Introduction to Supersymmetry

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Hinweise auf Tippfehler und andere Unzulänglichkeiten sind willkommen
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Kapitel 1

Introduction

Supersymmetric theories are highly symmetric and beautiful. They unify fermions (matter) with bosons (carrier of forces), either in flat space (supersymmetry) or in curved space-time (supergravity). Supergravity unifies the gravitational with other interactions. The energy at which gravity and quantum effects become of comparable strength can be estimated from the only expression with the dimension of energy that can be formed from the constants of nature $\hbar, c$ and G,

$$E_{\text{Pl}} = m_{\text{Pl}} c^2 = c^2 \sqrt{\hbar c / G} \sim 10^{19} \text{GeV}.$$

The SCHWARZSCHILD radius of a point particle with Planck mass is just twice its COMPTON wavelength,

$$r_{\text{S}} = \frac{2G}{c^2}\, m_{\text{Pl}} = \frac{2G}{c^2} \sqrt{\hbar c/G} = \frac{2\hbar}{c} \sqrt{G/\hbar c} = \frac{2\hbar}{m_{\text{Pl}}\, c} = 2\lambda_{\text{C}}.$$

Supersymmetry transformations relate bosons to fermions,

$$\mathcal{Q}\,|\text{Boson}\rangle \sim |\text{Fermion}\rangle \quad \text{and} \quad \mathcal{Q}\,|\text{Fermion}\rangle \sim |\text{Boson}\rangle,$$

and hence relate particles with different spins. The particles fall into multiplets and the supersymmetry transforms different members of such a super multiplet into each other. Each supermultiplet must contain at least one boson and one fermion whose spins differ by 1/2. All states in a multiplet (of unbroken supersymmetry) have the same mass.

So far no experimental observation has revealed particles or forces which manifestly show such a symmetry. Yet supersymmetry has excited great enthusiasm in large parts of the community and more recently in the context of superstring theories. It has even be said of the theory that it

IS SO BEAUTIFUL IT MUST BE TRUE.

Notation:

symbols	range	meaning
$i, j, k, \ldots$	$1, 2, \ldots, d-1$	space indices
$\mu, \nu, \rho, \sigma, \ldots$	$0, 1, \ldots, d-1$	space-time indices
$\alpha, \beta, \gamma, \delta \ldots$	$1, \ldots 2^{[d/2]}$	Dirac-spinor indices
$\alpha, \beta, \dot\alpha, \dot\beta \ldots$	$1, \ldots 2^{d/2-1}$	Weyl-spinor-indices (d even)
$A^\dagger, A^*, A^T$	A matrix	adjoint, complex conjugate and transpose of A

The symmetrization and anti-symmetrization of a tensor $A_{\mu_1 \ldots \mu_n}$ are

$$A_{(\mu_1 \ldots \mu_n)} = \frac{1}{n!} \sum_\sigma A_{\sigma(\mu_1) \ldots \sigma(\mu_n)}, \quad A_{[\mu_1 \ldots \mu_n]} = \frac{1}{n!} \sum_\sigma \text{sign}(\sigma) A_{\sigma(\mu_1) \ldots \sigma(\mu_n)}.$$

Reading

The following introductory books and review articles maybe useful:

1. J. Wess and J. Bagger, *Supersymmetry and Supergravity*,
   Princeton University Press, Princeton, 1983

2. Martin F. Sohnius, *Introducing Supersymmetry*,
   Physics Reports **128** (1985) 39

3. S.J. Gates, M.T. Grisaru, M. Rocek and W. Siegel, *Superspace: Or One Thousand and One Lessons in Supersymmetry*,
   Benjamin/Cummings, London, 1983

4. S. Weinberg, *The Quantum Theory of Fields*,
   Vol.3, Cambridge University Press, Cambridge, 1999

5. P. West, *Introduction to Supersymmetry and Supergravity*,
   Word Scientific, Singapore, 1987

6. M. Jacob, *Supersymmetry and Supergravity*,
   North Holland, Amsterdam, 1987

7. O. Piguet and K. Sibold, *Renormalized Supersymmetry*,
   Birkhäuser Boston Inc., 1986

8. M.B. Green, J.H. Schwartz and E. Witten, *Superstring Theory*,
   Cambridge University Press, 1987

9. Peter G.O. Freund, *Introduction to Supersymmetry*,
   Cambridge University Press, Cambridge, 1986

10. H.J.W. Müller-Kirsten and A. Wiedemann, *Supersymmetry: An Introduction with Conceptual and Calculational Details*
    World Scientific, Singapore, 1987

11. Antoine Van Proeyen, *Tools for Supersymmetry*, hep-th/9910030
12. Stephen P. Martin, *A Supersymmetry Primer*, hep-ph/9709356
13. Joseph D. Lykken, *Introduction to Supersymmetry*, hep-th/9612114
14. Manuel Drees, *An Introduction to Supersymmetry*, hep-ph/9611409

Kapitel 2

Supersymmetric Quantum Mechanics

In this chapter we examine simple toy models of supersymmetric field theories. These are quantum mechanical systems possessing supersymmetry [1]. Here there are no technical difficulties hiding the essential structures. Besides, such simple systems are interesting in their own right, since the dynamics of supersymmetric quantum field theories in finite volumes reduce to that of supersymmetric quantum mechanics in the infrared limit [3]. This observation maybe used to better understand non-perturbative features of field theories. A supersymmetric quantum mechanics with 16 supercharges appear in the matrix theory description of M theory [2]. In mathematical physics supersymmetric quantum mechanics has proved to be useful in proving index theorems for physically relevant differential operators [4].
Quantum mechanics can be thought of as quantum field theory in $0+1$ dimensions. For a point particle on the line the position x and momentum p play the role of a real scalar field ϕ and its conjugate momentum field π. The Poincaré algebra reduces to time translations generated by the HAMILTONian H. The ground state of quantum mechanics corresponds to the vacuum state of field theory.

2.1 Supersymmetric harmonic oscillator

Setting $2m=\hbar=1$ the ordinary harmonic oscillator in one dimension has Hamiltonian

$$H_{\mathrm{B}} = p^2 + \omega^2 x^2 - \omega, \tag{2.1}$$

where we have subtracted the zero-point energy $\hbar\omega$. As is well-known one can define *lowering* and *raising* operators,

$$a = \frac{1}{\sqrt{2\omega}}\,(p - i\omega x) \quad , \quad a^\dagger = \frac{1}{\sqrt{2\omega}}\,(p + i\omega x) \tag{2.2}$$

with commutation relations

$$[a,a] = [a^\dagger, a^\dagger] = 0 \quad \text{and} \quad [a, a^\dagger] = 1. \tag{2.3}$$

These allow us to rewrite the Hamiltonian as follows,

$$H_{\rm B} = \omega\left(a^\dagger a + aa^\dagger - 1\right) = 2\omega N_{\rm B}, \quad N_{\rm B} = a^\dagger a\,. \tag{2.4}$$

The *ground state* is annihilated by the lowering operator a and the excited states are gotten by applying the raising operator $a^\dagger$ several times to the ground state,

$$|n\rangle = \frac{1}{\sqrt{n!}}\,(a^\dagger)^n|0\rangle, \quad a|0\rangle = 0. \tag{2.5}$$

The *number operator* counts, how many times $a^\dagger$ has been applied to the ground state,

$$N_{\rm B}|n\rangle = a^\dagger a|n\rangle = n|n\rangle, \tag{2.6}$$

and with (2.4) this determines the discrete energies $E_n = 2n\omega$ of the harmonic oscillator. Now we 'supersymmetrize' this construction and consider the *super*-Hamiltonian

$$H = H_B + 2\omega b^\dagger b = 2\omega a^\dagger a + 2\omega b^\dagger b, \tag{2.7}$$

where the new operators b and $b^\dagger$ are *fermionic* annihilation and creation operators fulfilling anti-commutation relations,

$$\{b,b\} = \{b^\dagger,b^\dagger\} = 0 \quad \text{and} \quad \{b,b^\dagger\} = 1. \tag{2.8}$$

Since $a^\dagger a$ and $b^\dagger b$ are both non-negative, the state which is annihilated by a and b has minimal energy and hence is the ground state of H,

$$a|0\rangle = b|0\rangle = 0 \Longleftrightarrow H|0\rangle = 0. \tag{2.9}$$

The Fock space is generated by acting with the creation operators on this state. The raising operators $(a^\dagger, b^\dagger)$ increase the bosonic number operator $N_{\rm B}$ and fermionic number operator $N_{\rm F} = b^\dagger b$ by one,

$$[N_{\rm B}, a^\dagger] = a^\dagger \quad \text{and} \quad [N_{\rm F}, b^\dagger] = b^\dagger. \tag{2.10}$$

Of course, the lowering operators decrease these numbers by one unit. Because of the Pauli principle the only eigenvalues of $N_{\rm F}$ are 0 and 1. States with fermion number 0 are called *bosonic* and those with fermion number 1 *fermionic*. All excited states of

$$H = 2\omega(N_{\rm B} + N_{\rm F}) \tag{2.11}$$

come in pairs: the 'bosonic' eigenstate $(a^\dagger)^n|0\rangle$ and the 'fermionic' eigenstate $b^\dagger(a^\dagger)^{n-1}|0\rangle$ have the same energy $E_n = 2n\omega$, so that every bosonic state has a fermionic partner with the same energy.

In analogy with supersymmetric theories in higher dimensions we introduce the nilpotent *supercharge* $\mathcal{Q}$ and its adjoint,

$$\mathcal{Q} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Q}^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad A = \frac{d}{{\rm d}x} + \omega x. \tag{2.12}$$

The non-hermitean A is proportional to a and we obtain

$$\{\mathcal{Q},\mathcal{Q}^\dagger\} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \begin{pmatrix} H_\mathrm{B} & 0 \\ 0 & H_\mathrm{F} \end{pmatrix} \equiv H \tag{2.13}$$

where H_B and H_F are the restriction of the super-Hamiltonian H to the bosonic and fermionic subspaces, respectively,

$$H_\mathrm{B} = H|_{N_\mathrm{F}=0} \quad \text{and} \quad H_\mathrm{F} = H|_{N_\mathrm{F}=1}.$$

$\mathcal{Q}$ is nilpotent and commutes with the super-Hamiltonian

$$\{\mathcal{Q},\mathcal{Q}\} = 0 \quad \text{and} \quad [\mathcal{Q},H] = 0. \tag{2.14}$$

It generates the supersymmetry of the quantum mechanical system.

2.2 Pairing and ground states for SQM

The supersymmetric harmonic oscillator is the simplest example of a susy quantum mechanics. In general the raising and lowering operators are replaced by first order differential operators and one introduces the nilpotent supercharges (2.12) with

$$A = \frac{d}{\mathrm{d}x} + W(x) \quad \text{and} \quad A^\dagger = -\frac{d}{\mathrm{d}x} + W(x). \tag{2.15}$$

For a linear function $W(x)$ the operators A and $A^\dagger$ are proportional to the bosonic annihilation and creation operators of the supersymmetric harmonic oscillator. The arbitrary real function $W(x)$ is called *super potential*. The super-Hamiltonian as defined in (2.13) yields

$$\begin{aligned} H_\mathrm{B} &= A^\dagger A = -\frac{d^2}{\mathrm{d}x^2} + W^2(x) - W'(x) \\ H_\mathrm{F} &= AA^\dagger = -\frac{d^2}{\mathrm{d}x^2} + W^2(x) + W'(x). \end{aligned} \tag{2.16}$$

Since Hamiltonians in the bosonic and fermionic subspaces are non-negative, all energies of the super-Hamiltonian are zero or positive. A bosonic zero-energy state is annihilated by A and a fermionic zero-energy state is annihilated by $A^\dagger$,

$$H_\mathrm{B}|0\rangle = 0 \Longleftrightarrow A|0\rangle = 0 \quad , \quad H_\mathrm{F}|0\rangle = 0 \Longleftrightarrow A^\dagger|0\rangle = 0 \tag{2.17}$$

The excited modes come always in pairs, similarly as for the supersymmetric harmonic oscillator. To prove this statement we consider a bosonic eigenfunction with energy E,

$$H_\mathrm{B}|\psi_\mathrm{B}\rangle = A^\dagger A|\psi_\mathrm{B}\rangle = E|\psi_\mathrm{B}\rangle.$$

It follows, that $A|\psi_\mathrm{B}\rangle$ is a fermionic eigenfunction with the same energy,

$$H_\mathrm{F}(A|\psi_\mathrm{B}\rangle) = (AA^\dagger)A|\psi_\mathrm{B}\rangle = A(A^\dagger A)|\psi_\mathrm{B}\rangle = AH_\mathrm{B}|\psi_\mathrm{B}\rangle = E(A|\psi_\mathrm{B}\rangle).$$

The bosonic state $|\psi_{\rm B}\rangle$ and its partner state

$$|\psi_{\rm F}\rangle = \frac{1}{\sqrt{E}}\,A|\psi_{\rm B}\rangle \tag{2.18}$$

have identical norm,

$$\langle\psi_{\rm F}|\psi_{\rm F}\rangle = \frac{1}{E}\langle\psi_{\rm B}|A^\dagger A|\psi_{\rm B}\rangle = \langle\psi_{\rm B}|\psi_{\rm B}\rangle \tag{2.19}$$

and this proves, that the partner state of any excited state is never the null-vector. Likewise, the nontrivial partner state of any fermionic eigenstate $|\psi_{\rm F}\rangle$ with positive Energy E is

$$|\psi_{\rm B}\rangle = \frac{1}{\sqrt{E}}\,A^\dagger|\psi_{\rm F}\rangle. \tag{2.20}$$

This then proves that $H_{\rm B}$ and $H_{\rm F}$ have identical spectra, up to possible zero-modes.
We may calculate the ground state(s) of the super-Hamiltonian in position space explicitly. With (2.17) we must study the first order differential equations

$$\begin{aligned} A\psi_{\rm B}(x) &= \left(\frac{d}{{\rm d}x} + W(x)\right)\psi_{\rm B}(x) = 0 \\ A^\dagger\psi_{\rm F}(x) &= \left(-\frac{d}{{\rm d}x} + W(x)\right)\psi_{\rm F}(x) = 0\,. \end{aligned} \tag{2.21}$$

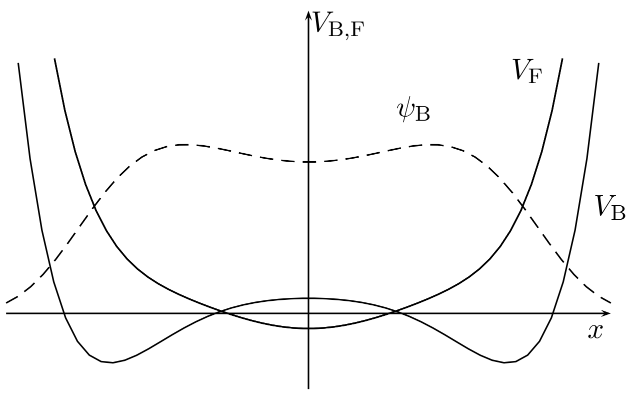
The solutions of these first order differential equations are

$$\psi_{\rm B}(x) \propto \exp\left(-\int^x W(x'){\rm d}x'\right) \quad \text{and} \quad \psi_{\rm F}(x) \propto \exp\left(\int^x W(x'){\rm d}x'\right). \tag{2.22}$$

If one of these two functions is normalizable, then Susy is *unbroken*. Since $\psi_{\rm B}(x)\cdot\psi_{\rm F}(x)$ is constant, there exists at most one *normalizable* state with zero energy.
As example for a superpotential with unbroken supersymmetry we choose $W = \lambda x(x^2 - a^2)$. The partner-potentials

$$\begin{aligned} V_{\rm B} &= \lambda^2 x^2(x^2-a^2)^2 - \lambda(3x^2-a^2) \\ V_{\rm F} &= \lambda^2 x^2(x^2-a^2)^2 + \lambda(3x^2-a^2) \end{aligned} \tag{2.23}$$

are plotted for $\lambda = a = 1$ in the following figure.



The corresponding partner Hamiltonians have the same positive eigenvalues. There is one normalizable bosonic state with zero energy,

$$\psi_{\rm B} = \text{const.} \cdot \exp\left(-\lambda x^2 \left(\tfrac{1}{4}\lambda x^2 - \tfrac{1}{3}a^2\right)\right). \tag{2.24}$$

This wave function is shown in the above figure.
A long time ago, Schrödinger asked the following question [5]: Given a general Hamiltonian in position space

$$\tilde{H}_B = -\frac{d^2}{\mathrm{d}x^2} + V(x). \tag{2.25}$$

Is there always a first order differential operator A, such that $\tilde{H}_B = A^\dagger A$? This is the co-called *factorization-problem.* Since the ground state energy E_0 of $\tilde{H}_B$ is in general not zero, in contrast to the ground state energy of $A^\dagger A$, we need to subtract E_0 from $\tilde{H}_B$ for the factorization to work. Hence we set

$$H_B = \tilde{H}_B - E_0 = A^\dagger A \tag{2.26}$$

Comparing with (2.16) this problem leads to the nonlinear differential equation of Ricatti,

$$V(x) - E_0 = W^2(x) - W'(x). \tag{2.27}$$

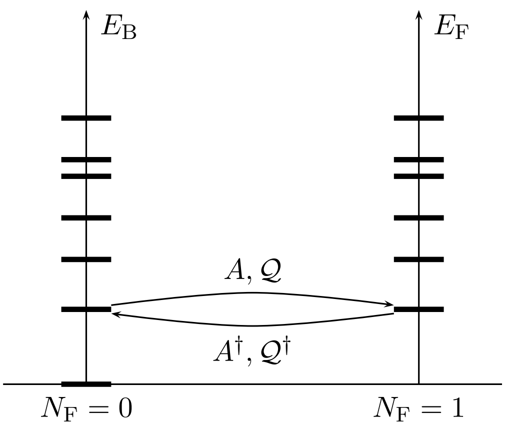
This equation is solved by the following well-known trick: setting

$$W(x) = -\frac{\psi_0'(x)}{\psi_0(x)} = -\frac{d}{\mathrm{d}x}\log\psi_0(x), \tag{2.28}$$

the Ricatti equation transforms into the linear Schrödinger equation for ψ_0,

$$-\psi_0'' + V\psi_0 = E_0\psi_0. \tag{2.29}$$

Since the ground state ψ_0 has no node, the superpotential W is real and regular, as required. Of course, the transformation (2.28) is just the relation (2.22) between the superpotential and the ground state wave function in the bosonic sector.



The pairing of the non-zero energies and eigenfunctions in supersymmetric quantum mechanics is depicted in the figure on the left. The supercharge $\mathcal{Q}$ maps bosonic eigenfunctions into fermion ones and $\mathcal{Q}^\dagger$ maps fermionic eigenfunctions into bosonic ones. For potentials with scattering states there is a corresponding relation between the transmission and reflection coefficients of $H_{\rm B}$ and $H_{\rm F}$. From the scattering data of $H_{\rm B}$ one can calculate those of its partner-Hamiltonian $H_{\rm F}$. The explicit formulas are given below.

2.3 SUSY breaking in SQM

Supersymmetry requires the existence of a fermion for every boson, and vice-versa. If supersymmetry were unbroken, we would expect the super-partners to have equal mass. However, in a system with broken Susy, this might not be the case, even if the partner particles did still exist.

The fact that Susy has not been observed in nature so far does not imply that there are no practical uses for supersymmetric theories. It could be that every occurring supersymmetry is a broken one. We still would have a supercharge and super-Hamiltonian obeying the super algebra. But the symmetry could be spontaneously broken, in which case there is no invariant vacuum state.

In order for supersymmetry to exist and be unbroken, we require a ground state such that $H_{\rm B}|0\rangle = H_{\rm F}|0\rangle = 0|0\rangle$. This requires that the ground state is annihilated by the generators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ of supersymmetry. Thus we have

$$\text{Susy unbroken} \Longleftrightarrow \text{exist normalizable } |0\rangle \text{ with } Q|0\rangle = Q^\dagger|0\rangle = 0.$$

The Witten Index: Witten defined an index to determine whether Susy was broken in supersymmetric field theories. This index is

$$\Delta = \mathrm{Tr}\,(-1)^{N_{\rm F}}, \tag{2.30}$$

where $N_{\rm F}$ is the *fermion number*. For simplicity we assume that the spectrum of H is discrete and use the energy eigenfunctions to calculate Δ. Let us first assume that supersymmetry is broken, which means, that there is *no* normalizable zero-energy state. Then all eigenstates of H have positive energies and come in pairs: one bosonic state with $N_{\rm F} = 0$ and one fermionic state with $N_{\rm F} = 1$ having the same energy. Their contribution to Δ cancel. Since all states with positive energy are paired we obtain $\Delta = 0$.

