The singular terms which arise for $n = 0$ cancel. The remaining terms are non singular in the coincidence limit $x \to y$, and one finds for the expectation value of the total chiral charge

$$\begin{aligned} \langle Q_i^5 \rangle &= \frac{im}{\pi} \int_0^L \sum_{n \neq 0} e^{-2\pi i \alpha n} \Big(\mathrm{Tr}_C g^\dagger(x) T_i g(x) D^{-n} \\ &\quad - \mathrm{Tr}_C T_i \Big) K_1(mnL). \end{aligned} \tag{6.139}$$

This is the non-Abelian version of our earlier result (6.107). In the limit of vanishing mass one finds, using (6.129) and $K_1(x) \sim 1/x$,

$$\begin{aligned} \langle Q_i^5 \rangle &\sim \frac{2m}{L} \int_0^L dx \mathrm{Tr}_C g^\dagger(x) T_i g(x) \sum_a e_a e_a^\dagger ([\phi_a] + \frac{1}{2} - \phi_a) \\ &\quad - \mathrm{Tr}_C T_i ([\alpha] + \frac{1}{2} - \alpha). \end{aligned} \tag{6.140}$$

Note that for semisimple groups the trace of the T_i vanishes. We emphasize that the currents in the non-Abelian theory are *not* gauge invariant quantities but instead transform under the adjoint representation of the gauge group.

We finally come to the calculation of the vacuum expectation value for the energy. This closely parallels the discussion of the Abelian case which was discussed in section 6.2 so that we can be brief in the present case. The point splitted version of the expectation value now reads, in analogy to (6.77),

$$\langle \Psi | H_\psi(x,y) | \Psi \rangle = \frac{1}{2} \mathrm{Tr} \int dx \exp\left(-i \int_x^y A \right) h_x(\delta(x-y) + \Omega(x,y). \tag{6.141}$$

We recall that the exponential stands for a path ordered product. Inspection of the explicit form of the covariance, Eq. (6.130), exhibits that, as in the Abelian case, the factors $g(x)$ and $g^\dagger(y)$ are exactly canceled by the exponential in (6.141). In analogy to (6.112) we then find, after the subtraction of the expectation value for vanishing external field,

$$\langle H_\psi \rangle = \frac{2m}{\pi} \sum_a \sum_{n>0} \frac{1}{n} K_1(nmL) \cos(2\pi n \alpha + n\mu_a). \tag{6.142}$$

In the limit of vanishing mass this becomes

$$\langle H_\psi \rangle_{m=0} = \frac{2\pi}{L} \sum_a \left(\alpha + \frac{\mu_a}{2\pi} - [\alpha + \frac{\mu_a}{2\pi}] - \frac{1}{2} \right)^2 - \frac{\pi}{6L} N, \tag{6.143}$$

where N is the number of flavors.

6.4 Particle Creation

6.4.1 Constant electric field in four dimensions

In this subsection we demonstrate how the well known expression for the creation of fermions in a constant external electric field [16] can be recovered in the functional Schrödinger picture. The physical picture is the following: We start with a fermionic vacuum state in the far past ("in - region") and let it evolve under the influence of the external field, using the Schrödinger equation, into the far future ("out - region"). There we calculate the overlap with the vacuum in the out - region and interpret the deviation from one as the probability for particle creation. The state remains, of course, Gaussian but its exact form (and thus the notion of the vacuum) changes under the evolution of the external field. It would be physically reasonable to switch on the field somewhere in the past and switch it off again in the future since no fields last infinitely long. In the present case of a constant electric field it will prove advantageous to treat an idealized situation by making use of the notion of an *adiabatic* vacuum state which is approached in the asymptotic regions. This is possible since $\dot{h}/h$, where $\dot{h}$ is the time-derivative of the first-quantized Hamiltonian h (6.4), approaches zero in both the asymptotic past and future. The concept of adiabatic states is also successfully applied in traditional discussions of particle creation [17] and finds in particular a fruitful application in quantum theory on curved spacetimes [18].

