**Friedrich-Schiller-University Jena** Summer term 2016
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Will be discussed: **24th week of year**

# Problems in Supersymmetry

## Sheet 9

### Problem 24: BPS Monopoles

In chapter 9.3.1. the supersymmetric Lagrangian for the $\mathcal{N}=2$ gauge theory has been introduced. We study solutions of the supersymmetric SU(2) theory with $\psi=0$ and $B=0$ and gauge-invariant potential $V(A)$. In this case the Lagrangian simplifies to

$$\mathcal{L}=-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}+\frac{1}{2}(D^\mu A)^a(D_\mu A)^a-V(A) \quad\text{with}\quad (D_\mu A)^a=\partial_\mu A^a+gf^a_{\ bc}A^b_\mu A^c$$

with structure constants $f^a_{\ bc}=\epsilon_{abc}$. The antisymmetric field strength tensor is defined as $F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu+gf^a_{\ bc}A^b_\mu A^c_\nu$. The potential depends on two parameters, $V(A)=\frac{\lambda}{4}(A^aA^a-v^2)^2$ with positive $\lambda$. The field equations read

$$D_\mu F^{a\mu\nu}=ig[A,D^\nu A]=gf^a_{\ bc}A^bD^\nu A^c,\quad D^\mu D_\mu A=-\lambda A^a(A^bA^b-v^2)\,,$$

and the symmetric stress-energy tensor takes the form

$$T_{\mu\nu}=-F^a_{\mu\rho}F^{a\rho}_\nu+(D_\mu A)^a(D_\nu A)^a-g_{\mu\nu}\mathcal{L}.$$

1. Express the total static energy of the system in terms of the chromoelectric and chromomagnetic fields $E^a_i=F^a_{0i}$ and $B^a_i=\frac{1}{2}\epsilon_{ijk}F^a_{jk}$. For which field configuration is the energy minimal?

2. Write the Higgs field as $A=(0,0,v+\chi)$ and expand the static energy up to second order in the fluctuating field $\chi$. What is the residual symmetry in the minimum of the energy? To determine this it is useful to consider $D_iA^aD_iA^a$.

3. Rewrite the static energy in the following form

$$E=\frac{1}{2}\int \mathrm{d}^3x\Big((E^a_i-D_iA^a\sin\alpha)^2+(B^a_i-D_iA^a\cos\alpha)^2+ \\ 2\sin\alpha\, E^a_iD_iA^a+2\cos\alpha\, B^a_iD_iA^a+2V\Big)$$

   with an arbitrary real parameter $\alpha$. Now discuss the case of a vanishing electrical field. Therefore set $\alpha=0$ and $E^a_i=0$. What is now the relation between $B^a_i$ and $D_iA$ in the minimum of the energy? Define in analogy to the electric charge the magnetic charge of the solution using the flux of $B_i=B^a_iA^a/v$ through a sphere at infinity. Relate the flux to the static energy. What is the minimum of the energy?

4. Make the following spherical symmetric ansatz for the fields (which mixes the internal group indices with the spatial ones)

$$A^a=\frac{r^a}{er^2}H(\xi),\quad A^a_i=\epsilon_{aji}\frac{r^j}{er^2}\big(1-K(\xi)\big),\quad A^a_0=0,\quad e=-g,\quad \xi=evr\,,$$

   and consider the limit $\lambda\to 0$. Keep the boundary conditions $|A(r\to\infty)|\to v$ in this limit. Look at the previously obtained equation relating $B^a_i$ and $D_iA^a$ in the minimum within ansatz given above. You should obtain two coupled first order differential equations. Solve these for the unknowns $K(\xi)$ and $H(\xi)$.

5. Check that this solution fulfills the field equations. Calculate the energy using the above found solution.

**Problem 25: Nonlinear O($N$)-model in two dimensions**

The supersymmetric off-shell Lagrangian of this model reads

$$\mathcal{L} = \frac{1}{2g^2}\left(\partial_\mu n_i \partial^\mu n_i + \mathrm{i}\bar{\psi}_i \not{\partial} \psi_i - f_i f_i\right), \quad i = 1, 2, \ldots, N,$$

and contains $N$ real fields $n_i$, $N$ Majorana-spinor fields $\psi_i$ and $N$ real auxiliary fields $f_i$. These fields fulfill the constraints

$$1 = n_i n_i, \quad 0 = n_i \psi_i, \quad n_i f_i = \frac{1}{2}\bar{\psi}_i \gamma_* \psi_i .$$

Why is the off-shell theory interacting? Find the corresponding on-shell action. What are the on-shell field equations?
The on-shell susy-transformations of the fields are

$$\delta_\alpha n_i = \bar{\alpha}\gamma_* \psi_i, \quad \delta_\alpha \psi_j = (\frac{1}{2}\bar{\psi}_i \gamma_* \psi_i n_j + i\gamma_* \not{\partial} n_j)\alpha .$$

Find the corresponding off-shell transformations.