Now we assume that there are ground states with zero energy, n_B bosonic ones and n_F fermionic ones. Their contribution to the Witten index is $n_B - n_F$. Since the contribution of the excited states cancel pairwise we obtain

$$\Delta = n_B - n_F. \tag{2.31}$$

This yields a efficient method to determine whether Susy is broken,

$$\Delta \neq 0 \Longrightarrow \text{supersymmetry is unbroken.} \tag{2.32}$$

The converse need not be true. It could be that Susy is unbroken but the number of zero-energy states in the bosonic and fermionic sectors are equal so that Δ vanishes. This does not happen in one-dimensional SQM, so that

$$\Delta \neq 0 \Longleftrightarrow \text{supersymmetry is unbroken in SQM.} \tag{2.33}$$

Already in SQM the operator $(-)^{N_{\rm F}}$ is not trace class and its trace must be regulated for the Witten index to be well defined. A natural definition is

$$\Delta = \lim_{\alpha\downarrow 0}\Delta(\alpha), \qquad \Delta(\alpha) = \mathrm{Tr}\,\left((-1)^{N_{\rm F}} \mathrm{e}^{-\alpha H}\right). \tag{2.34}$$

In SQM with discrete spectrum $\Delta(\alpha)$ does not depend on α, since the contribution of all super partners cancel in (2.34). The contribution of the zero-energy states is still $n_B - n_F$. In field theories the exited states should still cancel in $\Delta(\alpha)$ in which case it is α-independent. Since $\Delta(\alpha)$ is constant, is may be evaluated at small α. But for $\alpha \to 0$ one may use the asymptotic small-α expansion of the *heat kernel* of $\exp(-\alpha H)$ to actually calculate the Witten index.

2.4 Scattering states

Let us now see, how supersymmetry relates the *transmission* and *refection coefficients* of H_B and H_F for potentials supporting scattering states [6]. Thus we assume that the superpotential tends to constant values for large $|x|$,

$$\lim_{x\to\pm\infty} W(x) = W_\pm, \quad \text{such that} \quad \lim_{x\to\pm\infty} V_\mathrm{B}(x) = \lim_{x\to\pm\infty} V_\mathrm{F}(x) = W_\pm^2. \tag{2.35}$$

We consider an *incoming* plane wave from the left. The asymptotic wave function for scattering from the one-dimensional potential V_B is given by

$$\psi_\mathrm{B}(k,x) \longrightarrow \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R_\mathrm{B}\mathrm{e}^{-\mathrm{i}kx} & x\to-\infty \\ T_\mathrm{B}\mathrm{e}^{-\mathrm{i}kx} & x\to+\infty, \end{cases} \tag{2.36}$$

where R_B and T_B are the reflection and transmission coefficient in the bosonic sector. The properly normalized fermionic partner state has the asymptotic forms

$$\psi_\mathrm{F}(k,x) = \frac{1}{ik+W_-} A\psi_\mathrm{B}(x) \longrightarrow \begin{cases} \mathrm{e}^{\mathrm{i}kx} + R_\mathrm{F}\mathrm{e}^{-\mathrm{i}kx} & x\to-\infty \\ T_\mathrm{F}\mathrm{e}^{-\mathrm{i}kx} & x\to+\infty \end{cases}$$

with the following reflection and transmission coefficients,

$$R_\mathrm{F} = \frac{W_- - \mathrm{i}k}{W_- + \mathrm{i}k} R_\mathrm{B} \quad \text{and} \quad T_\mathrm{F} = \frac{W_+ + \mathrm{i}k}{W_- + \mathrm{i}k} T_\mathrm{B}. \tag{2.37}$$

The scattering data for the supersymmetric partners are not the same but they are related in this simple way. As example we consider the superpotential

$$W(x) = a\tanh(bx) \quad \text{with} \quad W_\pm = \pm a,$$

which gives rise to the two partner potentials

$$V_\mathrm{B}(a,b;x) = a\left(a - \frac{a+b}{\cosh^2 bx}\right) \quad \text{and} \quad V_\mathrm{F}(a,b;x) = a\left(a - \frac{a-b}{\cosh^2 bx}\right) \tag{2.38}$$

Supersymmetry, together with the socalled *shape-invariance* [7]

$$V_\mathrm{F}(a,b;x) = V_\mathrm{B}(a-b,b;x) + 2ab - b^2 \tag{2.39}$$

allows one to find the scattering data for an infinite tower of PÖSCHL-TELLER potentials. Let us assume, that we know the scattering data $R_{\rm B}(a,b)$ and $T_{\rm B}(a,b)$ for the parameters a and b. It follows that

$$\begin{aligned} R_{\rm B}(a-b,b;k)=R_{\rm F}(a,b;k) &= +\frac{a+{\rm i}k}{a-{\rm i}k}R_{\rm B}(a,b) \\ T_{\rm B}(a-b,b;k)=T_{\rm F}(a,b;k) &= -\frac{a+{\rm i}k}{a-{\rm i}k}T_{\rm B}(a,b). \end{aligned} \tag{2.40}$$

The iteration of this relations yields

$$\begin{aligned} R_{\rm B}(a,b;k) &= \prod_{n=0}^{N-1}\frac{a-nb-{\rm i}k}{a-nb+{\rm i}k}\,R_{\rm B}(a-Nb,b;k) \\ T_{\rm B}(a,b;k) &= (-)^N\prod_{n=0}^{N-1}\frac{a-nb-{\rm i}k}{a-nb+{\rm i}k}\,T_{\rm B}(a-Nb,b;k) \end{aligned} \tag{2.41}$$

Now we set $a=Nb$ in these relations. Then the coefficients $R_{\rm B}$ and $T_{\rm B}$ on the right hand sides are 0 and 1 and we read off the following scattering coefficients (after setting $N-n\equiv m$)

$$T_{\rm B}(Nb,b;k)=(-)^N\prod_{m=1}^{N}\frac{mb-{\rm i}k}{mb+{\rm i}k}\quad\text{and}\quad R_{\rm B}(Nb,b;k)=0. \tag{2.42}$$

for the PÖSCHL-TELLER potentials,

$$V_{\rm B}=b^2\left(N^2-\frac{N(N+1)}{\cosh^2 bx}\right),\qquad W=Nb\,\tanh(bx). \tag{2.43}$$

The poles $k_m=-mb$ of the transmission coefficient yield the energies of the bound states,

$$E_m=k_m^2+b^2N^2=b^2\left(N^2-m^2\right)\qquad m=1,\dots,N. \tag{2.44}$$

Supersymmetry is unbroken, since the ground state has a energy zero.

2.5 Isospectral deformations

Let us assume that $V_{\rm B}$ supports n bound states. By using supersymmetry one can easily construct an n-parameter family of potentials $V(\lambda_1,\dots,\lambda_n;x)$ for which the Hamiltonian has the same energies and scattering coefficients as $H=-\triangle+V_{\rm B}$. The existence of such families of isospectral potentials has been known for a long time from the inverse scattering approach [11], but the GELFAND-LEVITAN approach for constructing them is technically more involved than the supersymmetry approach described here. Here we show how a *one-parameter* isospectral family of potentials is obtained by first deleting and then re-inserting the ground state of $V_{\rm B}$ using the DARBOUX-procedure [12]. The generalization to an n-parameter family is described in [13].

Suppose that $|\psi_{\rm B}\rangle$ is a normalizable zero-energy ground state of the bosonic Hamiltonian with potential $V_{\rm B} = W^2 - W'$. Its explicit form in position space is

$$\psi_{\rm B}(x) = \exp\left(-\int^x W(x')\mathrm{d}x'\right). \tag{2.45}$$

Suppose that the partner potential $V_{\rm F} = W^2 + W'$ is kept fixed. A natural question is whether there are other superpotentials leading to the *same* potential $V_{\rm F}$. So let us assume that there exists a second solution $\hat{W} = W + \phi$ giving rise to the same potential. This requirement leads to

$$0 = (\hat{W}^2 + \hat{W}') - (W^2 + W') = \phi^2 + 2W\phi + \phi'\,. \tag{2.46}$$

The transformation $\phi = (\log F)'$ leads to the linear differential equation for F'

$$F'' + 2WF' = 0 \tag{2.47}$$

with solution

$$F'(x) = \exp\left(-2\int^x W(x')\mathrm{d}x'\right) = \psi_{\rm B}^2(x). \tag{2.48}$$

The integration constant is just the lower bound of the integral in the exponent or equivalently the norm of the bosonic ground state. A further integration yields F and hence $\phi = (\log F)'$ and introduces another integration constant λ which is identified with the deformation parameter,

$$\phi(x) = \frac{d}{\mathrm{d}x}\log\left(I(x) + \lambda\right), \qquad I(x) = \int_{-\infty}^{x} \psi_{\rm B}^2(x')\mathrm{d}x' \tag{2.49}$$

In this formula for ϕ we could change the lower integration bound or multiply I with any non-vanishing constant. This is equivalent to a redefinition of the constant λ.
By construction W and $\hat{W} = W + \phi$ lead to the same $V_{\rm F}$. But the corresponding partner potentials are different,

$$\hat{W}^2 - \hat{W}' = W^2 - W' + \phi^2 + 2W\phi - \phi' \overset{(2.46)}{=} V_{\rm B} - 2\phi' = V_{\rm B} + 2\phi^2 + 4W\phi. \tag{2.50}$$

Thus the bosonic Hamilton operators with superpotentials W and $W+\phi$ are unequal. But since they have the same partner Hamiltonian $H_{\rm F}$ they must have the same spectrum, up to possible zero modes. This then proves that the one-parameter family of Hamilton operators

$$H_{\rm B}(\lambda) = -\frac{d^2}{\mathrm{d}x^2} + V_{\rm B}(\lambda;x), \qquad V_{\rm B}(\lambda;x) = V_{\rm B}(x) - 2\frac{d^2}{\mathrm{d}x^2}\log\left(I(x)+\lambda\right) \tag{2.51}$$

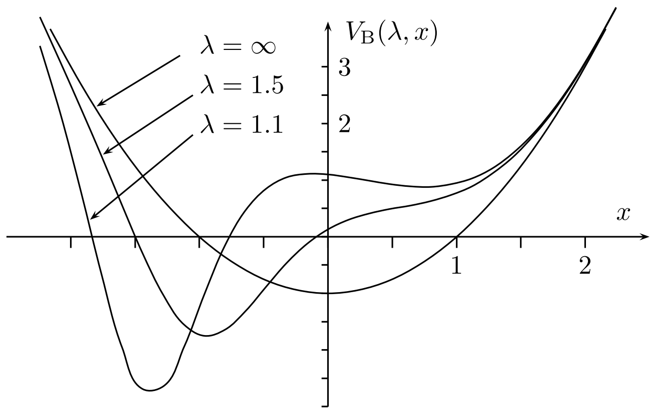
all have the same spectrum, up to possible zero modes. The deformation depends via $I(x)$ in (2.49) on the ground state wave function of the undeformed operator $H_{\rm B}$.

Deformation of the harmonic oscillator: Let us see how the deformation looks like for the simple harmonic oscillator with ground state wave function $\psi_{\rm B}(x) \propto \exp(-\omega x^2/2)$. We obtain

$$\phi(\lambda,x) = 2\sqrt{\frac{\omega}{\pi}}\,\frac{\mathrm{e}^{-\omega x^2}}{\mathrm{erf}(\sqrt{\omega}\,x)+\lambda}, \quad \text{where} \quad \mathrm{erf}(y) = \frac{2}{\sqrt{\pi}}\int_0^y \mathrm{e}^{-t^2}dt \tag{2.52}$$

is the error function, and this leads to the deformation

$$V_{\rm B}(\lambda;x) = \omega^2x^2 - \omega + 4\omega x\,\phi(\lambda,x) + 2\phi^2(\lambda,x) = V_{\rm B}(-\lambda;-x). \tag{2.53}$$



In the figure on the left we have plotted the potential of the harmonic oscillator and the two deformed potentials for $\lambda = 1.5$ and $\lambda = 1.1$. We have set $\omega = 1$. For the deformed potential to be regular we must assume $|\lambda| > 1$. For $\lambda \to \pm\infty$ the potential tends to the potential of the harmonic oscillator. For $|\lambda| \downarrow 1$ the deviation from the oscillator potential become significant near the origin.

Deformation of reflectionless Pöschl-Teller potentials: We deform the Potential

$$V_{\rm B} = b^2\left(1 - 2\cosh^{-2} bx\right), \tag{2.54}$$

that is the PÖSCHL-TELLER potential (2.43) with *one bound state*. The ground state

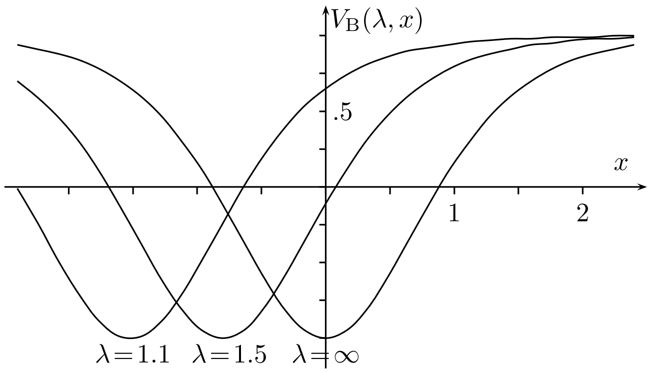
$$\psi_{\rm B}(x) = \frac{b}{\cosh bx} \tag{2.55}$$

has zero energy. Since $\int \psi_{\rm B}^2 = b\tan bx$ we obtain

$$\phi(x) = \frac{b^2}{\cosh^2 bx}\frac{1}{b\tanh bx + \lambda} \tag{2.56}$$

This leads to the deformation

$$V_{\rm B}(\lambda,x) = b^2\left(1 - 2\cosh^{-2} bx\right) + 4b\tanh(bx)\phi(x) + 2\phi^2(x) = V_{\rm B}(-\lambda,-x). \tag{2.57}$$



In the figure on the left we have plotted the refectionless PÖSCHL-TELLER potential with one bound state and two isospectral potentials for $\lambda = 1.5$ and 1.1. We have set $b = 1$. For the deformed potential to be regular we must assume $|\lambda| > 1$. For $\lambda \to \pm\infty$ the potential tends to the PÖSCHL-TELLER potential. For $\lambda \downarrow 1$ the minimum of the potential tends to $-\infty$ and for $\lambda \uparrow -1$ to ∞. The potential V_B maybe viewed as soliton with center at the minimum. For $\lambda = 1$ the soliton is at $x = -\infty$ and moves to the origin for $\lambda \to \infty$. For $\lambda = -1$ the soliton is centered at ∞ and moves with decreasing λ to the left. For $\lambda = -\infty$ it reaches the origin. Actually one show that after a change of variables, $\lambda = \lambda(t)$, the function $V_B(t, x)$ solves the KORTEWEG-DEVRIES equation.

2.6 SQM in higher dimensions

Supersymmetric quantum mechanical systems also exist in higher dimensions. The construction is based on the following rewriting of the supercharge (2.12):

$$\mathcal{Q} = \psi A \quad \text{and} \quad \mathcal{Q}^\dagger = \psi^\dagger A^\dagger \tag{2.58}$$

containing the *fermionic* operators

$$\psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \psi^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad \{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0, \quad \{\psi, \psi^\dagger\} = \mathbb{1} \tag{2.59}$$

and the operators $A, A^\dagger$ in (2.15). The super-Hamiltonian (2.13) takes the form

$$H = p^2 + W^2 + W'[\psi, \psi^\dagger]. \tag{2.60}$$

In [8] this construction has been generalized to higher dimensions. In d dimensions one has d fermionic annihilation operators ψ_i and d creation operators $\psi_i^\dagger$,

$$\{\psi_i, \psi_j\} = \{\psi_i^\dagger, \psi_j^\dagger\} = 0 \quad \text{and} \quad \{\psi_i, \psi_j^\dagger\} = \delta_{ij}, \quad i = 1, \dots, d. \tag{2.61}$$

For the supercharge we make the ansatz $Q = \sum_i \psi_i(\partial_i + W_i)$. It is nilpotent if $\partial_i W_j - \partial_j W_i = 0$. Locally this is equivalent to the existence of a potential χ with $W_i = \partial_i \chi$. Thus we are lead to the following *nilpotent* supercharge and its adjoint,

$$Q = \psi_i\,(\partial_i + \partial_i\chi) \quad \text{and} \quad Q^\dagger = \psi_i^\dagger\,(-\partial_i + \partial_i\chi)\,. \tag{2.62}$$

The super Hamiltonian takes the simple form

$$H = \{Q, Q^\dagger\} = -\triangle + (\nabla\chi, \nabla\chi) + [\psi_i, \psi_j^\dagger]\, \partial_i\partial_j\chi \tag{2.63}$$

and generalizes the operator (2.60) to higher dimensions. It commutes with the *number operator*

$$N_\mathrm{F} = \sum_i \psi_i^\dagger \psi_i \Longrightarrow [N_\mathrm{F}, \psi_i] = -\psi_i, \quad [N_\mathrm{F}, \psi_i^\dagger] = \psi_i^\dagger. \tag{2.64}$$

The most direct way to find a representation for the fermionic operators makes use of the FOCK construction over a 'vacuum'-state $|0\rangle$ which is annihilated by all ψ_i,

$$\psi_i|0\rangle = 0, \quad i = 1, \dots, d. \tag{2.65}$$

Acting with the $\psi_i^\dagger$ on $|0\rangle$ yields states with $N_\mathrm{F} = 1, 2, \dots$. When counting these states we should take into account, that the $\psi_i^\dagger$ anticommute such that

$$|ij\dots\rangle = \psi_i^\dagger \psi_j^\dagger \cdots |0\rangle \tag{2.66}$$

is antisymmetric in $i, j, \dots$. The states, the eigenvalues of N_F together with their degeneracies are listed in the following table:

states:	$\|0\rangle$	$\|i\rangle$	$\|i,j\rangle$	$\cdots$	$\|1,2,\dots,d\rangle$
N_F	0	1	2	$\dots$	d
# of states	$\binom{d}{0} = 1$	$\binom{d}{1} = d$	$\binom{d}{2}$	$\cdots$	$\binom{d}{d} = 1$

Since there are

$$\sum_{p=0}^{d} \binom{d}{p} = 2^d$$

independent states we obtain a 2^d-dimensional representation of the fermionic algebra (2.61). For example, for one-dimensional systems and the orthonormal basis

$$e_1 = |0\rangle \quad \text{and} \quad e_2 = \psi^\dagger|0\rangle$$

the annihilation operator reads

$$\psi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{2.67}$$

In 2 dimensions ψ_1 and ψ_2 are 4-dimensional matrices. For the orthonormal basis

$$\{|0\rangle, |1\rangle, |2\rangle, |12\rangle\}$$

they take the form

$$\psi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.68}$$

With the help of (2.61) and (2.65) one may calculate the matrix elements of ψ_i between any two Fock states (2.66) in arbitrary dimensions.
Taking into account the x-dependency of the states, the Hilbert space of SQM in d dimensions is

$$\mathcal{H} = L_2(\mathbb{R}^d) \otimes \mathbb{C}^{2^d} \tag{2.69}$$

and decomposes into sectors with different fermion numbers,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_d \quad \text{with} \quad N_{\mathrm{F}}|_{\mathcal{H}_p} = p\mathbb{1}. \tag{2.70}$$

An arbitrary element in $\mathcal{H}$ has the expansion

$$|\psi\rangle = f(x)|0\rangle + f_i(x)|i\rangle + \frac{1}{2} f_{ij}|ij\rangle + \frac{1}{3!} f_{ijk}|ijk\rangle + \ldots. \tag{2.71}$$

The supercharge and its adjoint in (2.62) decrease and increase the fermion number by one,

$$[N_{\mathrm{F}}, Q] = -Q \quad \text{and} \quad [N_{\mathrm{F}}, Q^\dagger] = Q^\dagger. \tag{2.72}$$

Since the super-Hamiltonian commutes with N_{F} it has block-diagonal form in the basis adapted to the decomposition (2.70) of the Hilbert space,

$$N_{\mathrm{F}} = \begin{pmatrix} 0_{\mathcal{H}_0} & & & & \\ & 1_{\mathcal{H}_1} & & & \\ & & 2_{\mathcal{H}_2} & & \\ & & & \ddots & \\ & & & & d_{\mathcal{H}_d} \end{pmatrix}, \quad H = \begin{pmatrix} H_0 & & & & \\ & H_1 & & & \\ & & H_2 & & \\ & & & \ddots & \\ & & & & H_d \end{pmatrix} \tag{2.73}$$

SQM in higher dimensions with a nilpotent supercharge defines a *complex* of the following structure:

$$\mathcal{H}_0 \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \mathcal{H}_1 \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \mathcal{H}_2 \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \bullet\ \bullet\ \bullet\ \bullet \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \mathcal{H}_{d-1} \underset{Q}{\overset{Q^\dagger}{\rightleftarrows}} \mathcal{H}_d \quad (Q^2 = 0)$$

It is very similar to the de Rham complex for differential forms. The nilpotent charge $Q^\dagger$ is the analog of the exterior differential d and Q of the co-differential δ. The super Hamiltonian $H = \{Q, Q^\dagger\}$ corresponds to the Laplace-Beltrami operator $-\triangle = d\delta + \delta d$.