We thus have for the in - vacuum state

$$\Psi_{in} = N \exp\left(u^\dagger \Omega^{in}_{(ad)} u \right), \tag{6.144}$$

and for the out - vacuum state

$$\Psi_{out} = N \exp\left(u^\dagger \Omega^{out}_{(ad)} u \right). \tag{6.145}$$

The "adiabatic" covariance $\Omega_{(ad)}$ can be obtained from the "free" covariance $\Omega_{(0)}$ (see 6.32) by replacing the momentum p with $p + eA$. It turns out to be convenient, in spite of the non-vanishing mass, to use the chiral representation for the Dirac matrices. The reason is that the mass terms in the expressions for the covariance become unimportant in the asymptotic regions. We thus have, instead of (6.32),

$$\Omega_{(ad)} = \frac{1}{\sqrt{\tilde{p}^2 + m^2}} \begin{pmatrix} -\sigma \cdot \tilde{p} & m \\ m & \sigma \cdot \tilde{p} \end{pmatrix}, \quad \tilde{p} \equiv (p_x, p_y, p_z + eA_z), \tag{6.146}$$

and the electric field points in z - direction, $\mathbf{E} = E\mathbf{e_z}$, so that $A_z = Et$. For simplicity we denote the transversal momentum by $p_\perp$ so that $p_\perp^2 = p_x^2 + p_y^2$. It will also be convenient to introduce the dimensionless quantity

$$\tau \equiv \sqrt{eE}\left(t + \frac{p_z}{eE}\right). \tag{6.147}$$

We now give the explicit expression for $\Omega_{(ad)}$ in both the asymptotic past and future. In the limit $\tau \to -\infty$, (6.146) reads (σ_i are the Pauli matrices)

$$\begin{aligned} \Omega_{(ad)} &= \frac{1}{\sqrt{A}} \begin{pmatrix} -\sigma_\perp \cdot p_\perp - \sigma_z \cdot \sqrt{eE}\tau & m \\ m & \sigma_\perp \cdot p_\perp + \sigma_z \cdot \sqrt{eE}\tau \end{pmatrix} \\ &\overset{\tau \to -\infty}{\longrightarrow} \begin{pmatrix} \sigma_z & 0 \\ 0 & -\sigma_z \end{pmatrix} \equiv \Omega_{(ad)}^{in}, \quad A = p_\perp^2 + eE\tau^2 + m^2 \end{aligned} \tag{6.148}$$

Analogously,

$$\Omega_{(ad)}^{out} = \begin{pmatrix} -\sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = -\Omega_{(ad)}^{in}. \tag{6.149}$$

Before we proceed to calculate the pair creation rate according to the general formula (6.51), we have to discuss one subtlety which arises through the use of asymptotic vacuum states. As can be immediately seen by comparing (6.148) and (6.149), the adiabatic covariances $\Omega_{(ad)}^{out}$ and $\Omega_{(ad)}^{in}$ differ in their sign. Consequently, from the general expression (6.25), the positive (negative) frequency eigenfunctions in the far future are the negative (positive) frequency eigenfunctions of the far past. An observer in the far future would replace the expansion (6.45) by

$$\chi_n(t) = \alpha_{nm}^f \chi_m^f + \beta_{nm}^f \psi_m^f = \alpha_{nm}^f \psi_m + \beta_{nm}^f \chi_m, \tag{6.150}$$

where the superscript f refers to "far future." Comparing (6.150) with (6.45) we see that $\alpha_{nm}^f = \beta_{nm}$ and $\beta_{nm}^f = \alpha_{nm}$. Nevertheless, one can still use the expression (6.51) to calculate the transition element. The reason is that one now has to use $\Omega_{(ad)}^{out} = -\Omega_{(ad)}^{in}$ instead of $\Omega_0 = \Omega_{(ad)}^{in}$ in (6.43). This would amount to replace β_{nm} in (6.51) by $\alpha_{nm} = \beta_{nm}^f$. Thus, the particle creation rate is still given by (6.51) with β_{nm} replaced by β_{nm}^f as it was introduced in (6.150) (in the following we will for simplicity omit the superscript f).