2.7 The supersymmetric hydrogen atom

For the simple function $\chi = -\lambda r$ we obtain the following super-Hamiltonian

$$\begin{aligned} H &= -\triangle + \lambda^2 - (d-1)\frac{\lambda}{r} + \frac{2\lambda}{r}N_{\mathrm{F}} - 2\lambda\frac{(x,\psi^\dagger)(x,\psi)}{r^3} \\ &= -\triangle + \lambda^2 + (d-1)\frac{\lambda}{r} - \frac{2\lambda}{r}\tilde{N}_{\mathrm{F}} + 2\lambda\frac{(x,\psi)(x,\psi^\dagger)}{r^3}\,, \end{aligned} \tag{2.74}$$

where we have introduced the operators

$$(x,\psi^\dagger) = \sum_i x_i\psi_i^\dagger, \quad (x,\psi) = \sum_i x_i\psi_i \quad \text{and} \quad \tilde{N}_{\mathrm{F}} = \sum_i \psi_i\psi_i^\dagger. \tag{2.75}$$

Since all states in $\mathcal{H}_0$ are annihilated by the ψ_i and all states in $\mathcal{H}_d$ by the $\psi_i^\dagger$ we find the following Hamilton operators in these subspaces,

$$\begin{aligned} H_0 &= -\triangle + \lambda^2 - (d-1)\frac{\lambda}{r} \\ H_d &= -\triangle + \lambda^2 + (d-1)\frac{\lambda}{r}. \end{aligned} \tag{2.76}$$

Hence, the Schrödinger operators for both the electron-proton and positron-proton systems are part of the super-Hamiltonian H. In [9] a detailed analysis of the supersymmetric hydrogen atom has been given. In this paper it is shown that the conserved RUNGE-LENZ vector of the COULOMB-problem can be supersymmetrized. Together with the angular momentum vector it generates an $SO(4)$ symmetry. This large *dynamical symmetry* group allows for a purely algebraic solution of the supersymmetric hydrogen atom, very similar to the algebraic solution of the ordinary hydrogen atom by W. PAULI [10].

Kapitel 3

Poincaré groups and algebras

A cornerstone of modern physics is that the fundamental laws are the same in all inertial frames. The transition between two inertial frames is given by a Poincaré transformation, depending on the relative spacetime displacement of the two systems, their relative velocity and orientation. In 4 spacetime dimensions it depends on 4+3+3 = 10 real parameters. The set of all transformations form a non-compact Lie group – the Poincaré group containing the Lorentz group as subgroup.

In these lectures we shall consider supersymmetric field theories in various dimensions. There are good reasons for doing that. For example, superstring theories can consistently be formulated in 10 spacetime dimensions only. Our 'real world' emerges after a dimensional reduction. Or one may easily construct theories with *extended supersymmetry* by a dimensional reduction of theories with simple supersymmetry.

In these lectures M denotes the d-dimensional Minkowski spacetime with metric coefficients $\eta_{\mu\nu} = \text{diag}(1, -1, \dots, -1)$. Points in M are events characterized by their coordinates with respect to some inertial system,

$$x = (x^\mu), \quad \mu = 0, \dots, d-1. \tag{3.1}$$

The bilinear scalar product between two tangential vectors is

$$(\xi, \eta) = \sum_{\mu\nu} \eta_{\mu\nu} \xi^\mu \eta^\nu \equiv \xi^\mu \eta_\mu .$$

Poincaré *transformations* are linear transformations

$$x^\mu \longrightarrow \tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad \text{or} \quad \tilde{x} = \Lambda x + a, \qquad d\tilde{x} = \Lambda \mathrm{d}x$$

with $ds^2 = (\mathrm{d}x, \mathrm{d}x) = (d\tilde{x}, d\tilde{x})$. They form the Poincaré group,

$$iL = \left\{ (\Lambda, a) |\, a \in \mathbb{R}^d, \Lambda \in L(\mathbb{R}^d), \Lambda^T \eta \Lambda = \eta \right\}, \tag{3.2}$$

also called inhomogeneous Lorentz group, with group multiplication

$$(\Lambda_2, a_2)(\Lambda_1, a_1) = (\Lambda_2 \Lambda_1, \Lambda_2 a_1 + a_2). \tag{3.3}$$

The unit element is $(\mathbb{1}, 0)$ and the inverse of (Λ, a) reads

$$(\Lambda, a)^{-1} = (\Lambda^{-1}, -\Lambda^{-1}a). \tag{3.4}$$

The Poincaré group has dimension

$$\tfrac{1}{2}d(d-1) + d = \tfrac{1}{2}d(d+1),$$

which is just the number of independent generators. For example, in 4 dimensions the Poincaré group has 10 generators and in 5 dimensions it has 15. 10 is also the number of generators of the anti-deSitter (AdS) group in 4 dimensions and of the conformal group in 3 dimensions. More generally, the AdS group in d dimensions is isomorphic to the *conformal group* in $d-1$ dimensions. This equivalence is at the heart of the so-called AdS-CFT-correspondence.
Note, that iL is the semi-direct product of space-time *translations* and Lorentz *transformations.* The normal subgroup of translations is Abelian and the subgroup of Lorentz transformations

$$L = \Big\{\Lambda \in L(\mathbb{R}^d) |\, \Lambda^T \eta \Lambda = \eta \Big\} \tag{3.5}$$

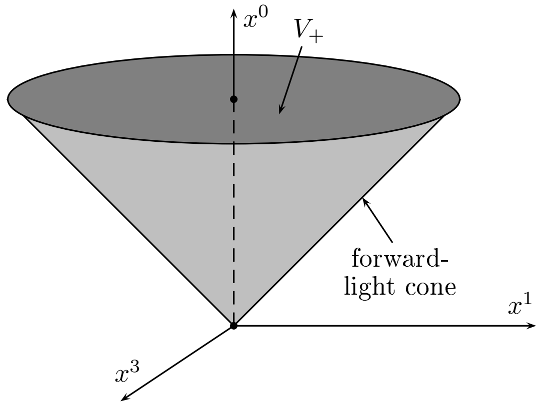
form the (non-compact) simple Lie group $O(1, d-1)$. From (3.5) it follows at once, that

$$\det \Lambda = \pm 1.$$

The transformations Λ with determinant one form the subgroup $SO(1, d-1)$ of *proper* Lorentz *transformations.* Every Λ maps the forward light cone

$$V_+ = \{\xi^0 > 0, (\xi, \xi) > 0\}$$

into itself or into the past light cone V_-. In the second case the time direction is reversed.



The Lorentz groups are not (simply) connected, e.g.

$$\begin{aligned} \pi_1(\mathrm{SO}_e(1,2)) &= \mathbb{Z} \quad , \quad \pi_0(\mathrm{O}(1,3)) = \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \pi_0(\mathrm{SO}_e(1,3)) &= \mathbb{Z}_2 \quad , \quad \pi_1(\mathrm{SO}_e(1,3)) = \mathbb{Z}_2. \end{aligned}$$

3.1 Poincaré Algebras

The Lorentz *group* in d dimensions is generated by $d(d-1)/2$ elements $M_{\mu\nu} = -M_{\nu\mu}$,

$$(\Lambda, 0) = \exp\left(\tfrac{i}{2}\omega^{\mu\nu} M_{\mu\nu}\right) = \exp\left(\tfrac{i}{2}(\omega, M)\right) = \mathbb{1} + \tfrac{i}{2}(\omega, M) + O(\omega^2). \tag{3.6}$$

In the defining representation each $M_{\mu\nu}$ is a d-dimensional imaginary matrix. The condition (3.5) on the Lorentz transformation implies

$$(\omega, M)^T \eta + \eta(\omega, M) = 0 \Longleftrightarrow (\omega, M)_{\rho\sigma} = -(\omega, M)_{\sigma\rho}$$

for all ω. Hence each $M_{\mu\nu}$ must be antisymmetric if both its indexes ρ and ρ are *lower* ones:

$$(M_{\mu\nu})_{\rho\sigma} = -(M_{\mu\nu})_{\sigma\rho}. \tag{3.7}$$

There are $\frac{1}{2}d(d-1)$ independent antisymmetric matrices and as independent ones we choose

$$(M_{\mu\nu})_{\rho\sigma} = -(M_{\nu\mu})_{\rho\sigma} = -i(\eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\nu\rho}\eta_{\mu\sigma}) \Longrightarrow \tfrac{i}{2}(\omega, M)^{\rho}{}_{\sigma} = \omega^{\rho}{}_{\sigma}. \tag{3.8}$$

These generators fulfill the commutation relations

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}]_{\lambda\xi} &= (M_{\mu\nu})_{\lambda\tau}(M_{\rho\sigma})^{\tau}{}_{\xi} - (M_{\rho\sigma})_{\lambda\tau}(M_{\mu\nu})^{\tau}{}_{\xi} \\ &= i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma})_{\lambda\xi}. \end{aligned}$$

It is the relativistic generalisation of the well-known angular momentum algebra and is generated by the infinitesimal rotations M_{ij} and boosts M_{0i}.
The subgroup of spacetime *translations*

$$(\mathbb{1}, a) = \exp(ia^{\mu}P_{\mu}) = \exp(i(a, P)) = \mathbb{1} + i(a, P) + O(a^2) \tag{3.9}$$

form a normal Abelian subgroup of the Poincaré group with commuting generators

$$[P_{\mu}, P_{\nu}] = 0, \qquad \mu, \nu = 0, \dots, d-1.$$

The commutation relations of infinitesimal Lorentz transformations and translations follow from

$$(\Lambda, 0)(\mathbb{1}, a)(\Lambda, 0)^{-1} = (\mathbb{1}, \Lambda a)$$

after inserting the infinitesimal transformations (3.6,3.9). The term linear in ω and in a reads

$$[(\omega, M), (a, P)] = \omega^{\mu\nu}(M_{\mu\nu}a, P).$$

This holds true for any a, ω and with (3.8) we conclude

$$[M_{\mu\nu}, P_{\rho}] = (M_{\mu\nu})^{\sigma}{}_{\rho}P_{\sigma} = i(\eta_{\mu\rho}P_{\nu} - \eta_{\nu\rho}P_{\mu}).$$

To summarize, the important Poincaré *algebra* is given by

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}) \\ [M_{\mu\nu}, P_\rho] &= i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \\ [P_\mu, P_\nu] &= 0 \end{aligned} \tag{3.10}$$

One of the main task will be to extend this Lie algebra to supersymmetric algebras containing commutators and anti-commutators. Now we turn to the particular and most important cases.

3.2 Lorentz-algebras in low dimensions

In the following we study the Lorentz algebras and groups in two, three and four spacetime dimensions in more detail. Particular emphasis is put on the four-dimensional case.

3.2.1 Two dimensions

In this simple case the Lorentz group is Abelian and has one independent generator

$$(M_{01}) = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = \mathrm{i}\sigma_1. \tag{3.11}$$

Thus a proper Lorentz transformation has the form

$$\Lambda = \exp\left(\tfrac{i}{2}\omega^{\mu\nu}M_{\mu\nu}\right) = \exp\left(-\alpha\sigma_1\right) = \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ -\sinh\alpha & \cosh\alpha \end{pmatrix}, \quad \text{where} \quad \alpha = \omega^{01}. \tag{3.12}$$

Since every irreducible representation of an Abelian groups is 1-dimensional, this representation should decompose into two irreducible part. Indeed, introducing the light-cone components

$$\xi^+ = \xi^0 + \xi^1 \quad \text{and} \quad \xi^- = \xi^0 - \xi^1 \tag{3.13}$$

of a vector ξ we obtain the simple transformation law,

$$\xi^+ \longrightarrow \mathrm{e}^{-\alpha}\xi^+ \quad , \quad \xi^- \longrightarrow \mathrm{e}^{\alpha}\xi^-. \tag{3.14}$$

These components transform according to different 1-dimensional irreducible representations of the 2-dimensional proper Lorentz group $SO(1,1)$. An important point which is worth making at this stage is that this finite dimensional representation is *non-unitary*, since the generator M_{01} is not hermitian. It is generally true that any finite-dimensional representation of a non-compact (semi-simple) Lie group is non-unitary.

3.2.2 Three dimensions

In 3 spacetime dimensions the Lorentz group in non-Abelian and has 3 independent generators,

$$(M_{\mu\nu}) = \begin{pmatrix} 0 & K_1 & K_2 \\ -K_1 & 0 & J \\ -K_2 & -J & 0 \end{pmatrix}. \tag{3.15}$$

K_1 and K_2 generate boosts in the two spatial directions and J generates rotations in 2-dimensional space. The Lorentz algebra takes the form

$$[K_1, K_2] = iJ \quad , \quad [J, K_1] = -iK_2 \quad \text{and} \quad [J, K_2] = iK_1. \tag{3.16}$$

The explicit representation (3.8) leads to the following form for these generators,

$$K_1 = \begin{pmatrix} 0 & \mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathrm{i} \\ 0 & -\mathrm{i} & 0 \end{pmatrix}. \tag{3.17}$$

In order to find the Lorentz transformations in (3.6) we need to exponentiate

$$\tfrac{i}{2}(\omega, M) = \omega = \begin{pmatrix} 0 & \omega^0_{\ 1} & \omega^0_{\ 2} \\ \omega^0_{\ 1} & 0 & \omega^1_{\ 2} \\ \omega^0_{\ 2} & -\omega^1_{\ 2} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ \alpha_1 & 0 & \theta \\ \alpha_2 & -\theta & 0 \end{pmatrix}. \tag{3.18}$$

Since ω^3 is proportional to ω we obtain the following form for the Lorentz transformations

$$\Lambda(\boldsymbol{\alpha}, \theta) = \mathrm{e}^{\omega} = \mathbb{1} + \frac{\omega}{\kappa} \sinh \kappa + \frac{\omega^2}{\kappa^2} (\cosh \kappa - 1), \qquad \kappa^2 = \boldsymbol{\alpha}^2 - \theta^2. \tag{3.19}$$

As expected, for vanishing $\boldsymbol{\alpha}$ they reduce to rotations in $SO(2) \subset SO(1,2)$ in space,

$$\Lambda(\boldsymbol{0}, \theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}.$$

On the other hand, for vanishing θ and α_2 we obtain a boost in x^1-direction,

$$\Lambda(\alpha_1, 0, 0) = \begin{pmatrix} \gamma & \beta\gamma & 0 \\ \beta\gamma & \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \gamma = \cosh \alpha_1 = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}. \tag{3.20}$$

In passing we note, that the the complexified Lorentz algebra is just A_2. To see that we set

$$K_1 = \mathrm{i}L_1, \quad K_2 = -\mathrm{i}L_2 \quad \text{and} \quad J = L_2,$$

and observe, that the hermitian operators L_i fulfill the well-known commutation relations of the angular momentum operators,

$$[L_i, L_j] = \mathrm{i}\epsilon_{ijk} L_k. \tag{3.21}$$

It follows that the finite dimensional (and hence non-unitary) representations of the Lorentz group $SO(1,2)$ are classified by an integer number which corresponds to the total angular momentum in quantum mechanics. If this number is half-integer, we would obtain a representation of the spin-group which is the double-cover of the Lorentz group. We refrain from further study these representations and instead turn to 4 spacetime dimensions.

3.2.3 Four dimensions

In this physically most relevant case there are 6 independent Lorentz generators $M_{\mu\nu}$. We denote the *infinitesimal boosts* by K_i and the infinitesimal rotations by J_i such that

$$(M_{\mu\nu}) = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & -J_3 & J_2 \\ -K_2 & J_3 & 0 & -J_1 \\ -K_3 & -J_2 & J_1 & 0 \end{pmatrix}. \tag{3.22}$$

The explicit form of the antihermitean boost generators is

$$K_1 = \begin{pmatrix} 0 & \mathrm{i} & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 & 0 \end{pmatrix} \tag{3.23}$$

and that of the infinitesimal hermitian rotations

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.24}$$

These generators fulfill the commutation relations

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk} J_k, \quad [K_i, K_j] = -\mathrm{i}\epsilon_{ijk} J_k \quad \text{and} \quad [J_i, K_j] = [K_i, J_j] = \mathrm{i}\epsilon_{ijk} K_k. \tag{3.25}$$

In a next step we introduce the following generators of the complexified Lorentz algebra,

$$S_i = \tfrac{1}{2}(J_i + \mathrm{i}K_i) \quad \text{and} \quad A_i = \tfrac{1}{2}(J_i - \mathrm{i}K_i). \tag{3.26}$$

They obey the simple commutation relations of the $A_1 \times A_1$ Lie algebra

$$[S_i, S_j] = \mathrm{i}\epsilon_{ijk} S_k, \quad [A_i, A_j] = \mathrm{i}\epsilon_{ijk} A_k \quad \text{and} \quad [S_i, A_j] = 0. \tag{3.27}$$

We conclude, that

$$\mathrm{so}(4,\mathbb{C}) \sim \mathrm{sl}(2,\mathbb{C}) \times \mathrm{sl}(2,\mathbb{C}),$$

so that every irreducible representation of $so(1,3)$ is uniquely determined by (n,m), where the half-integers n and m characterize the $sl(2,\mathbb{C})$ representations.

Note that the two $sl(2,\mathbb{C})$ subalgebras are not independent. They can be interchanged by the *parity operation*. Parity acts on the rotation - and boost - generators as follows:

$$J_i \to J_i \quad \text{and} \quad K_i \to -K_i, \tag{3.28}$$

and this shows that parity transforms S_i into A_i and A_i into S_i. If the generators J_i and K_i are represented by hermitian operators, then parity operation is equivalent to hermitian conjugation. As examples we consider the *smallest representations:*

- $(0,0)$: this trivial one-dimensional scalar representation has total spin zero.
- $(\frac{1}{2},0)$: this two-dimensional spinor representation is called the left-handed representation. *left-handed spinors* transform trivial under one subgroup $sl(2)$ and according to the spin $\frac{1}{2}$-representation with respect to the other subgroup $sl(2)$.
- $(0,\frac{1}{2})$: this two-dimensional spinor representation is called the right-handed representation and spinors transforming according to this representation are called *right handed.*
- Since parity exchanges S_i and A_i, the *handed spinors* have no fixed parity: parity transforms the representation $(\frac{1}{2},0)$ into $(0,\frac{1}{2})$ and vice-versa. To obtain a representation such that parity acts as a linear transformation, one needs to combine the two spinor representation to

$$(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2}) \tag{3.29}$$

  which yields the Dirac *spinor representation.*

- Any representation can be generated from the left- and right-handed spinor representations. For example, the tensor product of the right- and left-handed representations is

$$(\tfrac{1}{2},0)\otimes(0,\tfrac{1}{2})=(\tfrac{1}{2},\tfrac{1}{2}) \tag{3.30}$$

  and yields the 4-dimensional vector-representation, giving the transformations of a 4-vector. Also,

$$(\tfrac{1}{2},0)\otimes(\tfrac{1}{2},0)=(0,0)\oplus(1,0) \tag{3.31}$$

  and yields the scalar and spin-one representation, the second given by self-dual antisymmetric second rank tensors.

- More generally, for tensors of rank two we find

$$\begin{aligned} T^{\mu\nu} &\in (\tfrac{1}{2},\tfrac{1}{2})\otimes(\tfrac{1}{2},\tfrac{1}{2})=(\tfrac{1}{2},0)\otimes(0,\tfrac{1}{2})\otimes(0,\tfrac{1}{2})\otimes(\tfrac{1}{2},0)\\ &= [(1,0)\oplus(0,0)]\otimes[(0,1)\oplus(0,0)]\\ &= (1,1)\oplus(1,0)\oplus(0,1)\oplus(0,0). \end{aligned}$$

Every second rank tensor contains a scalar part $(0,0)$, a spin-1 part $(1,0) \oplus (0,1)$ and a spin-2 part $(1,1)$. This decomposition corresponds to

$$T^{\mu\nu} = a\eta^{\mu\nu} + A^{\mu\nu} + S^{\mu\nu} \quad \text{with} \quad a = \tfrac{1}{4}T^\rho_{\ \rho}, \quad A^{\mu\nu} = -A^{\nu\mu}, \quad S^{\mu\nu} = S^{\nu\mu}, \quad S^\rho_{\ \rho} = 0.$$

The antisymmetric part can be further decomposed into its selfdual and anti-selfdual pieces.

3.3 Representations of $4d$ Poincaré algebra

There are two cases to be distinguished:

1. There are massive particles. Then the symmetry consists of spacetime translations, Lorentz transformations and internal symmetries commuting with the spacetime symmetries.
2. There are only massless particles. Then there is the extra possibility of having conformal symmetry.

First we find two Casimir *operators* of the Poincaré algebra. It is not difficult to prove

Lemma 1 *$P^2 = P_\mu P^\mu$ is a Casimir operator for the Poincaré algebra.*

Proof: Clearly, $[P^2, P_\mu] = 0$. Using the Leibniz rule and Poincaré algebra (3.10) one has

$$[P^2, M_{\rho\sigma}] = P_\mu [P^\mu, M_{\rho\sigma}] + [P_\mu, M_{\rho\sigma}]P^\mu = 0$$

and this proves the lemma. The last equation just states that for any 4-vector V^μ the object $V^2 = V^\mu V_\mu$ is a Lorentz scalar.
The second Casimir-operator can be constructed from the Pauli-Ljubanski polarization vector, defined by

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}, \qquad \epsilon_{0123} = 1. \tag{3.32}$$

This 4-vector is orthogonal to the 4-momentum,

$$W_\mu P^\mu = 0. \tag{3.33}$$

In the $1+3$ split this vector takes the form

$$W_0 = -P^i J_i, \quad W_i = P^0 J_i - \epsilon_{ijk}P^j K_k. \tag{3.34}$$

For a particle at rest $P = (m, 0)$ and $W = (0, m\boldsymbol{J})$. Now we shall prove the

Lemma 2 *$W^2 = W_\mu W^\mu$ is a Casimir operator for the Poincaré algebra.*

Proof: Since

$$[W_\mu, P_\xi] = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu[M^{\rho\sigma}, P_\xi] = \mathrm{i}\epsilon_{\mu\nu\rho\xi}P^\nu P^\rho = 0 \tag{3.35}$$

holds true, W^2 commutes with the 4-momentum P_σ. Since W_μ is a 4-vector, its square W^2 is a Lorentz scalar and hence commutes with the infinitesimal Lorentz transformations $M_{\mu\nu}$. This then proves the lemma.