The general expression (6.44) for the covariance $\Omega(t)$ contains, via (2.61), the functions $\chi_n(t)$ which obey

$$i\dot{\chi}_n(t) = h\chi_n(t), \tag{6.151}$$

where the first-quantized Hamiltonian h is given explicitly by

$$h = \begin{pmatrix} \sigma \cdot \tilde{p} & -m \\ -m & -\sigma \cdot \tilde{p} \end{pmatrix}. \tag{6.152}$$

Note that $h^2 = (p_\perp^2 + m^2 + E\tau^2)\mathbf{I}$, and n has to be replaced by $\mathbf{p}$. Differentiating (6.151) by t and using (6.151) again, one arrives at a second order equation for the χ_n. The first and fourth component of the $\chi_\mathbf{p}$ obeys (we omit the index $\mathbf{p}$ in the following)

$$\left(\frac{d^2}{d\tau^2} + \tau^2 + \Lambda + i \right) \chi_{1,4} = 0, \tag{6.153}$$

while the second and third component obeys

$$\left(\frac{d^2}{d\tau^2} + \tau^2 + \Lambda - i \right) \chi_{2,3} = 0. \tag{6.154}$$

We have introduced in these expressions the quantity

$$\Lambda = \frac{p_\perp^2 + m^2}{|eE|}. \tag{6.155}$$

The discussion is greatly simplified if we treat the case of two spacetime dimensions first and recover the four-dimensional case by some simple manipulations from the final result. Instead of (6.153) and (6.154) we have then to deal with the equations

$$\left(\frac{d^2}{d\tau^2} + \tau^2 + \xi + i \right) \chi_1 = 0 \quad , \quad \left(\frac{d^2}{d\tau^2} + \tau^2 + \xi - i \right) \chi_2 = 0, \tag{6.156}$$

where, obviously,

$$\xi = \frac{m^2}{|eE|}. \tag{6.157}$$

Since χ obeys the first-order equation (6.151), the equations (6.156) and (6.156) cannot be solved independently. If we choose, say, for χ_1 the general solution of (6.156), we find from (6.151) that

$$\chi_2 = \frac{1}{\sqrt{\xi}} \left(i \frac{d\chi_1}{d\tau} - \tau \chi_1 \right). \tag{6.158}$$

The general solution of (6.156) is then given by a sum of parabolic cylinder functions [19]

$$\chi_1 = A_1 D_{-i\xi/2}[(1+i)\tau] + B_1 D_{-i\xi/2}[-(1+i)\tau]. \tag{6.159}$$

We now have to impose the boundary condition that χ approaches a *negative frequency eigenfunction* for $\tau \to -\infty$. For this we need the asymptotic expansion of (6.159) which reads [19]

$$\begin{aligned}\chi_1 \quad \overset{\tau\to-\infty}{\sim} \quad & A_1 \left(e^{-\frac{i\tau^2}{2}} [(1+i)\tau]^{-\frac{i\xi}{2}} \right. \\ & \left. - \frac{\sqrt{2\pi}}{\Gamma(\frac{i\xi}{2})} e^{-\frac{\pi\xi}{2}+\frac{i\tau^2}{2}} [(1+i)\tau]^{\frac{i\xi}{2}-1} \right) \\ & + B_1 e^{-\frac{i\tau^2}{2}} [-(1+i)\tau]^{-\frac{i\xi}{2}}.\end{aligned} \tag{6.160}$$

The usual definition of positive and negative frequencies involves the phase of the first-quantized eigenfunctions: For a positive frequency function the phase decreases with increasing time, while for a positive frequency function it increases [17]. The expression (6.160) thus should only contain terms proportional to $\exp(-i\tau^2/2)$. We thus have $A_1 = 0$ and one is left with

$$\chi_1 = B_1 D_{-i\xi/2} \left[-(1+i) \right]. \tag{6.161}$$

From (6.158) one then gets

$$\chi_2 = -\frac{B_1\sqrt{\xi}}{2}(1+i) D_{-i\xi/2-1} \left[-(1+i) \right]. \tag{6.162}$$

We want to normalize the solution $\chi = (\chi_1, \chi_2)^T$. Since the norm is conserved (h in (6.151) is hermitian), it is sufficient to perform the normalization in the asymptotic past where

$$\chi_1 \overset{\tau\to-\infty}{\longrightarrow} B_1 e^{-\frac{i\tau^2}{2}} |\tau|^{-\frac{i\xi}{2}} 2^{-\frac{i\xi}{4}} e^{\frac{\pi\xi}{8}} \quad , \quad \chi_2 \overset{\tau\to-\infty}{\longrightarrow} 0.$$

Thus, the choice

$$B_1 = \exp(-\pi\xi/8) \tag{6.163}$$

yields $\chi^\dagger \chi \equiv |\chi_1|^2 + |\chi_2|^2 = 1$.