We conclude that P^2 and W^2 are CASIMIR operators of the Poincaré algebra. Since the Lorentz algebra has rank 2 these are all CASIMIR operators. Thus we have constructed a maximal set of operators commuting with translations and Lorentz transformations[1]. Hence all physical states (fields, particles) in a quantum field theory may be classified by the eigenvalues of these two CASIMIR operators.

The infinitesimal Lorentz transformations consists of two parts, namely the external orbital and the internal spin part[2]:

$$M_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}, \qquad L_{\mu\nu} = -x_\mu P_\nu + x_\nu P_\mu, \quad [P_\mu, x^\nu] = i\delta_\mu^{\ \nu}. \tag{3.36}$$

Since the cyclic sum of $P^\nu L^{\rho\sigma}$ vanishes, the PAULI-LJUBANSKI vector simplifies to

$$W_\mu = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu\Sigma^{\rho\sigma}. \tag{3.37}$$

3.3.1 Massive particles

A massive particle has a non-vanishing rest mass m,

$$P^2 = m^2 > 0 \tag{3.38}$$

and we may transform to its rest frame, in which

$$P^\mu = (m, \boldsymbol{0}). \tag{3.39}$$

In this frame we have

$$W_0 = 0 \quad \text{and} \quad W_i = -\frac{m}{2}\epsilon_{ijk}\Sigma^{jk} = mS_i,$$

where the operators

$$S_i = -\tfrac{1}{2}\epsilon_{ijk}\Sigma^{jk}, \qquad [S_i, S_j] = \mathrm{i}\epsilon_{ijk}S_k \tag{3.40}$$

generate the spin rotations. Hence W^μ is the relativistic generalization of the spin. Since

$$W^2 = -\boldsymbol{W}^2 = -m^2\boldsymbol{S}^2,$$

we see that the massive representations are labeled by the mass m and the spin s:

$$P^2 = m^2 \quad \text{and} \quad W^2 = -m^2 s(s+1), \quad s = 0, \tfrac{1}{2}, 1, \dots. \tag{3.41}$$

The representations are characterized by the mass and spin of the particle. Massive particles fall into $(2s+1)$-dimensional irreducible multiplets the states of which are distinguished by the third component of the spin and the continuous eigenvalues of $\boldsymbol{P}$. Examples of massive states are the *electron* with spin $\frac{1}{2}$ and the *pion* with spin 0.

[1]and which are in the enveloping algebra of then Poincaré algebra

[2]See section (4.2).

3.3.2 Massless particles

A massless particle has a vanishing rest mass, $P^2 = 0$, and there is no system in which it is at rest. But we can always transform into a inertial frame in which

$$P_\mu = (P_0, 0, 0, P_3) \quad \text{with} \quad P_0^2 - P_3^2 = 0 \tag{3.42}$$

holds true. Since $P_\mu W^\mu = 0$ we conclude that in this system

$$W_\mu = (\lambda P_0, W_1, W_2, \lambda P_3), \quad \text{such that} \quad W^2 = -W_1^2 - W_2^2 \leq 0.$$

For $W^2 = 0$ the 4-momentum and PAULI-LJUBANSKI vector become linearly dependent,

$$W_\mu = \lambda P_\mu.$$

This tensor identity holds in any inertial system. The factor λ is a Lorentz scalar and must commute with all generators of the Poincaré group,

$$[\lambda, P^\mu] = [\lambda, M^{\mu\nu}] = 0. \tag{3.43}$$

Thus λ is a CASIMIR operator in the massless case. Since $W_0 = -P^i J_i$ we conclude, that

$$\lambda \frac{\boldsymbol{P} \cdot \boldsymbol{J}}{P_0} \tag{3.44}$$

which is the *helicity* of the massless particle. It must be half-integer[3]. Examples of particles falling into this category are the *photon* with helicities ± 1 and a massless *neutrino* with helicities $\pm\frac{1}{2}$.
The remaining case $P^2 = 0$ and $W^2 < 0$ describes a particle with zero rest mass and an infinite number of polarization states. These representations do not seem to be realized in nature.

3.4 Appendix: Anti-de Sitter algebra

In d-dimensional Minkowski spacetime the symmetry group has $\frac{1}{2}d(d+1)$ generators. There are only two other spaces with the same maximal number of symmetries, namely the deSitter and anti-deSitter spaces. An Anti-deSitter space is a consistent backgrounds for $\mathcal{N} = 1$ supergravity models. They play a prominent role in recent duality conjectures and hence it maybe useful to introduce these maximally symmetric spaces. The main reference on AdS spaces is the book of HAWKING and ELLIS [14].
The Anti-deSitter space maybe embedded as hyperboloid in a $d+1$-dimensional flat space with two time-like directions. The Cartesian coordinates of the embedding space $\mathbb{R}^{d+1}$ are denoted by ξ^m and the line element reads

$$ds^2 = \tilde{\eta}_{mn} d\xi^m d\xi^n, \qquad \tilde{\eta}_{mn} = (+, -, \dots, -, +). \tag{3.45}$$

[3]See S. Weinberg, The Quantum Theory of Fields, I, p.90

The d-dimensional AdS_d-submanifold is defined by the $SO(2,d-1)$-invariant condition

$$\tilde{\eta}_{mn}\xi^m\xi^n = R^2. \tag{3.46}$$

Note that every point is invariant under rotations around the vector pointing from the origin to this point. Hence the little group of every point is $SO(1,d-1)$ and we conclude

$$AdS_d \sim \frac{SO(2,d-1)}{SO(1,d-1)}. \tag{3.47}$$

Next we choose local coordinates $(x^\mu)=(y^\alpha,z)$ on AdS_d, where α is less or equal to $d-2$,

$$\xi^\alpha = \frac{Ry^\alpha}{z}, \qquad \xi^{d-1} = \frac{R^2+y^\alpha y_\alpha - z^2}{2z} \quad \text{and} \quad \xi^d = \frac{R^2-y^\alpha y_\alpha + z^2}{2z}, \tag{3.48}$$

where the α,β-indexes are lowered with the Minkowski metric $\eta_{\alpha\beta}=\mathrm{diag}(+1,-1,\dots,-1)$ in $d-1$ dimensions. The induced metric reads

$$\mathrm{d}s^2 = \tilde{\eta}_{mn}\mathrm{d}\xi^m\mathrm{d}\xi^n = \frac{R^2}{z^2}(\mathrm{d}y^\alpha\mathrm{d}y_\alpha - dz^2) \equiv g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \tag{3.49}$$

and has signature $(1,d-1)$. Note that it is conformally flat,

$$g_{\mu\nu} = \mathrm{e}^{2\sigma}\eta_{\mu\nu}, \qquad \mathrm{e}^\sigma = \frac{R}{z} \tag{3.50}$$

such that the Christoffel-symbols have the form

$$\Gamma^\rho_{\mu\nu} = \left(\delta^\rho_\mu\sigma_{,\nu} + \delta^\rho_\nu\sigma_{,\mu} - \eta_{\mu\nu}\sigma^{,\rho}\right). \tag{3.51}$$

From these one can read off the Ricci tensor and Ricci scalar,

$$\begin{aligned}\mathcal{R}_{\mu\nu} &= (2-d)\left(\sigma_{;\mu\nu} - \sigma_{,\mu}\sigma_{,\nu} + \eta_{\mu\nu}(\nabla\sigma)^2\right) - \eta_{\mu\nu}\triangle\sigma \\ \mathcal{R} &= -(d-1)\mathrm{e}^{-2\sigma}\left(2\triangle\sigma + (d-2)(\nabla\sigma)^2\right).\end{aligned} \tag{3.52}$$

In particular for the AdS-metric the Ricci tensor is proportional to the metric and the Ricci scalar is constant,

$$\mathcal{R}_{\mu\nu} = \frac{d-1}{R^2}g_{\mu\nu} \quad \text{and} \quad \mathcal{R} = \frac{d(d-1)}{R^2}. \tag{3.53}$$

as one might expect for a hyperboloid. For such Ricci *flat* space-times the Einstein tensor is proportional to the metric as well,

$$G_{\mu\nu} = \mathcal{R}_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\mathcal{R} = -\frac{(d-1)(d-2)}{2R^2}\,g_{\mu\nu},$$

such that AdS is as solution of Einstein' vacuum-equation with negative *cosmological constant*

$$G_{\mu\nu} = \Lambda g_{\mu\nu}, \qquad \Lambda = -\frac{(d-1)(d-2)}{2R^2}. \tag{3.54}$$

By construction AdS_d has isometry group $SO(2, d-1)$ and this group acts linearly in the embedding space with coordinates ξ^m. They transform under infinitesimal transformation as follows,

$$\delta\xi^m = \omega^m_{\ n}\xi^n. \tag{3.55}$$

Since (ω_{mn}) is antisymmetric this can be written as

$$\begin{aligned} \delta\xi^\alpha &= \omega^\alpha_{\ \beta}\xi^\beta + a^\alpha\xi^{d-1} + b^\beta\xi^d \\ \delta\xi^{d-1} &= a_\alpha\xi^\alpha + \omega\,\xi^d \\ \delta\xi^d &= -b_\alpha\xi^\alpha + \omega\,\xi^{d-1} \end{aligned} \tag{3.56}$$

For the local coordinates in (3.48) this reads

$$\begin{aligned} \delta y^\alpha &= \omega^\alpha_{\ \beta}y^\beta + \frac{1}{2R}\delta^\alpha\left(y^\beta y_\beta - z^2\right) - \frac{1}{R}y^\alpha\delta_\beta y^\beta - \omega y^\alpha + R\sigma^\alpha, \\ \delta z &= -\frac{1}{R}\delta_\beta y^\beta\, z - \omega z, \quad \text{with} \quad \delta^\alpha = (a-b)^\alpha, \;\; \sigma^\alpha = \tfrac{1}{2}(a+b)^\alpha \end{aligned} \tag{3.57}$$

This transformations are infinitesimal symmetry transformations generated Killing-vectorfields. To each parameter ω_{mn} there is one vector field. Their explicit form reads

$$\begin{aligned} X(\omega) &= -\left(z\partial_z + y^\alpha\partial_\alpha\right) \\ X(\delta^\alpha) &= \tfrac{1}{R}y_\alpha X(\omega) + \tfrac{1}{2R}\left(y^\beta y_\beta - z^2\right)\partial_\alpha \\ X(\sigma^\alpha) &= R\partial_\alpha \\ X(\omega^\alpha_{\ \beta}) &= \eta^{\beta\gamma}\left(y_\gamma\partial_\alpha - y_\alpha\partial_\gamma\right) \end{aligned} \tag{3.58}$$

They form a nonlinear realization of the $so(2, d-1)$ symmetry algebra on the Anti-deSitter spacetime, as expected. In particular, the last set of vector fields generate an $SO(1, d-2)$ subgroup. We leave the proof as an exercise.

Suppose that you only know the metric (3.49). To find the symmetries of space-time, you should solve the *Killing-equation*

$$(L_X g)_{\mu\nu} = X^\rho g_{\mu\nu,\rho} + g_{\rho\nu}X^\rho_{,\mu} + g_{\mu\rho}X^\rho_{,\nu} = 0. \tag{3.59}$$

One can prove, that the above vector fields are the only solutions of these equations.

In passing we note that Anti-deSitter spaces are not globally hyperbolic, and hence possess no *Cauchy-hypersurface*. The corresponding problems for (quantum)field theories on AdS-spaces have been discussed by Avis et.al [15]. In passing we note, that AdS_d spaces possess a boundary which maybe identified with the $d-1$-dimensional Minkowski spacetime. The isometry group $SO(d, 2)$ maps points on the boundary into points on the boundary. Its restriction to the boundary of AdS_d is isomorphic to the conformal group of $d-1$-dimensional Minkowski spacetime.

Kapitel 4

Spinors

In this chapter we study Clifford algebras, spinors and spin transformation in d dimensions. Most of the time we shall assume the metric to have Lorentzian signature,

$$(\eta_{\mu\nu}) = \mathrm{diag}(1, -1, \dots, -1). \tag{4.1}$$

Sometimes we consider spaces with Euclidean metrics, $\eta_{\mu\nu} = \delta_{\mu\nu}$.

4.1 Gamma matrices in d dimensions

The Clifford algebra is the free algebra generated by the d elements $\gamma_0, \dots, \gamma_{d-1}$, modulo the quadratic relation

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \equiv \{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\,. \tag{4.2}$$

We give explicit representations of these algebras for spaces with Lorentzian signature in terms of the Pauli-matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.3}$$

The relevant properties of these *hermitian* matrices are

$$\sigma_i\sigma_j = \delta_{ij}\sigma_0 + \mathrm{i}\epsilon_{ijk}\sigma_k, \quad \text{where} \quad \sigma_0 = \mathbb{1}_2.$$

We claim, that the following matrices furnish a representation,

$$\begin{array}{lll} \gamma_0 = \ \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \dots & \gamma_2 = \mathrm{i}\,\sigma_3 \otimes \sigma_1 \otimes \sigma_0 \otimes \dots & \gamma_4 = \mathrm{i}\,\sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes \dots \\ \gamma_1 = \mathrm{i}\,\sigma_2 \otimes \sigma_0 \otimes \sigma_0 \otimes \dots & \gamma_3 = \mathrm{i}\,\sigma_3 \otimes \sigma_2 \otimes \sigma_0 \otimes \dots & \quad \dots\,. \end{array} \tag{4.4}$$

To prove this assertion one uses

$$\{A_1 \otimes \dots \otimes A_p, B_1 \otimes \dots \otimes B_p\} = A_1B_1 \otimes \dots \otimes A_pB_p + B_1A_1 \otimes \dots \otimes B_pA_p$$

and that if $[A_i,B_i]=0$ for all i but $i=q$, then

$$\{A_1\otimes\ldots\otimes A_p, B_1\otimes\ldots\otimes B_p\}=A_1B_1\otimes\ldots\otimes\{A_q,B_q\}\otimes\ldots\otimes A_pB_p. \tag{4.5}$$

For even dimensions this yields a $2^{d/2}$-dimensional representation of the Clifford algebra. For odd dimensions we may choose the representation in $d-1$ dimensions, supplemented by the matrix $\gamma_{d-1}=\mathrm{i}\,\sigma_3\otimes\cdots\otimes\sigma_3$ ($d/2$–1 factors). This way we obtain *explicit* representations in all spacetime dimensions d and

$$\gamma^\mu\in \mathrm{GL}(2^{[d/2]},\mathbb{C}), \tag{4.6}$$

where $[a]$ is the biggest integer less or equal to a. We have the following hermiticity property of the γ-matrices,

$$\gamma_0^\dagger=\gamma_0 \quad\text{and}\quad \gamma_i^\dagger=-\gamma_i. \tag{4.7}$$

The above representation is not unique. Equivalent representations preserving (4.2) are

$$\gamma'_\mu=U\gamma_\mu U^{-1}\,. \tag{4.8}$$

To preserve (4.7) the transformation U should be *unitary.* It is clear, that $\pm\gamma^\mu$ can also be used as representations for the Clifford algebra. For even dimensions these are unitary equivalent to the original one. In odd dimensions one finds *two unitary inequivalent representations.* For example,

$$\gamma_0=\sigma_1,\quad \gamma_1=\mathrm{i}\sigma_2\quad\text{and}\quad \gamma_2=\pm\mathrm{i}\sigma_3$$

are two inequivalent representation of the Clifford algebra in 3 dimensions.
For applications the following observations are relevant:

- In *even dimensions* a complete set of $2^{d/2}\times 2^{d/2}$ matrices is provided by the antisymmetrized products of n gamma-matrices, where $n=0,1,\ldots,d$:

$$\gamma_{\mu_1\ldots\mu_n}\equiv\gamma_{[\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_n]}\quad \binom{d}{n}\quad\text{matrices for}\quad n=0,1,\ldots,d. \tag{4.9}$$

  The exist one matrix with $n=d$ and it is proportional to

$$\gamma_*=\mathrm{i}^{1+d/2}\gamma_0\ldots\gamma_{d-1}\quad\text{with}\quad\gamma_*\gamma_*=1. \tag{4.10}$$

  In the representation (4.4) it is just the $d/2$-fold tensor product $\sigma_3\otimes\sigma_3\otimes\cdots\otimes\sigma_3$. In addition, we have

$$\gamma_{\mu_1\ldots\mu_n}=\mathrm{i}^{1+d/2}\frac{1}{(d-n)!}\,\varepsilon_{\mu_1\ldots\mu_d}\,\gamma_*\gamma^{\mu_d\cdots\mu_{n+1}}\,. \tag{4.11}$$

  Note that in even dimensions γ_* anti-commutes with all γ_μ and thus can be viewed as γ_d in $d+1$ dimensions.

- In *odd dimensions* the product of all γ-matrices is a multiple of the identity. As basis for the $2^{(d-1)/2}$-dimensional matrices we can choose the antisymmetrized products $\gamma_{\mu_1\ldots\mu_n}$ with $n=0,1,\ldots,(d-1)/2$.

4.2 Spin transformations

In this section we study the transformation of spinors under Spin-transformations. For that aim we introduce the $d(d-1)/2$ matrices

$$\Sigma^{\mu\nu} = \frac{1}{2\mathrm{i}}\gamma^{\mu\nu}. \tag{4.12}$$

They possess the following commutators with the γ-matrices and themselves:

$$\begin{aligned} [\Sigma^{\mu\nu}, \gamma^\rho] &= \mathrm{i}\left(\eta^{\mu\rho}\gamma^\nu - \eta^{\nu\rho}\gamma^\mu\right) \\ [\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] &= \mathrm{i}\left(\eta_{\mu\rho}\Sigma_{\nu\sigma} + \eta_{\nu\sigma}\Sigma_{\mu\rho} - \eta_{\mu\sigma}\Sigma_{\nu\rho} - \eta_{\nu\rho}\Sigma_{\mu\sigma}\right). \end{aligned} \tag{4.13}$$

Hence, the Σ furnish a $2^{[d/2]}$-dimensional *representation* of the Lorentz algebra. It is the socalled *spin-representation* to be analyzed below.
Let $S(s)$ be the following one-parameter family of transformations

$$S(s) = e^{i\frac{s}{2}(\omega,\Sigma)} \quad \text{with} \quad (\omega, \Sigma) = \omega_{\mu\nu}\Sigma^{\mu\nu}$$

and consider the related one-parameter family of matrices

$$\Gamma^\rho(s) = S^{-1}(s)\gamma^\rho S(s)$$

with 'initial value' $S(0) = \mathbb{1}$. Making use of (4.13) one obtains the differential equation

$$\frac{d}{ds}\Gamma^\rho(s) = -\frac{\mathrm{i}}{2}\omega_{\mu\nu}S^{-1}(s)\left[\Sigma^{\mu\nu}, \gamma^\rho\right]S(s) = \omega^\rho_{\ \sigma}\Gamma^\sigma(s)$$

the solution of which reads

$$\Gamma^\rho(s) = S^{-1}(s)\gamma^\rho S(s) = \left(e^{s\omega}\right)^\rho_{\ \sigma}\gamma^\sigma.$$

Recall that for antisymmetric $(\omega_{\mu\nu})$ the matrix $(e^\omega)^\rho_{\ \sigma}$ is a Lorentz transformation. Thus, setting $s = 1$ we conclude

$$S^{-1}\gamma^\rho S = \Lambda^\rho_{\ \sigma}\gamma^\sigma, \quad \text{where} \quad S = e^{\frac{\mathrm{i}}{2}(\omega,\Sigma)}, \quad \Lambda = e^{\frac{\mathrm{i}}{2}(\omega,M)} = e^\omega. \tag{4.14}$$

To find the action of infinitesimal Lorentz transformation on spinors, we recall that a Dirac spinor transforms as

$$\psi(x) \longrightarrow S\psi(\Lambda^{-1}x) \equiv e^{\frac{\mathrm{i}}{2}(\omega,\Sigma)}\psi\left(e^{-\omega}x\right). \tag{4.15}$$

This should be compared with the transformation of scalar and vector fields,

$$\phi(x) \longrightarrow \phi(\Lambda^{-1}x) \quad \text{and} \quad A^\mu(x) \longrightarrow \Lambda^\mu_{\ \nu}A^\nu(\Lambda^{-1}x). \tag{4.16}$$