To make use of (6.150) we have to find the positive and negative frequency functions in the asymptotic future, i.e. for $\tau \to \infty$. The correctly normalized negative frequency solution χ_f to (6.156) and (6.158) reads

$$\chi_1^f = \sqrt{\frac{\xi}{2}} e^{-\frac{\pi\xi}{8}} D_{i\xi/2-1} \left[(1-i)\tau \right] \quad , \quad \chi_2^f = -\frac{i+1}{\sqrt{2}} e^{-\frac{\pi\xi}{8}} D_{i\xi 2} \left[(1-i) \right].$$

5.26 This is easily seen from the asymptotic expansion of the parabolic cylinder functions [19]. Similarly, the positive frequency functions are found to read

$$\psi_1^f = e^{-\frac{\pi\xi}{8}} D_{-i\xi/2} \left[(1+i)\tau \right] \quad , \quad \psi_2^f = \frac{\sqrt{\xi}}{2}(i+1) e^{-\frac{\pi\xi}{8}} D_{-i\xi/2-1} \left[(1+i)\tau \right].$$

Making now use of the identity [19]

$$D_\lambda(z) = e^{\lambda\pi i} D_\lambda(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\lambda)} e^{\pi(\lambda+1)i/2} D_{-\lambda-1}(-iz), \qquad (6.164)$$

we can expand the solution (6.161), (6.162), (6.163) according to (6.150) into the asymptotic positive and negative frequency solutions, respectively:

$$\chi(\tau) = \frac{\sqrt{\pi\xi}}{\Gamma(\frac{i\xi}{2}+1)} e^{-\frac{\pi\xi}{4}} \chi^f + e^{-\frac{\pi\xi}{2}} \psi^f. \qquad (6.165)$$

The Bogolubov coefficients can be easily read off from this equation,

$$\alpha = \frac{\sqrt{\pi\xi}}{\Gamma(\frac{i\xi}{2}+1)} e^{-\frac{\pi\xi}{4}}, \ \beta = e^{-\frac{\pi\xi}{2}}, \qquad (6.166)$$

and it is easily checked that $|\alpha|^2 + |\beta|^2 = 1$. Finally, one then finds for the matrix element (6.51)

$$\begin{aligned} |\langle\Psi_1|\Psi_2\rangle|^2 &= \det(1 - |\beta|^2) \\ &= \exp \operatorname{Tr} \ln(1 - e^{-\pi\xi}) \\ &= \exp\left(-\operatorname{Tr} \sum_n \frac{1}{n} e^{-\pi n\xi}\right). \end{aligned} \qquad (6.167)$$

In two dimensions the trace reads

$$\operatorname{Tr} \longrightarrow \frac{L}{2\pi} \int_{eEt_{in}}^{eEt_{out}} dp = \frac{eELT}{2\pi},$$

where $T \equiv t_{out} - t_{in}$ is the time difference between two asymptotic times t_{out} and t_{in}. This, as well as the length L, has been introduced as an infrared regulator [17], [7]. Thus,

$$|\langle\Psi_1|\Psi_2\rangle|^2 = \exp\left(-\frac{eELT}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{n\pi m^2}{eE}}\right). \qquad (6.168)$$

(If eE is negative, one has to take its absolute value.) To find the corresponding expression in four spacetime dimensions, we have to replace ξ by Λ, see (6.155). One thus has

$$|\beta|^2 = e^{-\pi\Lambda} = e^{-\frac{\pi(m^2+p_\perp^2)}{eE}} \qquad (6.169)$$

and

$$\operatorname{Tr} \longrightarrow \frac{V}{(2\pi)^3} \int_{eEt_{in}}^{eEt_{out}} dp_z \int 2\pi p_\perp dp_\perp.$$

Moreover, one gets an additional factor of 2 from the discrete part of the determinant in (6.167) over the spinor indices since one now deals with four spinors instead of two spinors. Thus,

$$\begin{aligned} |\langle\Psi_1|\Psi_2\rangle|^2 &= \exp\left(-2\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\pi n\Lambda}\right) \\ &= \exp\left(-\frac{2(eE)^2VT}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-\frac{n\pi m^2}{eE}}\right). \end{aligned} \tag{6.170}$$

This is in agreement with the classical result of Schwinger [16].

6.4.2 Arbitrary external fields for massless QED_2

We now proceed to calculate the vacuum - to - vacuum transition rate (6.42) in the case of massless fermions for arbitrary external electromagnetic fields in two spacetime dimensions. In contrast to the previous section we shall assume that the electric field is switched off for some time $t < t_1$ in the past and $t > t_2$ in the future. While one can consistently assume that the vector potential vanishes for $t < t_1$, this is *not* possible for $t > t_2$ since the flux

$$\int_0^L dx \int_{t_1}^{t_2} dtE = \int dxdt\dot{A} = \int dx\,(A(x,t_2) - A(x,t_1)) = 2\pi\varphi(t_2)$$

need not vanish. In fact, this will give rise to the nontrivial features which will be discussed in this section. We can, however, assume that A does not depend on x for $t > t_2$.

To determine the covariances Ω_1 and Ω_2 in (6.42) we need to solve the time-dependent Dirac equation,

$$i\dot{\psi} = h\psi = -i\gamma_5(\partial_x + iA)\psi. \tag{6.171}$$

We make the ansatz

$$\psi(x,t) = \exp(i\lambda(x,t) + i\delta(x,t)\gamma_5)\psi_0(x,t) \tag{6.172}$$

and choose λ and δ such that ψ_0 obeys the free Dirac equation (without A-field). Inserting (6.172) into (6.171) one recognizes that this can be achieved if

$$\dot{\lambda} + \delta' = 0 \quad , \quad \lambda' + \dot{\delta} = -A. \tag{6.173}$$

The formal solution reads

$$\lambda = \frac{1}{\partial^2}A' \quad , \quad \delta = -\frac{1}{\partial^2}E. \tag{6.174}$$

The solution of the free equation for ψ_0,

$$i\dot{\psi}_0 = -i\gamma_5\partial_x\psi_0, \tag{6.175}$$

can of course be immediately written down by making use of (6.55) - (6.58) (we choose $\beta = 0$ for simplicity):

$$\psi_{0,n} = \begin{pmatrix} \varphi_{0,n} \\ \chi_{0,n} \end{pmatrix} \tag{6.176}$$

with

$$\varphi_{0,n} = \frac{1}{\sqrt{L}} \exp\left(-ik_n(x+t)\right) \quad , \quad \chi_{0,n} = \frac{1}{\sqrt{L}} \exp\left(-ik_n(x-t)\right),$$

where $k_n = 2\pi(n-\alpha)/L$. The positive energy (negative energy) solutions are obtained for $k_n > 0$ ($k_n < 0$) in the ϕ's and and for $k_n < 0$ ($k_n > 0$) in the χs (recall (6.57) and (6.58)). The solutions of (6.171) thus read

$$\psi_n(x,t) = \exp(i\lambda + i\delta\gamma_5)\psi_{0,n}. \tag{6.177}$$

The components of the covariance are calculated in full analogy to Eq. (6.61). One finds

$$\Omega_+(x,y,t) = e^{i\lambda(x,t)-i\delta(x,t)}\,\Omega_+^{(0)}(x,y)\,e^{-i\lambda(y,t)+i\delta(y,t)} \tag{6.178}$$

and

$$\Omega_-(x,y,t) = e^{i\lambda(x,t)+i\delta(x,t)}\,\Omega_-^{(0)}(x,y)\,e^{-i\lambda(y,t)-i\delta(y,t)}, \tag{6.179}$$

where $\Omega_+^{(0)}$ and $\Omega_-^{(0)}$ are obtained from (6.62) and (6.63) by setting the A-field equal to zero:

$$\Omega_+^{(0)}(x,y) = -\Omega_-^{(0)}(x,y) = \frac{i}{L}\, e^{\frac{2\pi i}{L}(\alpha-[\alpha]-\frac{1}{2})(x-y)} \frac{1}{\sin\frac{\pi}{L}(x-y)}. \tag{6.180}$$

Since $A = 0$ for $t < t_1$ one can choose $\lambda = \delta = 0$ for $t < t_1$. This corresponds to the choice of the retarded Green function in (6.174). We thus have $\Omega = \Omega^{(0)}$ for $t < t_1$.