The components of a scalar field (it could be a doublet as in the electroweak theory) do not transform at all, the components of a vector field transform with the Lorentz

transformation Λ and the components of a spinor field with the spin-transformation S. The mapping $S \to \Lambda(S)$ defined in (4.14) is a group homomorphism (a representation) from the simply connected spin group into the non-simply connected Lorentz group.
The infinitesimal form of the transformation (4.15) reads

$$\delta\psi(x) = \tfrac{\mathrm{i}}{2}\omega_{\mu\nu}\left(L^{\mu\nu} + \Sigma^{\mu\nu}\right)\psi(x), \tag{4.17}$$

where the $\Sigma_{\mu\nu}$ generate spin rotations and the

$$L_{\mu\nu} = \frac{1}{\mathrm{i}}(x_\mu\partial_\nu - x_\nu\partial_\mu)$$

orbital transformations. $L_{\mu\nu}$ and $\Sigma_{\mu\nu}$ both satisfy the commutation relations of the Lorentz algebra and so does their sum

$$J_{\mu\nu} = L_{\mu\nu} + \Sigma_{\mu\nu}. \tag{4.18}$$

The $J_{\mu\nu}$ generate the spin-transformations of Dirac spinors,

$$S\psi(\Lambda^{-1}x) = \left(e^{\frac{\mathrm{i}}{2}(\omega,J)}\psi\right)(x) \tag{4.19}$$

and hence generalize the total angular momentum in non-relativistic quantum mechanics.
To construct *tensor fields* which are bilinear in the spinor fields we use that $\gamma^0 = (\gamma^0)^{-1}$ conjugates the γ and Σ-matrices into their adjoints,

$$\gamma^0\gamma_\mu\gamma^0 = \gamma_\mu^\dagger \quad , \quad \gamma^0\Sigma_{\mu\nu}\gamma^0 = \Sigma_{\mu\nu}^\dagger. \tag{4.20}$$

It follows at once that γ^0 conjugates the adjoint of S into the inverse of S,

$$\gamma^0 S^\dagger\gamma^0 = \gamma^0 e^{-\frac{\mathrm{i}}{2}(\omega,\Sigma)^\dagger}\gamma^0 = e^{-\frac{\mathrm{i}}{2}(\omega,\Sigma)} = S^{-1} \tag{4.21}$$

Now we are ready to define the Dirac *conjugate spinor*,

$$\bar\psi \equiv \psi^\dagger\gamma^0 \Longrightarrow \bar\psi^T = \gamma^{0T}\psi^* = \gamma^{0*}\psi^* \tag{4.22}$$

which enter the expression for the fermionic bilinears. Under spin-transformations it transforms with the inverse spin rotation,

$$\bar\psi \longrightarrow (S\psi)^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0 \stackrel{(4.21)}{=} \psi^\dagger\gamma^0 S^{-1} = \bar\psi S^{-1}. \tag{4.23}$$

With the help of (4.14) it is now easy to prove that the bilinear objects

$$A^{\mu_1\cdots\mu_n} = \bar\psi\gamma^{\mu_1\cdots\mu_n}\psi \tag{4.24}$$

are *antisymmetric tensor fields.* The transformation of these objects follow from that of ψ and $\bar\psi$,

$$\begin{aligned} A^{\mu_1\cdots\mu_n} \longrightarrow \bar\psi\, S^{-1}\gamma^{\mu_1\cdots\mu_n} S\,\psi &= \Lambda^{\mu_1}_{\ \nu_1}\cdots\Lambda^{\mu_n}_{\ \nu_n}\,\bar\psi\gamma^{\nu_1\cdots\nu_n}\psi \\ &= \Lambda^{\mu_1}_{\ \nu_1}\cdots\Lambda^{\mu_n}_{\ \nu_n}\,A^{\nu_1\cdots\nu_n}. \end{aligned} \tag{4.25}$$

In particular in 4 dimensions there are 5 tensor fields

$$\bar\psi\psi \quad , \quad \bar\psi\gamma_*\psi \quad , \quad \bar\psi\gamma^\mu\psi, \quad , \quad \bar\psi\gamma_*\gamma^\mu\psi \quad , \quad \bar\psi\gamma^{\mu\nu}\psi,$$

that is a scalar, pseudo-scalar, vector, pseudo-vector and antisymmetric 2-tensor field.

4.3 Charge conjugation

The best way to see how charge conjugation emerges is to consider the Dirac equation

$$\mathrm{i}\gamma^\mu D_\mu(e)\psi - m\psi = 0, \qquad D_\mu(e) = \partial_\mu - ieA_\mu \tag{4.26}$$

in d dimensions. We multiply this equation from the left with γ^0

$$\mathrm{i}\gamma^0\gamma^\mu\gamma^0 D_\mu(e)\gamma^0\psi - m\gamma^0\psi = \mathrm{i}\gamma^{\mu\dagger}D_\mu(e)\gamma^0\psi - m\gamma^0\psi = 0.$$

With (4.22) the complex conjugate of this equation takes the form

$$-\mathrm{i}\gamma^{\mu T}D_\mu(-e)\bar\psi^T - m^*\bar\psi^T = 0. \tag{4.27}$$

Let us now assume that there exists a *charge conjugation matrix* $\mathcal{C}$ which fulfills

$$\mathcal{C}\gamma_\mu^T\mathcal{C}^{-1} = \eta\,\gamma^\mu \Longrightarrow \eta = \pm 1, \tag{4.28}$$

in which case we define the *charge conjugated spinor*

$$\psi_{\mathrm{c}} = \mathcal{C}\bar\psi^T = \mathcal{C}\gamma_0^T\psi^*. \tag{4.29}$$

Now we multiply (4.27) with $\mathcal{C}$ from the left and obtain

$$\mathrm{i}\gamma^\mu D_\mu(-e)\psi_{\mathrm{c}} + \eta m^*\psi_{\mathrm{c}} = 0. \tag{4.30}$$

We see that for a real mass the charge conjugated spinor ψ_{c} fulfills the Dirac equation with reversed electric charge and $m \to -\eta m$. This justifies the name charge conjugation[1]. Since γ_0 is hermitian and the γ_i are antihermitian the condition (4.28) is equivalent to

$$\gamma_0^* = \eta\,\mathcal{C}^{-1}\gamma_0\mathcal{C} \quad \text{and} \quad \gamma_i^* = -\eta\,\mathcal{C}^{-1}\gamma_i\mathcal{C}. \tag{4.31}$$

Thus, in case there is a representation with only *real* or only *imaginary* $\{\gamma^\mu\}$ then we may choose $\mathcal{C} = \gamma_0^T$,

$$\begin{aligned} \text{real } \gamma^\mu \quad &\Longrightarrow \quad \gamma_0^T = \gamma_0 = \mathcal{C}, \qquad \eta = 1, \qquad \psi_{\mathrm{c}} = \psi^* \\ \text{imaginary } \gamma^\mu \quad &\Longrightarrow \quad \gamma_0^T = -\gamma_0 = \mathcal{C}, \quad \eta = -1, \quad \psi_{\mathrm{c}} = \psi^*. \end{aligned} \tag{4.32}$$

Such representations are called *Majorana representations*. For a Majorana representation charge conjugation becomes complex conjugation. A *spinor* which is invariant under charge conjugations is called *Majorana spinor* – such a spinor is real in a Majorana representation. Now we are going to prove

Lemma 1 *In any dimension there exists a* symmetric *or* antisymmetric *matrix* $\mathcal{C}$ *with*

$$\gamma_\mu^T = \eta\,\mathcal{C}^{-1}\gamma_\mu\mathcal{C}, \qquad \eta = \pm 1. \tag{4.33}$$

A formal proof (in arbitrary dimensions and for any signature) is found in [20, 21], but we may as well give explicit solutions in the representation (4.4). If one changes the representation as in (4.8), then, to preserve (4.33), the charge conjugation matrix must transform as

$$\mathcal{C}' = U\mathcal{C}U^T. \tag{4.34}$$

Hence $\mathcal{C}$ in *unitary and (anti)symmetric in any representation.*

[1] The sign of m is irrelevant and thus we may allow for $\eta = 1$.

Even dimension: To proceed we use

$$\sigma_1\sigma_1\sigma_1 = \sigma_1^T \quad , \quad \sigma_1\sigma_2\sigma_1 = \sigma_2^T \quad , \quad \sigma_1\sigma_3\sigma_1 = -\sigma_3^T \quad \text{and} \quad \sigma_2\sigma_i\sigma_2 = -\sigma_i^T \tag{4.35}$$

to show that in the representation (4.4) the symmetric or antisymmetric matrices

$$\begin{aligned} \mathcal{C}_+ &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \dots \quad (\eta = +1) \\ \mathcal{C}_- &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \dots \propto \mathcal{C}_+\gamma_* \qquad (\eta = -1) \end{aligned} \tag{4.36}$$

are solutions of (4.33) with $\eta = \pm 1$ in even dimensions. This then proves that there exist charge conjugation matrices in even dimensions. These particular solutions are hermitian, unitary and hence square to $\mathbb{1}$.

Odd dimension: Let $\gamma_0, \dots, \gamma_{d-1}$ be the γ-matrices for even dimension d. As γ-matrices in $d+1$ dimensions we may take these γ-matrices supplemented by $\gamma_d = \alpha\gamma_0 \cdots \gamma_{d-1}$. The phase α is chosen such that γ_d squares to $-\mathbb{1}$. This last matrix must fulfill

$$\mathcal{C}^{-1}\gamma_d\,\mathcal{C} = \eta\gamma_d^T .$$

The left hand side of this relation can be rewritten as

$$\alpha\mathcal{C}^{-1}\gamma_0 \cdots \gamma_{d-1}\mathcal{C} = \alpha\eta^d\gamma_0^T \cdots \gamma_{d-1}^T \overset{\eta^d=1}{=} \alpha\,(\gamma_{d-1}\cdots\gamma_0)^T = (-)^{d/2}\gamma_d^T , \tag{4.37}$$

which implies $\eta = (-)^{d/2}$. Hence $\mathcal{C}_+$ is a charge conjugation matrix in $1+4n$ dimensions and $\mathcal{C}_-$ in $3+4n$ dimensions. We conclude that there are two charge conjugation matrices in even dimensions and one charge conjugation matrix in odd dimensions. They are symmetric or antisymmetric. For example, there is an antisymmetric $\mathcal{C}_-$ solution but no $\mathcal{C}_+$-solution in 3 dimensions. Since the results are identical in d and $d+8n$ dimensions, it is sufficient to give the results for $d = 1, \dots, 8$. The various possibilities are summarized in table 4.3.

d	1	2	3	4	5	6	7	8
$\mathcal{C}_+$	S	S		A	A	A		S
$\mathcal{C}_-$		A	A	A		S	S	S

Tabelle 4.1: *In even dimensions there exist two charge conjugation matrices $\mathcal{C}_\pm$ with $\eta = \pm 1$, and in odd dimensions there is only one. These matrices are either symmetric (S) oder anti-symmetric (A).*

4.3.1 Explicit Majorana representations

Sometimes it is useful to have an explicit Majorana representation for the γ-matrices at hand. As will be demonstrated below, such representations only exist in dimensions d for which

$$d \bmod 8 \in \{0, 1, 2, 3, 4\}. \tag{4.38}$$

We give Majorana representations in $2, 3$ and 4 spacetime dimensions.

- In 2 dimensions we may choose imaginary γ-matrices

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1 \qquad \text{with} \quad \gamma_* = -\gamma_0\gamma_1 = \sigma_3, \quad \mathcal{C}_- = -\gamma^0, \tag{4.39}$$

  or real γ-matrices

$$\gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2 \qquad \text{with} \quad \gamma_* = -\gamma_0\gamma_1 = \sigma_1, \quad \mathcal{C}_+ = \gamma^0 \tag{4.40}$$

- In 3 dimensions there exists the imaginary representation

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \gamma^2 = i\sigma_3 \quad \text{with} \quad \mathcal{C}_- = -\gamma^0 \tag{4.41}$$

- In 4 dimensions there is again an imaginary representation

$$\begin{gathered}\gamma^0 = \sigma_0 \otimes \sigma_2, \quad \gamma^1 = i\sigma_0 \otimes \sigma_3, \quad \gamma^2 = i\sigma_1 \otimes \sigma_1, \quad \gamma^3 = i\sigma_3 \otimes \sigma_1 \\ \text{with} \quad \gamma_* = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \sigma_2 \otimes \sigma_1 \quad \text{and} \quad \mathcal{C}_- = -\gamma^0.\end{gathered} \tag{4.42}$$

4.4 Irreducible spinors

We do not know whether the spinor representations (4.15) are *irreducible*. In general they are not and there are two types of projections onto invariant subspaces that one can envisage:

Chiral spinors: They can be defined in even dimensions where one has an anticommuting γ_*. This matrix can be used to define *left and right spinors*,

$$\psi_L = \tfrac{1}{2}(1 - \gamma_*)\,\psi \equiv P_L\psi\,, \qquad \psi_R = \tfrac{1}{2}(1 + \gamma_*)\,\psi \equiv P_R\psi\,. \tag{4.43}$$

Since γ_* commutes with the Lorentz generators $\Sigma_{\mu\nu}$ it commutes with the spin rotations S generated by the $\Sigma_{\mu\nu}$ such that a chiral spinor is chiral in any inertial frame,

$$P_L\psi = 0 \Longrightarrow P_L S\psi = SP_L\psi = 0. \tag{4.44}$$

Majorana spinors: The second projection consists of the "reality condition"

$$\psi = \psi_{\rm c} = \mathcal{C}\bar{\psi}^T, \tag{4.45}$$

which defines a Majorana spinor. Let us assume that this condition can be solved in a fixed inertial frame. Then it holds in any other inertial frame provided

$$S\psi = (S\psi)_{\rm c} = \mathcal{C}\bar{\psi}^T S^{-1T}$$

Using (4.45) this condition for the charge conjugation matrix reads

$$\mathcal{C}^{-1}S\mathcal{C} = S^{-1T} \Longleftrightarrow \mathcal{C}^{-1}\Sigma_{\mu\nu}\mathcal{C} = -\Sigma^T_{\mu\nu}. \tag{4.46}$$

In the last step we used that S is generated by the $\Sigma_{\mu\nu}$. The last conditions are automatically fulfilled, since they follow from (4.28), irrespective of the value of η. Thus, if a spinor is Majorana in one inertial frame then it is Majorana in any other inertial frame.
It remain to check whether we can solve the Majorana condition in a fixed inertial frame. Let us take the complex conjugate of the Majorana condition $\psi = \mathcal{C}\gamma_0^T\psi^*$ and use the Majorana condition to rewrite the resulting expression,

$$\psi^* = \mathcal{C}^*\gamma_0^\dagger\psi = \mathcal{C}^*\gamma_0\psi = \mathcal{C}^*\gamma_0\mathcal{C}\gamma_0^T\psi^* = \eta\,\mathcal{C}^*\mathcal{C}\psi^*$$

This should hold for any Majorana spinor and thus we conclude that

$$\mathcal{C}^*\mathcal{C} = \eta \Longrightarrow \mathcal{C}^* = \eta\,\mathcal{C}^{-1}. \tag{4.47}$$

Since $\mathcal{C}$ is unitary this condition is equivalent to $\mathcal{C}^T = \eta\,\mathcal{C}$. This means that only symmetric solutions $\mathcal{C}_+$ and antisymmetric solutions $\mathcal{C}_-$ are admissible. Comparing with table 4.3 shows, that Majorana fermions only exist in dimensions where there exist Majorana representations, see (4.38). The results are listed in table 4.4.

d	1	2	3	4	5	6	7	8
$\mathcal{C}_+$	S	S						S
$\mathcal{C}_-$		A	A	A				
		MW				SM		

Tabelle 4.2: *Majorana spinors exist in dimensions where there exist Majorana representations. In $d = 2 + 8n$ dimensions there exist Majorana fermions which are chiral.*

In cases where no Majorana spinors exist, there is still another possibility for theories with extended supersymmetry. One can define *symplectic* Majorana *spinors*

$$\psi_i = \mathcal{C}\gamma_0^T\,\Omega_{ij}\psi_j^*\,, \tag{4.48}$$

where Ω is some antisymmetric matrix, with $\Omega\Omega^* = -1$. Symplectic Majorana spinors exist in $6, 14, \dots$ dimensions. The existence of symplectic Majorana spinors is marked with SM in table 4.4.
Having two projections, to chiral spinors and to Majorana spinors, one may ask whether one can define a reality condition respecting the chiral projection. For the left handed spinors this means

$$P_L\mathcal{C}\gamma_0^T = \mathcal{C}\gamma_0^T P_L^* \quad \text{or} \quad \gamma_*\mathcal{C}\gamma_0^T = \mathcal{C}\gamma_0^T\gamma_*^*. \tag{4.49}$$

On the other hand, the definition (4.10) of γ_* together with (4.45) imply

$$\mathcal{C}\gamma_0^T\gamma_*^* = -\mathrm{i}^d\gamma_*\mathcal{C}\gamma_0^T$$

and comparing with (4.49) leads to

$$d \bmod 8 \in \{2, 6\}$$

Since there are no Majorana fermions in 6 dimensions the second solutions does not exist. Hence there are only Majorana-Weyl-spinors in $2, 10, 18, \dots$ dimensions.

4.5 Chiral representations

When dealing with chiral fermions, which exist in all even dimensions, it is convenient to use a chiral representation for the γ-matrices. These are representations for which γ_* is diagonal, $\gamma_* = -\sigma_3 \otimes \mathbb{1}$. Then the chiral projectors onto the left- and right handed spinors (often called *chiral* and *anti-chiral fermions*) are diagonal as well.

$$P_L = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad P_R = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix}. \tag{4.50}$$

Left-handed spinors have only upper and right-handed spinors only lower components in these chiral representations. The left-handed object λ_α with lower components is called *chiral spinor* while the right-handed object $\bar{\lambda}^{\dot{\alpha}}$ with upper components is called *anti-chiral spinor*. Sometimes one calls them left-handed and right-handed Weyl-spinors.
The following representations in even dimensions are chiral:

$$\gamma_0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma_j = \begin{pmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{pmatrix}, \quad j = 1, \dots, d-1, \tag{4.51}$$

with hermitean α_j generating an Euclidean Clifford algebra in one dimension less,

$$\{\alpha_i, \alpha_j\} = 2\delta_{ij}. \tag{4.52}$$

Without loss of generality we may assume that the hermitian matrix $\mathrm{i}^{1-d/2}\alpha_1 \cdots \alpha_{d-1}$ is the identity matrix. Then we find

$$\gamma_* = \mathrm{i}^{1+d/2}\gamma_0 \cdots \gamma_{d-1} = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = -\sigma_3 \otimes \mathbb{1}. \tag{4.53}$$

4.5.1 4 dimensions

In this physically most relevant case we choose the chiral representation

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma_5 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix} \tag{4.54}$$

with two-dimensional matrices

$$\sigma_\mu = (\sigma_0, -\sigma_i), \quad \tilde{\sigma}_\mu = (\sigma_0, \sigma_i). \tag{4.55}$$

The infinitesimal spinor-rotations are block-diagonal in this representation,

$$\gamma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde{\sigma}_{\mu\nu} \end{pmatrix}, \qquad \sigma_{\mu\nu} = \tfrac{1}{2}(\sigma_\mu\tilde{\sigma}_\nu - \sigma_\nu\tilde{\sigma}_\mu), \quad \tilde{\sigma}_{\mu\nu} = \tfrac{1}{2}(\tilde{\sigma}_\mu\sigma_\nu - \tilde{\sigma}_\nu\sigma_\mu). \tag{4.56}$$

Note that the σ_{oi} are hermitian, whereas the σ_{ij} are antihermitian, $\tilde{\sigma}_{\mu\nu} = -\sigma^\dagger_{\mu\nu}$. It follows that the spin rotations have the form

$$S = \exp\left(\tfrac{1}{4}\omega^{\mu\nu}\gamma_{\mu\nu}\right) = \begin{pmatrix} A & 0 \\ 0 & A^{\dagger -1} \end{pmatrix} \quad \text{with} \quad A = \exp\left(\tfrac{1}{4}\omega^{\mu\nu}\sigma_{\mu\nu}\right). \tag{4.57}$$

The complex 2-dimensional matrix A has determinant one, $A \in SL(2,\mathbb{C})$, since the $\sigma_{\mu\nu}$ are traceless. A Dirac spinor consists of a left-handed and a right-handed part,

$$\psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot\alpha} \end{pmatrix} \tag{4.58}$$

The left-handed spinors transform with A and the right-handed with $A^{\dagger -1}$

$$\lambda_\alpha \longrightarrow A_\alpha^{\ \beta}\lambda_\beta \quad , \quad \bar{\chi}^{\dot\alpha} \longrightarrow (A^{\dagger -1})^{\dot\alpha}_{\ \dot\beta}\bar{\chi}^{\dot\beta} \tag{4.59}$$

Next we define the ε-tensors

$$(\varepsilon_{\alpha\beta}) = (\varepsilon_{\dot\alpha\dot\beta}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (\varepsilon^{\alpha\beta}) = (\varepsilon^{\dot\alpha\dot\beta}) \quad = \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{4.60}$$

which obey the relations,

$$\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma \quad , \quad \varepsilon_{\alpha\beta}\varepsilon^{\delta\gamma} = \delta_\alpha^\gamma\delta_\beta^\delta - \delta_\alpha^\delta\delta_\beta^\gamma. \tag{4.61}$$

Because $A^T\varepsilon A = \varepsilon$ for any matrix A with determinant one, the following bilinears are Lorentz-invariant,

$$\lambda\chi = \lambda^\alpha\chi_\alpha \quad \text{and} \quad \bar\lambda\bar\chi = \bar\lambda_{\dot\alpha}\bar\chi^{\dot\alpha}, \tag{4.62}$$

where the raising and lowering of indexes are done with ε,

$$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta, \quad \lambda_\alpha = \varepsilon_{\alpha\beta}\lambda^\beta \quad \text{and} \quad \bar\lambda^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\lambda_{\dot\beta}, \quad \bar\lambda_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\lambda^{\dot\beta}. \tag{4.63}$$