We now proceed to calculate the overlap (6.42) between the out - vacuum and the out - state which results from evolving the in - vacuum (which is the free state) with the Schrödinger equation. In the out - region ($t \to \infty$) we can choose A to be constant. From (6.173) we can choose $\lambda = 0$ and $\delta = -At$. The one particle wave functions (6.177) then read

$$\psi_n(x,t) = \exp(-iAt\gamma_5)\psi_{0,n}(x,t). \tag{6.181}$$

The out - vacuum is calculated from the wave functions (6.57) and (6.58) for $A = constant$. As can be recognized from these expressions, A drops out and one is left with the *free* wave functions $\psi_{0,n}$. Does this also mean that the out - vacuum state is identical with the free vacuum state? This is *not* the case since in the general expression for the covariance, Eq. (6.25), one has to distinguish between positive and negative energy solutions. For non-vanishing (even constant) A- field this distinction is field-dependent since the energy values are given by

$$E_n = \pm \frac{2\pi}{L}(n - \phi), \tag{6.182}$$

where the upper sign is for the right- handed part and the lower sign for the left- handed part (compare (6.57) and (6.58)). Let us focus in the following on the right-hand part. In the expression (6.42) for the overlap we choose for Ω_1 the covariance which corresponds to the out - vacuum, i. e.,

$$\Omega_1(x, y) = \sum_{n \le \phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(y) - \sum_{n > \phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(y), \tag{6.183}$$

where we have included the zero energy eigenfunction in the first sum. Since t has dropped out in this expression, we have skipped it in the arguments for the wave functions. Since the phase factor in (6.172) is space-independent, the time-evolved in - covariance (which plays the role of Ω_2) is just given by

$$\Omega_2(x, y) = \sum_{n \le \alpha} \psi_{0,n}(x)\psi^\dagger_{0,n}(y) - \sum_{n > \alpha} \psi_{0,n}(x)\psi^\dagger_{0,n}(y). \tag{6.184}$$

It is clear that this satisfies the time-dependent Schrödinger equation (6.38) trivially with the correct boundary condition at $t < t_1$. We then find for the operator product $\Omega_1\Omega_2$ in (6.42)

$$\Omega_1\Omega_2 = \int dz \Bigg(\sum_{n \le \phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(z) \sum_{l \le \alpha} \psi_{0,l}(z)\psi^\dagger_{0,l}(y)$$
$$+ \sum_{n > \phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(z) \sum_{l > \alpha} \psi_{0,l}(z)\psi^\dagger_{0,l}(y) - \sum_{n > \phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(z) \sum_{l \le \alpha} \psi_{0,l}(z)\psi^\dagger_{0,l}(y)$$
$$- \sum_{n \le \phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(z) \sum_{l > \alpha} \psi_{0,l}(z)\psi^\dagger_{0,l}(y) \Bigg).$$

We may assume without loss of generality that $\phi > \alpha$. The first and second term in (6.176) give together

$$\left(\sum_{n \le \alpha} + \sum_{n > \phi} \right) \psi_{0,n}(x)\psi^\dagger_{0,n}(y) = \delta(x - y) - \sum_{\alpha < n \le \phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(y).$$

The third term vanishes for $\phi > \alpha$, and the last term gives

$$- \sum_{\alpha < n \le \phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(y).$$

We thus have

$$\Omega_1\Omega_2 = \delta(x-y) - 2\sum_{\alpha<n\leq\phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(y).$$

The determinant in the overlap (6.42) thus contains the operator

$$\mathcal{A} \equiv \frac{1}{2}(1+\Omega_1\Omega_2) = \delta(x-y) - \sum_{\alpha<n\leq\phi} \psi_{0,n}(x)\psi^\dagger_{0,n}(y).$$

By acting with $\mathcal{A}$ on $\psi_{0,k}$ one recognizes that $\mathcal{A}$ has a zero eigenvalue if $\alpha < n \leq \phi$. In this case, therefore, the overlap in (6.42) *vanishes*! This means that the probability for the vacuum to remain a vacuum is zero – particles are always created. Since both states Ψ_1 and Ψ_2 are, however, Gaussians it follows that these states belong to different Hilbert spaces – in the case of infinitely many degrees of freedom the overlap between Gaussians can vanish [1]. How can one cope with this situation? The key to a proper treatment is provided by the observation that the energy eigenvalues E_n of the first-quantized eigenfunctions exhibit a *spectral flow* – some of them pass through zero between the in- and out - region. This is peculiar to the massless case since the energy values E_n do not change sign for $m \neq 0$, see (6.88). As a consequence of the spectral flow the time - evolved in - state contains, in the out - region, either occupied positive energy states or empty negative energy states (for definiteness we assume that there exist occupied positive energy states). Our original filling prescription says, however, that for the vacuum state all positive energy states are empty. To have all states in the *same* Hilbert space (Fock space), one has thus to define the out - vacuum state by applying as many annihilation operators on the out - Gaussian as there are occupied energy states, i.e.,

$$|0, out\rangle \equiv N \prod_{k=1}^{[\varphi]} a_k \exp(u^\dagger\Omega_1 u). \tag{6.185}$$

Again, $\varphi = (\int_0^L A)/(2\pi)$ is the flux. The time - evolved in - state can thus be written as

$$\Psi_{in} \stackrel{t\to\infty}{\longrightarrow} N\exp(u^\dagger\Omega_1 u) = \prod_{k=1}^{[\varphi]} a_k^\dagger |0, out\rangle. \tag{6.186}$$

This state thus contains $[\varphi]$ particles with respect to the out - vacuum, a result which is of course well known (see, e. g., [20]). The particle creation rate expressed by (6.186) is directly related to the anomaly in the axial current, and there is a general relationship between the spectral flow of the first - quantized Dirac Hamiltonian, the topological charge, and the anomalous particle production. This is very clearly discussed, for example, in [21]. The important difference to the previous subsection is the fact that in the present case a *definite* number of particles has been produced (as given by the flux of the external field), whereas in the previous case there is a non-vanishing probability for the production of any number of particles. The Schrödinger picture thus provides us with an intuitive explanation for the anomaly: The filling prescription, which is crucial for the specification of the ground state, changes in dependence on the external field. Consequently, the notions of vacuum and excited states change under the influence of the external field.

6.5 The Gauss constraint

So far we have restricted ourselves to the case where the external electromagnetic field can be treated semiclassically. This is formally expressed by neglecting terms containing $\delta/\delta\mathbf{A}(x)$ in the full Hamiltonian (6.3). We want to relax this restriction now and conclude our paper with a brief discussion of some subtleties which arise when the Gauss constraint (6.5) is realized on wave functionals $\Psi[A,u,u^\dagger]$ in the full theory. Applying the Gauss operator

$$\mathcal{G}(x) = \nabla\mathbf{E} - e\psi^\dagger\psi \tag{6.187}$$

on states Ψ we find, using the realization (6.8) - (6.8) for the field operators,

$$\begin{aligned} \mathcal{G}(x)\Psi &= \Big(\frac{1}{i}\nabla\frac{\delta}{\delta\mathbf{A}} - \frac{e}{2}[u^\dagger u + \frac{\delta^2}{\delta u\delta u^\dagger} \\ &\quad +u^\dagger\frac{\delta}{\delta u^\dagger} - u\frac{\delta}{\delta u}]\Big)\,\Psi[A,u,u^\dagger] = 0. \end{aligned} \tag{6.188}$$

Classically, the Gauss operator generates local gauge transformations. This also holds in the quantum theory, in the sense that

$$\Big[\int dx\lambda(x)\mathcal{G}(x),\psi(y)\Big] = e\lambda(y)\psi(y),\ \ etc. \tag{6.189}$$

with an appropriate test class function $\lambda(x)$. The surprise comes if one evaluates the expression (6.188) for the Gaussian state (6.11). This yields

$$\begin{aligned} \mathcal{G}(x)\Psi &= -\frac{1}{2}\int dydzu^\dagger_\alpha(y)[\delta(y-x)\delta_{\alpha\beta} + \Omega_{\alpha\beta}(y,x)]\times \\ &\quad [\delta(x-z)\delta_{\beta\gamma} - \Omega_{\beta\gamma}(x,z)]u_\gamma(z)\Psi \neq 0. \end{aligned} \tag{6.190}$$