The condition $A^T\varepsilon A = \varepsilon$ translates into

$$(A^{T-1})^\alpha_{\ \beta} = \varepsilon^{\alpha\rho}A_\rho^{\ \sigma}\varepsilon_{\sigma\beta} \quad \text{and} \quad (A^{\dagger -1})^{\dot\alpha}_{\ \dot\beta} = \varepsilon^{\dot\alpha\dot\rho}\bar{A}_{\dot\rho}^{\ \dot\sigma}\varepsilon_{\dot\sigma\dot\beta} \tag{4.64}$$

which means, that A can be conjugated into A^{T-1} and $\bar A$ into $A^{\dagger -1}$. Using these formulas one can show that the spin-transformation in (4.59) are equivalent to

$$\lambda^\alpha \longrightarrow (A^{T-1})^\alpha_{\ \beta}\lambda^\beta \quad , \quad \bar\chi_{\dot\alpha} \longrightarrow \bar{A}_{\dot\alpha}^{\ \dot\beta}\bar\chi_{\dot\beta} \tag{4.65}$$

The components λ^α transform as the complex conjugate of $\bar\lambda^{\dot\alpha}$ and the components $\bar\chi_{\dot\alpha}$ as the complex conjugate of the χ_α. The index structure of the Dirac-conjugate spinor is

$$\bar\psi = (\chi^\alpha, \bar\lambda_{\dot\alpha}) \quad \text{such that} \quad \bar\psi\psi = \chi\lambda + \bar\lambda\bar\chi. \tag{4.66}$$

A mass term vanishes for left- or for right-handed spinors. Since σ_μ maps right- into left-handed spinors and $\tilde\sigma_\mu$ left- into right-handed ones they have the index structure

$$(\sigma_\mu)_{\alpha\dot\beta} \quad \text{and} \quad (\tilde\sigma_\mu)^{\dot\alpha\beta}. \tag{4.67}$$

The generator $\sigma_{\mu\nu}$ maps left- into left-handed spinors and $\tilde{\sigma}_{\mu\nu}$ right- into right-handed ones and they have the index structure

$$(\sigma_{\mu\nu})_\alpha{}^\beta \quad \text{and} \quad (\tilde{\sigma}_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}. \tag{4.68}$$

Since spinor components are Grassmann-variables we find the following symmetry properties,

$$\begin{aligned} \lambda\chi &= \lambda^\alpha\chi_\alpha = -\chi_\alpha\lambda^\alpha = \chi^\alpha\lambda_\alpha = \chi\lambda \\ \bar\lambda\bar\chi &= \bar\lambda_{\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\chi^{\dot\alpha}\bar\lambda_{\dot\alpha} = \bar\chi_{\dot\alpha}\bar\lambda^{\dot\alpha} = \bar\chi\bar\lambda \end{aligned} \tag{4.69}$$

The vector current can be written as

$$\bar\psi\gamma^\mu\psi = \bar\lambda_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\beta}\lambda_\beta + \chi^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\chi^{\dot\beta} = \bar\lambda\,\tilde\sigma^\mu\lambda + \chi\sigma^\mu\bar\chi, \tag{4.70}$$

so that the 'kinetic term' for fermions reads

$$\bar\psi\partial\!\!\!/\psi = \bar\lambda\,\tilde\sigma^\mu\partial_\mu\lambda + \chi\sigma^\mu\partial_\mu\bar\chi. \tag{4.71}$$

4.5.2 SL(2,ℂ)-representations

Every finite-dimensional representation of $SL(2,\mathbb{C})$ can be gotten by tensoring the two fundamental 2-dimensional representations. The knowledge of the representation theory of $SL(2,\mathbb{C})$ is necessary to construct Lorentz invariant field equations.
The components λ_α and $\bar\lambda_{\dot\alpha}$ (with lower indexes) are called covariant. The two inequivalent fundamental representations of $SL(2,\mathbb{C})$ are

$$\lambda_\alpha \to A_\alpha{}^\beta\lambda_\beta \quad \text{and} \quad \bar\lambda_{\dot A} \to \bar A_{\dot\alpha}{}^{\dot\beta}\bar\lambda_{\dot\beta}. \tag{4.72}$$

The transformation of the different components are summarized in the following table:

spinor	λ_α	$\lambda^\alpha = \varepsilon^{\alpha\beta}\lambda_\beta$	$\bar\lambda_{\dot\alpha}$	$\bar\lambda^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\lambda_{\dot\beta}$
transforms with	A	$\varepsilon A\varepsilon^{-1} = (A^T)^{-1}$	$\bar A$	$\varepsilon\bar A\varepsilon^{-1} = (A^\dagger)^{-1}$

The representations A^{T-1} and $A^{\dagger-1}$ are equivalent to the representations A and $\bar A$. One uses the following notations

$$\begin{aligned} &\text{representation} \quad D^{(1/2,0)}: \quad && \lambda \to A\lambda \\ &\text{representation} \quad D^{(0,1/2)}: \quad && \bar\lambda \to \bar A\bar\lambda. \end{aligned} \tag{4.73}$$

The tensor product of n un-dotted and m dotted spinors yield a spinor of rank $n+m$ with n un-dotted and m dotted indexes. Such a spinor transforms as follows under $SL(2,\mathbb{C})$ transformations:

$$\lambda_{\alpha_1\ldots\alpha_n\dot\alpha_1\ldots\dot\alpha_m} \longrightarrow A_{\alpha_1}^{\beta_1}\cdots A_{\alpha_n}^{\beta_n}\bar A_{\dot\alpha_1}^{\dot\beta_1}\cdots\bar A_{\dot\alpha_m}^{\dot\beta_m}\,\lambda_{\beta_1\ldots\beta_n\dot\beta_1\ldots\dot\beta_m}.$$

These representations are *reducible* if $(n,m) \neq (\frac{1}{2},0)$ or $\neq (0,\frac{1}{2})$.

4.6 Composite fields

In this section we collect and partly prove some useful properties and formulas for fermionic bilinears and other composites. These technical results will later be of use in our study of supersymmetric field theories.

4.6.1 Bilinears for charge-conjugated fields

The Dirac-conjugate of the charge conjugated spinor $\psi_c = \mathcal{C}\bar\psi^T$ has the form

$$\psi_c = \mathcal{C}\bar\psi^T, \quad \bar\psi_c = \psi_c^\dagger\gamma_0 = \psi^T\gamma_0^T\mathcal{C}^{-1}\gamma_0 = \eta\psi^T\mathcal{C}^{-1} \tag{4.74}$$

and is used to compute the bilinears for the charge conjugated fields,

$$\begin{aligned}\bar\psi_c\gamma^{(n)}\chi_c &= \eta\,\psi^T\mathcal{C}^{-1}\gamma^{(n)}\mathcal{C}\bar\chi^T = \eta^{1+n}(-1)^{n(n-1)/2}\psi^T\gamma^{(n)T}\bar\chi^T \\ &= -\eta^{1+n}(-1)^{n(n-1)/2}\bar\chi\gamma^{(n)}\psi.\end{aligned}$$

In the second last step we used, that the components of fermion fields *anticommute.* For *Majorana spinors* we find

$$\bar\psi\gamma^{(n)}\chi = -\eta^{1+n}(-1)^{n(n-1)/2}\,\bar\chi\gamma^{(n)}\psi. \tag{4.75}$$

In 4 dimensions $\eta = -1$ and we get for *Majorana spinors*

$$\begin{aligned}&\bar\psi\chi = \bar\chi\psi, \quad \bar\psi\gamma_5\chi = \bar\chi\gamma_5\psi, \quad \bar\psi\gamma_5\gamma_\mu\chi = \bar\chi\gamma_5\gamma_\mu\psi \\ &\bar\psi\gamma_\mu\chi = -\bar\chi\gamma_\mu\psi \quad \text{and} \quad \bar\psi\gamma_{\mu\nu}\chi = -\bar\chi\gamma_{\mu\nu}\psi.\end{aligned} \tag{4.76}$$

4.6.2 Hermitean conjugation

We *define* the hermitian conjugate as if the spinor-components are operators in a Hilbert-space (which they are in QFT). For example $(\bar\psi\chi)^\dagger \equiv \chi^\dagger\bar\psi^\dagger$. It follows, that

$$(\bar\psi M\chi)^\dagger = \chi^\dagger M^\dagger\bar\psi^\dagger \overset{T}{=} -\bar\psi^* M^*\chi^* = -\eta\,\bar\psi_c M_c\chi_c, \quad M_c = \mathcal{C}\gamma_0^T M^*\gamma_0^T\mathcal{C}^{-1}. \tag{4.77}$$

This formula shows that we can hermitian conjugate by using the $\mathcal{C}$ operation. With

$$\mathcal{C}\gamma_0^T\gamma^*_{\mu_1\dots\mu_n}\gamma_0^T\mathcal{C}^{-1} = \eta^n\gamma_{\mu_1\dots\mu_n}$$

we conclude that

$$(\bar\psi\gamma_{\mu_1\dots\mu_n}\chi)^\dagger = -\eta^{1+n}\,\bar\psi_c\gamma_{\mu_1\dots\mu_n}\chi_c. \tag{4.78}$$

Note that for Majorana spinors the bilinear tensor fields are hermitian or antihermitian. In four dimensions $\eta = -1$ and with $\gamma_5 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ we have

$$\begin{aligned}&\bar\psi\chi,\ \bar\psi\gamma^{\mu\nu}\chi,\ \bar\psi\gamma_5\gamma^\mu\chi \quad \text{hermitean} \\ &\bar\psi\gamma^\mu\chi,\ \bar\psi\gamma_5\chi \quad \text{antihermitean.}\end{aligned} \tag{4.79}$$

4.7 Fierz identities

In d dimensions the γ^μ are Δ-dimensional matrices, where $\Delta = 2^{[d/2]}$. As *orthonormal basis* in the linear space of $\mathrm{Mat}(\Delta,\mathbb{C})$ with scalar product $(M,N) = \mathrm{Tr}\, M^\dagger N$ we may choose

$$\frac{1}{\sqrt{\Delta}}\{\mathbb{1},\gamma_\mu,\gamma_{\mu_1\mu_2},\dots,\gamma_{\mu_1\dots\mu_D}\},\quad \text{with}\qquad \begin{cases} D=d & \text{for even } d\\ D=\frac{1}{2}(d-1) & \text{for odd } d\,,\end{cases} \tag{4.80}$$

and we assumed $\mu_1<\mu_2<\dots$. With $\gamma_\mu^\dagger=\gamma^\mu$ we have

$$\gamma^\dagger_{\mu_1\mu_2\dots\mu_n} = (\gamma_{\mu_1}\gamma_{\mu_2}\cdots\gamma_{\mu_n})^\dagger = \gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1} = (-)^{n(n-1)/2}\gamma^{\mu_1\mu_2\dots\mu_n}. \tag{4.81}$$

Therefore every $\Delta\times\Delta$-matrix M can be expanded as

$$M = \frac{1}{\Delta}\sum_{n=0}^{D}\frac{1}{n!}\,(-)^{n(n-1)/2}\gamma_{\mu_1\dots\mu_n}\,\mathrm{Tr}\,(\gamma^{\mu_1\dots\mu_n}M)\ . \tag{4.82}$$

Since the sums extend over all indexes $\mu_1,\dots,\mu_n$ and not only the ordered ones we corrected this over counting by the factor $n!$ in the denominator. Now we take two spinors ψ and χ whose components anticommute and choose

$$M=\psi\bar\chi \quad\text{or}\quad M_\alpha^{\ \beta}=\psi_\alpha\bar\chi^\beta \quad\text{such that}\quad \mathrm{Tr}\,(\gamma^{\mu_1\dots\mu_n}M) = -\bar\chi\gamma^{\mu_1\dots\mu_n}\psi.$$

The minus sign originates from the anticommuting nature of the spinor components. The identity (4.82) implies the following general *Fierz-identity*

$$\psi\bar\chi = -\frac{1}{\Delta}\sum_n\frac{1}{n!}\,(-)^{n(n-1)/2}\,\gamma_{\mu_1\dots\mu_n}\,(\bar\chi\gamma^{\mu_1\dots\mu_n}\psi). \tag{4.83}$$

These general identity allows for the expansion of the matrix $\psi\bar\chi$ as linear combination of the basis elements $\gamma^{(n)}$. The expansion coefficients are the antisymmetric tensor fields $\bar\chi\gamma^{(n)}\psi$. For example, in 4 dimensions

$$4\psi\bar\chi = -(\bar\chi\psi) - \gamma_\mu(\bar\chi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\chi\gamma^{\mu\nu}\psi) + \tfrac{1}{3!}\gamma_{\mu\nu\rho}(\bar\chi\gamma^{\mu\nu\rho}\psi) - \tfrac{1}{4!}\gamma_{\mu\nu\rho\sigma}(\bar\chi\gamma^{\mu\nu\rho\sigma}\psi). \tag{4.84}$$

Now we may use (4.11)

$$\gamma_5 = -\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3,\quad \gamma_{\mu\nu\rho} = i\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma \quad\text{and}\quad \gamma_{\mu\nu\rho\sigma} = i\epsilon_{\mu\nu\rho\sigma}\gamma_5$$

to rewrite the Fierz-identity as follows

$$4\psi\bar\chi = -(\bar\chi\psi) - \gamma_\mu(\bar\chi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\chi\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\bar\chi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\chi\gamma_5\psi). \tag{4.85}$$

In particular, we find

$$\begin{aligned} 4\psi\bar\psi &= -(\bar\psi\psi) - \gamma_\mu(\bar\psi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\bar\psi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\psi\gamma_5\psi)\\ 4\gamma_5\psi\bar\psi\gamma_5 &= -(\bar\psi\psi) + \gamma_\mu(\bar\psi\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) - \gamma_5\gamma_\mu(\bar\psi\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\psi\gamma_5\psi)\\ \psi\bar\psi + \gamma_5\psi\bar\psi\gamma_5 &= -\tfrac{1}{2}(\bar\psi\psi) + \tfrac{1}{4}\gamma_{\mu\nu}(\bar\psi\gamma^{\mu\nu}\psi) - \tfrac{1}{2}\gamma_5(\bar\psi\gamma_5\psi). \end{aligned} \tag{4.86}$$

For Majorana *spinors* the second last term vanishes because of (4.76). Multiplying with ψ from the right, we find

$$\psi(\bar\psi\psi)+\gamma_5\psi(\bar\psi\gamma_5\psi)=0,\qquad(\psi\ \text{Majorana}).\tag{4.87}$$

More useful Fierz identities for Dirac- and Majorana spinors can be found in [23].

Fierz identities for Weyl-spinors The following table contains a rather useful list of Fierz-identities for chiral fermions.

$$\begin{array}{rcl}
(\theta\lambda)(\theta\chi) &=& -\frac12(\theta\theta)(\lambda\chi)\\
(\bar\theta\bar\lambda)(\bar\theta\bar\chi) &=& -\frac12(\bar\lambda\bar\chi)(\bar\theta\bar\theta)\\
\lambda\sigma_\mu\bar\chi &=& -\bar\chi\tilde\sigma_\mu\lambda\\
(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &=& \frac12\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta)\\
(\sigma^\mu\bar\theta)_\alpha(\theta\sigma^\nu\bar\theta) &=& \frac12\eta^{\mu\nu}\theta_\alpha(\bar\theta\bar\theta)-i(\sigma^{\mu\nu}\theta)_\alpha(\bar\theta\bar\theta)\\
(\theta\lambda)(\bar\theta\bar\chi) &=& \frac12(\theta\sigma^\mu\bar\theta)(\lambda\sigma_\mu\bar\chi)\\
(\bar\theta\bar\chi)(\theta\lambda) &=& \frac12(\theta\sigma^\mu\bar\theta)(\lambda\sigma_\mu\bar\chi)=(\theta\lambda)(\bar\theta\bar\chi)\\
(\theta\lambda)(\bar\chi\tilde\sigma^\mu\theta) &=& -\frac12\theta^2(\bar\chi\tilde\sigma^\mu\lambda)\\
(\bar\theta\bar\lambda)(\chi\sigma^\mu\bar\theta) &=& -\frac12\bar\theta^2(\chi\sigma^\mu\bar\lambda)\\
(\theta\lambda)(\chi\sigma^\mu\bar\theta) &=& -\frac12(\theta\sigma^\nu\bar\theta)(\lambda\sigma_\nu\tilde\sigma^\mu\chi)\\
(\bar\theta\bar\lambda)(\bar\chi\tilde\sigma^\mu\theta) &=& \frac12(\theta\sigma^\nu\bar\theta)(\bar\lambda\tilde\sigma_\nu\sigma^\mu\bar\chi)\\
(\bar\lambda\tilde\sigma^\mu\theta)(\theta\sigma^\nu\bar\theta) &=& -\frac12\theta^2\bar\theta(\tilde\sigma^\nu\sigma^\mu)\bar\lambda\\
(\lambda\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) &=& \frac12\bar\theta^2\theta(\sigma^\nu\tilde\sigma^\mu)\lambda
\end{array}\tag{4.88}$$

Most of these identities are easily proven by using the explicit representation

$$\begin{array}{l}
\lambda\chi=\lambda^\alpha\chi_\alpha=-\lambda^1\chi^2+\lambda^2\chi^1=-\lambda_1\chi_2+\lambda_2\chi_1,\quad \theta^1=\theta_2,\ \theta^2=-\theta_1\\
\bar\lambda\bar\chi=\bar\lambda_{\dot\alpha}\bar\chi^{\dot\alpha}=+\bar\lambda^{\dot1}\bar\chi^{\dot2}-\bar\lambda^{\dot2}\bar\chi^{\dot1}=+\bar\lambda_{\dot1}\bar\chi_{\dot2}-\bar\lambda_{\dot2}\bar\chi_{\dot1},\quad \bar\theta^{\dot1}=\bar\theta_{\dot2},\ \bar\theta^{\dot2}=-\bar\theta_{\dot1},
\end{array}$$

together with the anticommutation relations.

4.8 Spinors in Euclidean spaces

"Lorentz transformations" on a d-dimensional space with Euclidean metric are rotations. They form the compact Lie-group $SO(d)$. The γ^μ are hermitean and so are the generators $\Sigma_{\mu\nu}$ of the spin group. It follows that the spin transformation S in

$$\psi(x)\longrightarrow S\psi(\Lambda^{-1}x)\tag{4.89}$$

is unitary. As explicit representation for the Euclidean γ's we may take the matrices (4.4) without the factors i. It follows, that the charge conjugation matrices $\mathcal{C}_\pm$ in (4.36) are still solutions of

$$\gamma_\mu^T=\eta\,\mathcal{C}^{-1}\gamma_\mu\mathcal{C}\quad\text{with}\quad\eta=\pm1.\tag{4.90}$$

Since the γ^μ are hermitian, the charge conjugation of a spinor in Euclidean spaces has a different form as in Minkowski space times,

$$\psi_c = \mathcal{C}\psi^*. \tag{4.91}$$

The Majorana condition $\psi_c = \psi$ is again consistent with rotational invariance. Now the consistency condition $\psi^{**} = \psi$ implies the *symmetry* of the charge-conjugation matrix, $\mathcal{C} = \mathcal{C}^T$. Comparing with table (4.3) yields the following solutions

d	1	2	3	4	5	6	7	8
$\mathcal{C}_+$	S	S						S
$\mathcal{C}_-$						S	S	S
				SM				MW

(4.92)

The existence of Majorana-Weyl fermions in $8, 16, 24, \dots$ dimensions can be proved similarly as in Minkowski space times. Symplectic Majorana fermions exist only in $4, 12, 20, \dots$ dimensions.
Since spin transformations are unitary, the antisymmetric tensor fields in Euclidean spaces are

$$A_{\mu_1\mu_2\dots\mu_n} = \psi^\dagger\gamma_{\mu_1\mu_2\dots\mu_n}\psi. \tag{4.93}$$

Since the hermiticity relations (4.81) hold true for all signatures, the general Fierz-identity remain unaltered,

$$\psi\chi^\dagger = -\frac{1}{\Delta}\sum_n \frac{1}{n!}\,(-)^{n(n-1)/2}\,\gamma_{\mu_1\dots\mu_n}\,(\chi^\dagger\gamma^{\mu_1\dots\mu_n}\psi). \tag{4.94}$$

Note that for the metric $\delta_{\mu\nu}$ we have $\gamma^{\mu_1\mu_2\cdots} = \gamma_{\mu_1\mu_2\dots}$. The identity (4.84) in 4 dimensions still holds true (of course, after setting $\bar\chi = \chi^\dagger$). The definition of γ_5 is slightly modified in order to get a hermitian matrix which squares to $\mathbb{1}$,

$$\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3 \quad\text{and}\quad \gamma_{\mu\nu\rho} = -\epsilon_{\mu\nu\rho\sigma}\gamma_5\gamma^\sigma \quad\text{and}\quad \gamma_{\mu\nu\rho\sigma} = -\epsilon_{\mu\nu\rho\sigma}\gamma_5.$$

Nevertheless, the Fierz-identity (4.85) remains unchanged,

$$4\psi\chi^\dagger = -(\chi^\dagger\psi) - \gamma_\mu(\chi^\dagger\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\chi^\dagger\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\chi^\dagger\gamma_5\gamma^\mu\psi) - \gamma_5(\bar\chi\gamma_5\psi). \tag{4.95}$$

When one computes the fermionic bilinears for the charge conjugated fields one obtains

$$\psi^\dagger\gamma^{(n)}\chi = -\eta^n(-1)^{n(n-1)/2}\,\chi^\dagger\gamma^{(n)}\psi, \tag{4.96}$$

since in the relation $\psi_c^\dagger = \psi^T\mathcal{C}^{-1}$ there is no η, contrary to the corresponding identity (4.74) in Minkowski space time.
Let us finally investigate the hermiticity properties of the bilinears. The relation (4.77) becomes

$$(\psi^\dagger M\chi)^\dagger = \chi^\dagger M^\dagger\psi = -\psi^T M^*\chi^* = -\psi_c^\dagger M_c\chi_c, \qquad M_c = \mathcal{C}M^*\mathcal{C}^{-1}. \tag{4.97}$$

In a Majorana representation with real or imaginary γ^μ all γ's are symmetric or antisymmetric. Then $\mathcal{C}$ is proportional to $\mathbb{1}$ and $M_c = M^*$. For Majorana spinors in a Majorana representation we obtain the simple relation

$$(\psi^\dagger M\chi)^\dagger = -\psi^\dagger M^*\chi. \tag{4.98}$$

In passing we note that there are no Majorana spinors in 4-dimensional Euclidean space. Strictly speaking there is no Euclidean version of a theory with Majorana fermions in 4 dimensions. There have been suggestions to overcome this obstacle and there have been numerical simulations of such Euclidean models on lattices.

4.9 Appendix: Representations of spin and Lorentz groups

The fields furnish a representation of the homogeneous Lorentz groups $SO(1, d-1)$ or, strictly speaking, of their universal covering groups spin$(1, d-1)$. Assuming that they transform according to a finite dimensional representation of spin$(1, d-1)$, all these representation can be obtained from the finite-dimensional unitary representation of the corresponding orthogonal group spin(d) by setting $\tau = ix^0$.

4.9.1 Young-tableaus of O(n)

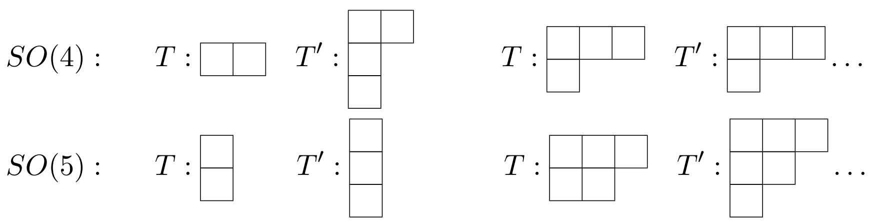
Every irreducible representation of $O(n)$ can be obtained from the tensor product of the defining representation

$$\begin{array}{rcll} T_\mu & \longrightarrow & R_\mu^{\ \nu} T_\nu, \qquad R \in O(n) & \text{defining repr.} \\ T_{\mu_1\dots\mu_r} & \longrightarrow & R_{\mu_1}^{\ \nu_1}\dots R_{\mu_r}^{\ \nu_r} T_{\nu_1\dots\nu_r} & \text{tensor product repr.} \end{array} \tag{4.99}$$

by suitable symmetrization and anti-symmetrization in groups of indexes. They are in one-to-one corresponding to admissible *Young tableaus*. Each tableau T contains *at most* $[n/2]$ *rows*. If ℓ_i is the number of boxes of the i'th row, then

$$\ell_1 \geq \ell_2 \geq \dots \geq \ell_{[n/2]}.$$

To get all irreducible representations of $O(n)$ we associate a diagram T' to T as follows: The length a of the first column in T is less or equal to $n/2$, $a \leq n/2$, the length of the first column in T' is $n-a$, and all other columns in T and T' have the same length. Examples



When n is even and $a = n/2$, then the diagrams T and T' coincide (they transform differently under improper transformations) and T is said to be *self-associate*. If we restrict ourselves to the proper subgroup $SO(n)$ with determinant one, the representations corresponding to the associate diagrams T and T' become equivalent. For even n the representation corresponding to a selfassociate pattern T splits into two non-equivalent irreducible representations: the *selfdual* and *antiselfdual* antisymmetric tensors of rank $n/2$,

$$T_{\mu_1\dots\mu_{n/2}}=\pm\frac{1}{(n/2)!}\,\epsilon_{\mu_1\dots\mu_{n/2}\nu_1\dots\nu_{n/2}}T_{\nu_1\dots\nu_{n/2}},\qquad T\quad\text{totally skew symmetric.}\tag{4.100}$$

4.9.2 Roots, weights and all that

Our Lie algebra conventions are as follows: Let $H_k,\ k=1,\dots,r$ be an orthogonal basis of the Cartan-subalgebra $\mathcal{H}$, which we diagonalize in a given representation,

$$H_k|\mu\rangle=\mu_k|\mu\rangle\quad\text{and}\quad[H_k,E_\alpha]=\alpha_kE_\alpha.\tag{4.101}$$

Let

$$\alpha_{(i)}\ ,\quad\mu_{(i)}\ ,\quad\alpha^\vee_{(i)}=\frac{2\alpha_{(i)}}{(\alpha_{(i)},\alpha_{(i)})}\quad\text{and}\quad\mu^\vee_{(i)}=\frac{2\mu_{(i)}}{(\alpha_{(i)},\alpha_{(i)})}\ ,\qquad i=1,\dots,r\tag{4.102}$$

be the *simple roots, fundamental weights, co-roots* and *co-weights*, respectively:

$$(\alpha_{(i)},\alpha^\vee_{(j)})=K_{ij},\quad(\alpha^\vee_{(i)},\mu_{(j)})=(\alpha_{(i)},\mu^\vee_{(j)})=\delta_{ij},\quad(\mu_{(i)},\mu^\vee_{(j)})=(K^{-1})_{ij}.\tag{4.103}$$

We used that the simple roots and fundamental weights are related by the Cartan-matrix,

$$\alpha_{(i)}=\sum_{j=1}^rK_{ij}\ \mu_{(j)}.\tag{4.104}$$

The fundamental weight-states (which are the highest weight states of the r fundamental representations) and states in the adjoint representation obey

$$\alpha^\vee_{(i)}\cdot H\ |\mu_{(j)}\rangle=\delta_{ij}|\mu_{(j)}\rangle\quad\text{and}\quad\mu^\vee_{(i)}\cdot H\ |\alpha_{(j)}\rangle=\delta_{ij}|\alpha_{(j)}\rangle.\tag{4.105}$$

To determine the dimension of the representation with highest weight μ we may use *Weyl's dimension formula*

$$\dim V_\mu=\prod_{\alpha>0}\frac{\langle\mu+\delta,\check\alpha\rangle}{\langle\delta,\check\alpha\rangle},\tag{4.106}$$

where the Weyl vector δ is half the sum over all positive roots or the sum over all fundamental weights,

$$\delta=\tfrac12\sum_{\alpha>0}\alpha=\sum\mu_{(i)}\tag{4.107}$$

A positive root and highest weight is a linear combination of the simple roots and fundamental weights with non-negative integer coefficients, respectively,

$$\alpha = \sum_i m_i \alpha_{(i)} \quad \text{and} \quad \mu = \sum_i k_i \mu_{(i)}, \qquad m_i, k_i \in \mathbb{N}_0. \tag{4.108}$$

Inserting this into Weyl's dimension formula and using $(\mu_{(i)}, \alpha_{(j)}) = \delta_{ij} {\alpha_{(j)}}^2/2$ yields

$$\dim V_\mu = \prod_{\alpha>0} \frac{\sum_i (1+k_i) m_i {\alpha_{(i)}}^2}{\sum_i m_i {\alpha_{(i)}}^2}. \tag{4.109}$$

To find the *center elements* we find conditions on $\rho \cdot H \in \mathcal{H}$ such that $\exp(2\pi i \rho \cdot H)$ is in the center $\mathcal{Z}$. Center elements are the identity in the adjoint representation. Because of the second set of equations in (4.105) they must be powers of

$$z_i = \exp\left(2\pi i \mu^\vee_{(i)} \cdot H\right) \Longrightarrow z_i |\mu\rangle = \exp\left(2\pi i (kK^{-1})_i\right) |\mu\rangle, \qquad \mu = \sum_i k_i \mu_{(i)}. \tag{4.110}$$

In an irreducible representation a center element acts the same way on all states. Hence, a necessary and sufficient condition for $z_i \neq \mathbb{1}$ is that

$$z_i \, |\mu_{(j)}\rangle = \exp\left(2\pi i K^{-1}_{ji}\right) |\mu_{(j)}\rangle \neq |\mu_{(j)}\rangle, \quad \text{or that} \quad K^{-1}_{ji} \notin \mathbb{Z} \tag{4.111}$$

for at least one fundamental weight $\mu_{(j)}$. Here we have used that the inner products of the weights with the co-weights yield the inverse Cartan-matrix, see (4.103). The order of the center group is just $\det(K)$.

4.9.3 The spin groups

Let us now turn to the spin groups, i.e. the universal (double) covering groups of the $SO(n)$-groups[2]. In the Cartan classification spin$(2n) \sim D_n$ and spin$(2n+1) \sim B_n$:

group	dim	dim. of fund. reps
spin$(4) = D_2 \sim A_1 \times A_1$	6	$2, 2$
spin$(5) = B_2 \sim C_2$	10	$5, 4$
spin$(6) = D_3 \sim A_3$	15	$6, 4, 4$
spin$(7) = B_3$	21	$7, 21, 8$
spin$(8) = D_4$	28	$8, 28, 8, 8$
spin$(9) = B_4$	36	$9, 36, 84, 16$
spin$(10) = D_5$	45	$10, 45, 120, 16, 16$

The *Dynkin-diagrams* of the spin groups are depicted in the following figure: The repre-

[2] we use the conventions as implemented by LIE 2.1

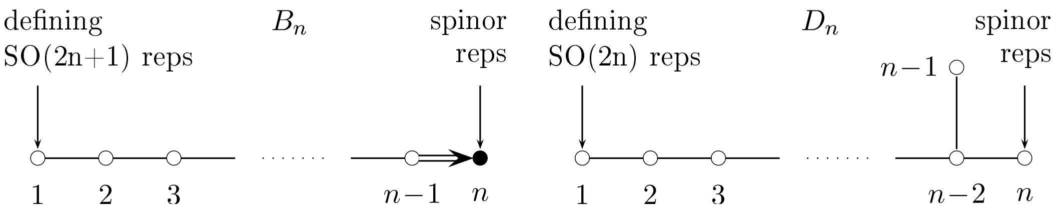


Abbildung 4.1: *Dynkin diagrams of the B_n and D_n series. $\bullet$ is a short root.*

sentation with highest weight $\mu_{(1)}$ is always the defining representation of $SO(n)$ and has dimension n. The representation with highest weight $\mu_{(n)}$ is a spinor representation for all groups. For the D_n the second last representation belonging to $\mu_{(n-1)}$ is the second spinor representation.
The Cartan-matrices of the B_n and D_n groups have the form

$$K_B = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}, \quad K_D = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & -1 & 0 & 2 \end{pmatrix}$$

The centers and their generators are

group	center	generators	action of center element(s)
D_n, n even	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$z_1 = e^{2\pi i \mu^\vee_{(1)}}$ $z_2 = e^{2\pi i \mu^\vee_{(n)}}$	$z_1 \vert\mu_{(j)}\rangle = +\vert\mu_{(j)}\rangle, \quad j \le n-2$ $z_1 \vert\mu_{(j)}\rangle = -\vert\mu_{(j)}\rangle, \quad j \ge n-1$ $z_2 \vert\mu_{(j)}\rangle = (-)^j \vert\mu_{(j)}\rangle, \quad j \le n-2$ $z_2 \vert\mu_{(n-1)}\rangle = -(-)^{n/2} \vert\mu_{(n-1)}\rangle$ $z_2 \vert\mu_{(n)}\rangle = (-)^{n/2} \vert\mu_{(n)}\rangle$
D_n, n odd	$\mathbb{Z}_4$	$z = e^{2\pi i \mu^\vee_{(n)}}$	$z \vert\mu_{(j)}\rangle = (-)^j \vert\mu_{(j)}\rangle, \quad j \le n-2$ $z \vert\mu_{(n-1)}\rangle = -i^n \vert\mu_{(n-1)}\rangle$ $z \vert\mu_{(n)}\rangle = i^n \vert\mu_{(n)}\rangle$
B_n	$\mathbb{Z}_2$	$z = e^{2\pi i \mu^\vee_{(1)}}$	$z \vert\mu_{(j)}\rangle = \vert\mu_{(j)}\rangle, \quad j \le n-1$ $z \vert\mu_{(n)}\rangle = -\vert\mu_{(n)}\rangle$

The group spin(5) $\sim$ B$_2$

This group is related to the isometry groups of the 5-dimensional deSitter and anti-deSitter spaces and to the conformal group of the 4-dimensional flat spaces (with different signatures). Hence it is useful to discuss the corresponding representations in some detail. The discussion for the higher spin groups is very similar. The Cartan-matrix of spin(5) and its inverse are

$$K = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}, \quad K^{-1} = \tfrac{1}{2}\begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix}. \tag{4.112}$$

The center element $z = \exp(\pi i \mu_1^\vee)$ is the identity on the defining $SO(5)$-representation with highest weight μ_1. It is $-\mathbb{1}$ on the spinor representation with highest weight μ_2. Hence, a representation with highest weight $\mu = k_1\mu_1 + k_2\mu_2$ is a faithful representation of

$$SO(5) = \text{spin}(5)/\mathbb{Z}_2 = B_2/\mathbb{Z}_2$$

if and only if

$$z|\mu\rangle = |\mu\rangle \Longleftrightarrow k_2 \in 2\mathbb{N}_0.$$

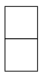
The positive roots are

$$\text{short roots:} \quad m = (0,1),(1,1) \qquad \text{long roots:} \quad m = (1,0),(1,2). \tag{4.113}$$

Using Weyl's formula we find that the dimension of the representation with highest weight μ is

$$\dim V_\mu = \frac{1}{6}(1+k_1)(1+k_2)(3+2k_1+k_2)(2+k_1+k_2). \tag{4.114}$$

The quantum numbers can easily be related to the length (ℓ_1, ℓ_2) of the rows in the Young-tableau. We need

 : 5 and  : 10 .

The 5 and 10-dimensional representation have

$$5: \quad \ell = (1,0),\ k = (1,0) \quad \text{and} \quad 10: \quad \ell = (1,1),\ k = (0,2).$$

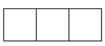
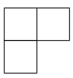
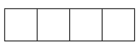
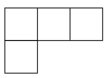
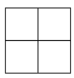
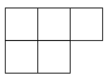
From this follows that

$$\ell_1 = k_1 + k_2/2 \quad \text{and} \quad \ell_2 = k_2/2$$

In terms of the lengths ℓ_1, ℓ_2 of the Young-tableau we obtain the following formula for the dimensions for the faithful $SO(5)$-representations

$$\text{dim}V_\ell = (\ell_1 - \ell_2 + 1)(2\ell_2 + 1)(\ell_1 + \ell_2 + 2)(2\ell_1 + 3)/6. \tag{4.115}$$

Examples:

 : 14  : 35  : 55  : 91  : 35  : 105.

The group spin$(6) \sim \mathbf{D}_3$

The simple laced group D_3 has Cartan-matrix

$$K = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad \text{and} \quad K^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

The center $\mathbb{Z}_4$ is generated by

$$z = \exp\left(2\pi i \mu^\vee_{(3)}\right) \quad \text{with} \quad z|\mu\rangle = \exp\left(\frac{i\pi}{2}(2k_1 + k_2 + 3k_3)\right)|\mu\rangle. \tag{4.116}$$

Since on the defining representation of $SO(6)$ it is $-\mathbb{1}$, a representation of spin(6) is a faithful representation of

$$SO(6) = \text{spin}(6)/\mathbb{Z}_2 = D_3/\mathbb{Z}_2 \tag{4.117}$$

if $z = \pm\mathbb{1}$ on this representation. This is equivalent to

$$2k_1 + k_2 + 3k_3 \in 2\mathbb{N}_0 \Longleftrightarrow k_2 + k_3 \in 2\mathbb{N}_0.$$

The positive roots are

$$m = (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (1,1,1), \tag{4.118}$$

and Weyl's formula yields

$$\dim V_\mu = \frac{1}{12}(1+k_1)(1+k_2)(1+k_3)(2+k_1+k_2)(2+k_1+k_3)(3+k_1+k_2+k_3) \tag{4.119}$$

The lowest representations of $SO(6)$ are

$$6 : \begin{array}{|c|}\hline \\ \hline\end{array} \quad 15 : \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} \quad \text{and} \quad 20 : \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}\,.$$

Using LIE it easy to relate the highest weights to these representations (recall, that the 20 splits into $10_S \oplus 10_A$).

dim	ℓ	k
5	$(1,0,0)$	$(1,0,0)$
15	$(1,1,0)$	$(0,1,1)$
20	$(1,1,1)$	$(0,2,0), (0,0,2)$

(4.120)

The group spin(7) $\sim$ B$_3$

A representation of spin(7) is a faithful representation of $SO(7)$ if k_3 is an even integer. Since B_3 has no self-associate diagrams it is easy to relate the lengths of the rows in a Young-tableau to a highest weight:

$$k_1 = \ell_1 - \ell_2 \quad k_2 = \ell_2 - \ell_3 \quad \text{and} \quad k_3 = 2\ell_3.$$

The dimensions of the various representations and the corresponding results for the smallest spin-groups are listed in the following table

group	dimension
spin(4)	$(k_1+1)(k_2+1)$
SO(5)	$(\ell_1-\ell_2+1)(2\ell_2+1)(\ell_1+\ell_2+2)(2\ell_1+3)/6$
spin(6)	$(1+k_1)(1+k_2)(1+k_3)$
	$\cdot(2+k_1+k_2)(2+k_1+k_3)(3+k_1+k_2+k_3)/12$
SO(7)	$(2\ell_1+5)(2\ell_2+3)(2\ell_3+1)(\ell_1+\ell_2+4)(\ell_1-\ell_2+1)$
	$\cdot(\ell_1-\ell_3+2)(\ell_1+\ell_3+3)(\ell_2+\ell_3+2)(\ell_2-\ell_3+1)/720$

(4.121)

Kapitel 5

Symmetries

Supersymmetry is an extension of Poincaré symmetry and hence is intimately related to the geometry of our spacetime. A supersymmetric *field theory* in flat space time is a particular theory compatible with the principles of special relativity. Thus it is useful to review the implications of internal and spacetime symmetries for relativistic field theories. All fundamental relativistic field equations are Euler-Lagrange equation of an Poincaré invariant action integral S. In a local field theory the action is the space-time integral of a local Lagrangean density $\mathcal{L}(x)$,

$$S = \int d^d x \; \mathcal{L}(x) \equiv \int dt d\boldsymbol{x} \mathcal{L}(t, \boldsymbol{x}), \tag{5.1}$$

where $\mathcal{L}$ is a function of the fields and their first space-time derivatives. Again the space-time dimension is left open. The volume element of space is denoted by $d\boldsymbol{x}$.
The action should be invariant under Poincaré *transformations*,

$$\tilde{x} = \Lambda(A)x + a \tag{5.2}$$

under which the field transforms as

$$\tilde{\phi}(\tilde{x}) = S(A)\phi(x) \quad \text{or} \quad \phi(x) \longrightarrow S(A)\phi\left(\Lambda^{-1}(A)(x-a)\right), \tag{5.3}$$

where $A \to S(A)$ is a representations of the spin group and $\Lambda(A)$ it the Lorentz transformation corresponding to A.
In addition the action may be invariant under *global gauge transformations*

$$\phi \longrightarrow U\phi \sim \phi + iX\phi = \phi + \delta_X \phi, \qquad X^\dagger = X. \tag{5.4}$$

They are called global since the transformation matrix U it the same for all space-time points.

5.1 Noether theorem and conserved charges

According to a theorem of Emmy Noether, to each parameter of the symmetry group there corresponds a conserved current. The global gauge transformations (5.4) leave the

Lagrangean density invariant so that

$$0 = \delta_X \mathcal{L} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\, \partial_\mu \left(\delta_X \phi\right) + \frac{\partial \mathcal{L}}{\partial \phi}\, \delta_X \phi = \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta_X \phi\right), \tag{5.5}$$

where we used the Euler-Lagrange equation (field equation, equation of motion)

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\right) - \frac{\delta \mathcal{L}}{\delta \phi} = 0 \tag{5.6}$$

in the last step. Thus the conserved Noether *current* for an internal symmetry takes the form

$$J^\mu_X = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} \delta_X \phi, \qquad \partial_\mu J^\mu_X = 0. \tag{5.7}$$

Integrating the last equation over the space-time region $[t_0, t] \times \mathbb{R}^{d-1}$ and converting the volume- into a surface integral, shows that the Noether *charges*

$$Q_X = \int_{x^0} d\boldsymbol{x}\, J^0_X = \int_{x^0} d\boldsymbol{x}\, \pi(x) \delta_X \phi(x), \tag{5.8}$$

are time-independent. Here we introduced the momentum density conjugate to the field ϕ,

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}. \tag{5.9}$$

To every internal symmetry there is always one conserved Noether charge. The dimension of the symmetry group equals the number of independent vector fields X in (5.8) and hence equals the number of independent Noether charges.