Thus, although Ψ is explicitly gauge - invariant, it is *not* annihilated by the Gauss operator. This can also be recognized from a different perspective. Under an infinitesimal gauge transformation a state Ψ changes as follows:

$$\Psi[\mathbf{A}, u, u^\dagger] \to \Psi[\mathbf{A}, u, u^\dagger] - \int dx \lambda(x) \Big(\nabla \frac{\delta}{\delta \mathbf{A}} + ieu \frac{\delta}{\delta u} - ieu^\dagger \frac{\delta}{\delta u^\dagger} \Big) \Psi. \tag{6.191}$$

The state therefore remains invariant if

$$\left(\frac{1}{i} \nabla \frac{\delta}{\delta \mathbf{A}} + eu \frac{\delta}{\delta u} - eu^\dagger \frac{\delta}{\delta u^\dagger} \right) \Psi \equiv \tilde{\mathcal{G}}(x) \Psi = 0. \tag{6.192}$$

Obviously, $\tilde{\mathcal{G}}$ differs from $\mathcal{G}$. The formal reason is the fermionic character of the matter fields which allows the realization of the field operators as in (6.8) and (6.8). In fact, in the bosonic case one has $\tilde{\mathcal{G}} \equiv \mathcal{G}$ [7]. Note that the integrated Gauss operator annihilates Ψ, i. e.,

$$\int dx \mathcal{G}(x) \Psi = \int dx \tilde{\mathcal{G}}(x) \Psi = 0. \tag{6.193}$$

The interpretation of (6.190) was given by Floreanini and Jackiw [8]. The Gauss operator $\mathcal{G}$ may produce states which lie outside the original Fock space from which one started, since the space spanned by u and $u^\dagger$ is much bigger than the space obtained from the ground state through application of the field operators ψ and $\psi^\dagger$. They can only produce polynoms in

$$(1+\Omega) u \equiv u_+, \quad u^\dagger (1-\Omega) \equiv u_-^\dagger, \tag{6.194}$$

whereas in (6.190) one recognizes their adjoints u_- and $u_+^\dagger$:

$$\mathcal{G}(x) \Psi = -\frac{1}{2} u_+^\dagger(x) u_-(x) \Psi. \tag{6.195}$$

The prescription we impose here is to *project* the action of the Gauss operator back onto the original Fock space,

$$\mathcal{G} \to P_F \mathcal{G} \equiv \frac{1}{4} u_+ u_-^\dagger \mathcal{G}.$$

Since the state (6.195) is orthogonal to each state in this space, one has of course

$$P_F \mathcal{G}(x) \Psi = 0. \tag{6.196}$$

In particular, one finds that the expectation value of the Gauss operator vanishes, $\langle \Psi | \mathcal{G}(x) \Psi \rangle = 0$.

There is only one possible obstruction to this prescription: it may happen that the presence of an anomaly spoils the commutativity of two Gauss operator (this anomaly should not be confused with the anomaly of the axial current). In this case our prescription would lead to a contradiction since the projected Gauss operators always commute with each other. An example where such anomalies occur are chiral fermions in an external electromagnetic field [8]. In such a case one *cannot* identify a state Ψ with its projected state, $u_+u_-^\dagger\Psi/4$. Here, however, we deal with Dirac fermions where the anomaly connected with the left - handed part cancels the corresponding anomaly of the right - handed part. It is thus perfectly consistent to identify states with their projected version.

In this respect the situation is analogous to the Gupta - Bleuler quantization of electrodynamics where one can get rid of negative norm states by identifying states with zero norm.

We have thus shown that the Gauss operator for fermions can be consistently interpreted in the functional Schrödinger picture if no gauge violating anomalies are present.

Outlook: The use of wave functionals gives an intuitive picture of the physics involved, in particular with regard to conceptual questions. This became especially clear in our discussion of particle creation and anomalies. Second, this picture may possess technical advantages in some applications, such as the calculation of expectation values or anomalous particle production rates. One might therefore expect this picture to be of some use in other branches of quantum field theory where less results are known than in QED, e.g. fermions in a gravitational background as well as coupled to a quantized gravitational field, especially in the framework of the new variables in canonical general relativity [5]. This could shed some light on the final stages of black hole evaporation. Further possible applications include non-Abelian fields in four dimensions [22], decoherence, the semiclassical approximation, bosonization, as well as the extension to problems where non-Gaussian states play a role.

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