The fundamental Poisson-bracket between field and conjugated momentum density has the form

$$\{\phi(x), \pi(y)\}_{x^0 = y^0} = \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{5.10}$$

and can be used to calculate the Poisson-brackets between the conserved charges and the field,

$$\{\phi(x), Q_X\} = \int d\boldsymbol{y}\, \{\phi(x), \pi(y)\, \delta_X \phi(y)\} = \delta_X \phi(x), \tag{5.11}$$

where we assumed, that $\delta_X \phi$ contains no time-derivatives of the field. We see that the conserved Noether charge generates the symmetry from which it has been derived.

5.1.1 $U(1)$ gauge transformations

Gauge transformations in non-Abelian gauge theories are generalizations of phase transformations in electrodynamics. As an example consider the Lagrangean of a charged scalar field,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2. \tag{5.12}$$

It is invariant under the global phase transformations

$$\phi \longrightarrow e^{i\lambda}\phi \sim \phi + \delta_\lambda \phi\,, \quad \text{with} \quad \delta_\lambda \phi = i\lambda\phi, \quad \delta_\lambda \phi^\dagger = -i\lambda\phi^\dagger, \tag{5.13}$$

and the corresponding Noether current has the form

$$J^\mu_\lambda = \lambda j^\mu, \qquad j_\mu = i(\partial_\mu \phi^\dagger \phi - \phi^\dagger \partial_\mu \phi). \tag{5.14}$$

Using the equation of motion $\Box\phi + m^2\phi + 2\lambda(\phi^\dagger\phi)\phi$ one can check that j^μ is conserved. The conserved charge is to be identified with the electric charge carried by the field,

$$Q = i\int_{x^0} d\boldsymbol{x}(\dot\phi^\dagger \phi - \phi^\dagger \dot\phi) = i\int_{x^0} d\boldsymbol{x}\left(\pi_\phi \phi - \pi_{\phi^\dagger}\phi^\dagger\right). \tag{5.15}$$

Again the charge generates the symmetry from which it has been derived,

$$\{\phi, Q\} = i\phi \quad \text{and} \quad \{\phi^\dagger, Q\} = -i\phi^\dagger. \tag{5.16}$$

5.1.2 The energy-momentum as Noether current

Under space-time translations the field transforms as

$$\phi(x+a) \sim \phi(x) + a^\mu \partial_\mu \phi(x) = \phi(x) + \delta_a \phi(x). \tag{5.17}$$

Since the space-time point is changing the Lagrangean is not invariant. But translations are symmetries and the *action* must be invariant. It is invariant since (off-shell) the Lagrangean only changes by total divergence,

$$\delta_a \mathcal{L} = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\,\partial_\mu(\delta_a\phi) + \frac{\delta\mathcal{L}}{\delta\phi}\,\delta_a\phi = a^\nu \partial_\nu \mathcal{L}. \tag{5.18}$$

Using the field equation (5.6) the on-shell identity (5.5) is now modified to

$$\delta_a \mathcal{L} = a^\nu \partial_\nu \mathcal{L} = \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} a^\nu \partial_\nu \phi\right), \quad \text{or} \quad \partial_\mu J^\mu_a = 0 \tag{5.19}$$

with covariantly conserved *canonical energy-momentum* tensor as Noether *current*

$$J^\mu_a = a^\nu T^\mu_{\ \nu}, \qquad T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta(\partial^\mu\phi)}\,\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}. \tag{5.20}$$

The conserved charges are the total energy-momentum of the field,

$$P^\mu = \int_{x^0} d\boldsymbol{x}\, T^{0\mu}, \qquad \dot{P}^\mu = 0, \tag{5.21}$$

and for the scalar field with Lagrangean (5.12) the *energy* and *momentum* have the form

$$P^0 \equiv H = \int_{x^0} d\boldsymbol{x}\, (\pi\dot{\phi} - \mathcal{L}) \quad \text{and} \quad P^i = \int_{x^0} d\boldsymbol{x}\, \pi\, \partial^i \phi. \tag{5.22}$$

These Noether charges generate the infinitesimal spacetime translations,

$$\{\phi, P_\mu\} = \partial_\mu \phi. \tag{5.23}$$

For the scalar field theory with Lagrangean (5.12) we find

$$T_{\mu\nu} = \partial_\mu \phi^\dagger \partial_\nu \phi + \partial_\nu \phi^\dagger \partial_\mu \phi - \eta_{\mu\nu}\mathcal{L} \tag{5.24}$$

which leads to the conserved energy

$$H = \int d\boldsymbol{x} \left(\pi^\dagger \pi + \nabla\phi^\dagger \nabla\phi + m^2 \phi^\dagger \phi + \lambda(\phi^\dagger\phi)^2\right) \tag{5.25}$$

and conserved momentum

$$P_i = \int d\boldsymbol{x} \left(\pi \partial_i \phi + \partial_i \phi^\dagger \pi^\dagger\right). \tag{5.26}$$

In *electrodynamics* we introduce the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ which is derived from the 4-potential $A^\mu = (A^0, \boldsymbol{A})$. The entries of the field strength are the components of the electric and magnetic fields,

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix} = (\boldsymbol{E}, \boldsymbol{B}). \tag{5.27}$$

Raising both indexes amount to changing the sign of the electric field, $(F^{\mu\nu}) = (-\boldsymbol{E}, \boldsymbol{B})$. The Lagrangean density in electromagnetism is

$$\mathcal{L} = \tfrac{1}{2}\left(\boldsymbol{E}^2 - \boldsymbol{B}^2\right) = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu} = -\tfrac{1}{4} F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) \tag{5.28}$$

From the last representation it follows at once that

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\rho)} = -F^{\mu\rho} \tag{5.29}$$

holds true, and this result is used when calculating the energy-momentum tensor,

$$T_{\mu\nu} = \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\rho)}\, \partial_\nu A_\rho - \eta_{\mu\nu}\mathcal{L} = -F_\mu^{\ \rho} \partial_\nu A_\rho + \tfrac{1}{4}\eta_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}. \tag{5.30}$$

This then leads to the following formula for the conserved energy-momentum in electromagnetism,

$$P_\nu = \int_{x^0} d\boldsymbol{x}\, T^0_{\ \nu} = \int_{x^0} d\boldsymbol{x}\, \left(F^{\rho 0}\,\partial_\nu A_\rho - \delta^0_{\ \nu}\mathcal{L}\right)$$

or written in components

$$H = \int_{x^0} d\boldsymbol{x}\, \left(E_i \dot{A}_i - \mathcal{L}\right) \quad \text{and} \quad P_i = \int_{x^0} d\boldsymbol{x}\, \left(E_j \partial_i A_j\right). \tag{5.31}$$

Although we succeeded in constructing a conserved second rank tensor which is to be interpreted as energy-momentum tensor, there is still some improvement necessary. The main reason for our unhappiness with the tensor (5.30) is its lack of symmetry and invariance. It is not gauge invariant and hence not observable. Also we would like the energy-momentum to be a symmetric tensor for several reasons. The most convincing one is that it appears on the right hand side of Einstein's field equation and then it must be both symmetric and conserved [28]

5.1.3 Improving Noether currents

Only for scalar fields is the canonical energy-momentum tensor arising from Noether's theorem symmetric. It is possible to correct the non-symmetric ones arising in non-scalar theories through the Belinfante symmetrization procedure [27]. This way one may arrive at the socalled metric energy-momentum tensor which is automatically symmetric and conserved.
Let the Lagrangean be invariant under a symmetry transformation $\phi \to \phi + \delta\phi$, up to a total divergence,

$$\delta\mathcal{L} = \partial_\mu V^\mu. \tag{5.32}$$

As we have seen, the corresponding conserved Noether current has the form

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\delta\phi - V^\mu. \tag{5.33}$$

It is important to note that the vector field V^μ in (5.32) is determined only up to terms $\partial_\nu A^{\mu\nu}$ with antisymmetric $A_{\mu\nu}$ and hence the Noether current J^μ is not unique,

$$J^\mu \longrightarrow J^\mu - \partial_\nu A^{\mu\nu}. \tag{5.34}$$

For localised fields the conserved charges are not affected by this ambiguity since

$$J^0 \to J^0 - \partial_i A^{0i} \quad \text{and hence} \quad Q = \int_{x^0} d\boldsymbol{x}\, J^0 \longrightarrow Q.$$

The possible improvement terms have been studied widely in the literature. For example, in electrodynamics the conserved momentum P^μ is gauge invariant, since $\partial_\rho F^{0\rho} = 0$, and we expect that there exists an improved gauge invariant energy momentum tensor. Indeed, we can add an improvement term $-\partial_\rho(F^{\mu\rho}A^\nu)$ to $T^{\mu\nu}$ in (5.30) to obtain a gauge invariant and symmetric tensor. The improved energy-momentum tensor takes the well-known form

$$T_{\mu\nu} = \tfrac{1}{2}F_\mu^{\ \rho}F_{\rho\nu} + \tfrac{1}{4}\eta_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \tag{5.35}$$

giving rise to

$$H = \tfrac{1}{2}\int_{x^0} d\boldsymbol{x}\,\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \quad \text{and} \quad \boldsymbol{P} = \int_{x^0} d\boldsymbol{x}\,\boldsymbol{E}\wedge\boldsymbol{B}. \tag{5.36}$$

The conserved tensor (5.35) may alternatively be gotten by coupling the Maxwell field minimally to gravity and varying the resulting Lagrangean with respect to the metric.

5.1.4 The conserved angular momentum

Deriving the conserved quantities belonging to the Lorentz transformations is a bit more elaborate. Under Lorentz transformation a field transforms as follows,

$$\phi(x) \longrightarrow e^{\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu}}\phi\left(e^{-\omega}x\right) \sim \phi(x) + \tfrac{i}{2}\,\omega^{\mu\nu}J_{\mu\nu}\phi(x) = \phi(x) + \delta_\omega\phi(x), \tag{5.37}$$

where the infinitesimal generators of Lorentz transformations $J_{\mu\nu}$ are the sum of orbital and spin terms,

$$J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} = \frac{1}{\mathrm{i}}\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) + S_{\mu\nu}. \tag{5.38}$$

The $L_{\mu\nu}$ commute with the $S_{\mu\nu}$ and both fulfil the Lorentz algebra (3.10). Again the Lagrangean is off-shell invariant up to a total divergence,

$$\delta_\omega\mathcal{L} = \tfrac{i}{2}\,(\omega, M)\,\mathcal{L} = \partial_\mu V^\mu \quad \text{with} \quad V^\mu = -\omega^{\mu\rho}x_\rho\mathcal{L}. \tag{5.39}$$

The first term in (5.33) reads

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\omega\phi = \frac{\mathrm{i}}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\omega^{\rho\sigma}\left(L_{\rho\sigma} + S_{\rho\sigma}\right)\phi$$

and subtracting V^μ in (5.39) yields the Noether current

$$J^\mu_\omega = \frac{\omega^{\rho\sigma}}{2}\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(x_\rho\partial_\sigma - x_\sigma\partial_\rho)\phi - (\delta^\mu_\sigma x_\rho - \delta^\mu_\rho x_\sigma)\mathcal{L} + i\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,S_{\rho\sigma}\phi\right)$$

The spin-independent terms between the brackets are $x_\rho T^\mu_{\ \sigma}$ and $-x_\sigma T^\mu_{\ \rho}$ and thus we obtain

$$J^\mu_\omega = \omega_{\rho\sigma}M^{\mu\rho\sigma}, \quad M^{\mu\rho\sigma} = \tfrac{1}{2}x^\rho T^{\mu\sigma} - \tfrac{1}{2}x^\sigma T^{\mu\rho} + \tfrac{i}{2}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,S^{\rho\sigma}\phi = -M^{\mu\sigma\rho} \tag{5.40}$$

and the corresponding conserved charges

$$J^{\rho\sigma} = -J^{\sigma\rho} = \int_{x^0} d\boldsymbol{x}\, M^{0\rho\sigma}. \tag{5.41}$$

The space-time and space-space components

$$\begin{aligned} J^{i0} &= \tfrac{1}{2}\int_{x^0} d\boldsymbol{x}\, \left(x^i\mathcal{H} - \pi x^0\partial^i\phi - i\pi S^{0i}\phi\right) \\ J^{ij} &= \tfrac{1}{2}\int_{x^0} d\boldsymbol{x}\, \pi\left(x^i\partial^j\phi - x^j\partial^i\phi + iS^{ij}\phi\right). \end{aligned} \tag{5.42}$$

A simple calculation reveals that

$$\{\phi, J^{\mu\nu}\} = \tfrac{i}{2}(L^{\mu\nu} + S^{\mu\nu})\phi, \tag{5.43}$$

or that the Noether charges J^{0i} generate boosts and the charges J^{ij} rotations. For example, for electrodynamics

$$M^{\mu\rho\sigma} = \left(\tfrac{1}{4}\eta^{\mu\rho}x^\sigma F^2 - F^{\mu\xi}x^\sigma\partial^\rho A_\xi - F^{\mu\rho}A^\sigma\right) - (\rho \leftrightarrow \sigma). \tag{5.44}$$

In passing we note, that for theories with spinor fields things may become tricky. The canonical energy-momentum tensor is generically not symmetric and must be improved. For bosonic fields the most efficient way to do this is to couple the fields to gravity and vary the resulting Lagrangean with respect to the metric. However, when coupling fermions to gravity one needs to introduce the vielbein. When one varies the action with respect to the vielbein one gets a conserved but not necessarily symmetric energy momentum tensor.

5.2 Conformal symmetry

Under conformal transformations

$$x^\mu \to y^\mu = y^\mu(x^\nu) \tag{5.45}$$

the metric tensor is unchanged up to a factor e^φ,

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = \eta_{\mu\nu}\frac{\partial x^\mu}{\partial y^\rho}\frac{\partial x^\nu}{\partial y^\sigma}dy^\rho dy^\sigma = e^{\varphi(y)}\eta_{\rho\sigma}dy^\rho dy^\sigma. \tag{5.46}$$

These transformation leave the light cone invariant and a spacelike vector can be mapped into a timelike vector. The apparent problem with causality has been solved by Lüscher and Mack [24].

5.2.1 The conformal algebra

Lets look at the infinitesimal conformal transformations,

$$y^\mu \sim x^\mu + X^\mu(x) \quad \text{and} \quad e^\varphi \sim 1 + \delta\varphi \tag{5.47}$$

for which

$$(\eta_{\mu\nu}+X_{\mu,\nu}+X_{\nu,\mu})(1+\delta\varphi)dx^\mu dx^\nu \sim \eta_{\mu\nu}dx^\mu dx^\nu$$

holds true. The linearision of this equation leads to

$$\delta\varphi\,\eta_{\mu\nu}+X_{\mu,\nu}+X_{\nu,\mu}=0.$$

Taking the trace implies that $\delta\varphi$ is proportional to the covariant divergence of the vector field X, so that X satisfies the *conformal Killing-equation*

$$X_{\mu,\nu}+X_{\nu,\mu}=\frac{2}{d}\eta_{\mu\nu}\partial_\rho X^\rho. \tag{5.48}$$

Hence the conformal coordinate transformations are generated by conformal Killing vector fields.
In 3 and more dimensions there are $\frac{1}{2}(d+1)(d+2)$ conformal Killing fields. These are

$$\begin{array}{rlcl} d\ \text{translations} & y^\mu=x^\mu+a^\mu & \Longrightarrow & X^\mu=a^\mu\\ \frac{1}{2}d(d-1)\ \text{Lorentztransf.} & y^\mu=\Lambda^\mu{}_\nu x^\nu & \Longrightarrow & X^\mu=\omega^\mu{}_\nu x^\nu\\ 1\ \text{dilatation} & y^\mu=\lambda x^\mu & \Longrightarrow & X^\mu=\lambda x^\mu\\ d\ \text{special conf. transf.} & y^\mu=\dfrac{x^\mu-x^2c^\mu}{1-2c\cdot x+c^2x^2} & \Longrightarrow & X^\mu=2(c\cdot x)x^\mu-x^2c^\mu. \end{array} \tag{5.49}$$

The corresponding hermitian generators are

$$iX^\mu\partial_\mu=\{a^\mu P_\mu, 2\omega^{\mu\nu}L_{\mu\nu}, \lambda D, c^\mu K_\mu\} \tag{5.50}$$

where

$$P_\mu=i\partial_\mu,\quad L_{\mu\nu}=\frac{1}{\mathrm{i}}(x_\mu\partial_\nu-x_\nu\partial_\mu),\quad D=\mathrm{i}x^\mu\partial_\mu\quad\text{and}\quad K_\mu=2\mathrm{i}x_\mu x^\rho\partial_\rho-\mathrm{i}x^2\partial_\mu \tag{5.51}$$

generate translations, Lorentz transformations, dilatations and special conformal transformations. The conformal algebra is

$$\begin{array}{rcl} [P_\mu,P_\nu] &=& [K_\mu,K_\nu]=[D,D]=0\quad,\quad [P_\mu,D]=iP_\mu\quad,\quad [K_\mu,D]=-iK_\mu\\ {}[P_\mu,K_\nu] &=& 2iL_{\mu\nu}+2\mathrm{i}\eta_{\mu\nu}D\quad,\quad [L_{\mu\nu},D]=0\\ {}[P_\rho,L_{\mu\nu}] &=& -\mathrm{i}(\eta_{\rho\mu}P_\nu-\eta_{\rho\nu}P_\mu)\quad,\quad [K_\rho,L_{\mu\nu}]=-\mathrm{i}(\eta_{\rho\mu}K_\nu-\eta_{\rho\nu}K_\mu)\\ {}[L_{\mu\nu},L_{\rho\sigma}] &=& \mathrm{i}(\eta_{\mu\rho}L_{\nu\sigma}+\eta_{\nu\sigma}L_{\mu\rho}-\eta_{\mu\sigma}L_{\nu\rho}-\eta_{\nu\rho}L_{\mu\sigma}). \end{array} \tag{5.52}$$

These are just the commutation relations of the generators of $SO(2,d)$. To see that, one introduces the coordinates

$$\xi_m=(\xi_{-1},x_\mu,\xi_d)$$

in a $d+2$-dimensional space with signature $(2,d)$:

$$d\xi^2=d\xi_{-1}^2+\eta_{\mu\nu}dx^\mu dx^\nu-d\xi_d^2=\eta_{mn}d\xi^m d\xi^n. \tag{5.53}$$

We consider infinitesimal transformations in this space and decompose the generators as follows,

$$2L_{mn} = -2L_{nm} = \begin{pmatrix} 0 & P_\mu & D \\ -P_\mu & L_{\mu\nu} & P_\mu \\ -D & -P_\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & K_\mu & D \\ -K_\mu & L_{\mu\nu} & -K_\mu \\ -D & K_\mu & 0 \end{pmatrix}. \tag{5.54}$$

The $SO(2,d)$ commutation relations

$$[L_{mn}, L_{pq}] = \mathrm{i}(\eta_{mp}L_{nq} + \eta_{nq}L_{mp} - \eta_{mq}L_{np} - \eta_{np}L_{mq})$$

are equivalent to commutation relations (5.52) of the conformal algebra. The number of generators equals the dimension of $SO(2,d)$ which is $\frac{1}{2}(d+1)(d+2)$.

5.2.2 The conformal group

Th relation between conformal transformations in $\mathbb{R}^d$ and $SO(2,d)$-transformations in $(\mathbb{R}^{d+2}, \eta_{mn})$ exists not only on the algebraic level. To see that one introduces the *projective* coordinates

$$\xi^m = (\psi - \chi, \xi^\mu, \psi + \chi), \quad \text{where} \quad \xi^\mu = 2\psi x^\mu, \qquad \chi = \psi x^2 \quad \text{and} \quad \xi^m \xi_m = 0. \tag{5.55}$$

The nonlinear conformal transformations are then represented by linear $SO(2,d)$ transformations in $\mathbb{R}^{d+2}$ as follows:

$$\xi'^m = \zeta \tilde{\Lambda}^m_{\ n} \xi^n \sim \tilde{\Lambda}^m_{\ n} \xi^n \quad \text{with} \quad \tilde{\Lambda}^m = e^\Omega \in SO(2,d).$$

The infinitesimal transformations have the form

$$\Omega_T = \begin{pmatrix} 0 & -a_\mu & 0 \\ a^\mu & 0 & a^\mu \\ a & a_\mu & 0 \end{pmatrix} \quad , \quad \Omega_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega^\mu_{\ \nu} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\Omega_D = -\begin{pmatrix} 0 & 0 & \log\lambda \\ 0 & 0 & 0 \\ \log\lambda & 0 & 0 \end{pmatrix} \quad , \quad \Omega_S = \begin{pmatrix} 0 & -c_\mu & 0 \\ c^\mu & 0 & -c^\mu \\ 0 & -c_\mu & 0 \end{pmatrix}.$$

For completeness we note that

$$\zeta_T = \xi_L = \frac{\psi'}{\psi}, \quad \zeta_D = \lambda\frac{\psi'}{\psi} \quad \text{and} \quad \zeta_S = \frac{\psi'/\psi}{1 - 2c\cdot x + c^2x^2}.$$

5.2.3 Transformation of fields and conformal weights

Conformal transformations are particular coordinate transformations and tensor fields transform under small transformations as

$$\delta_X T_{\rho\sigma\dots} = \frac{d}{d\epsilon}\left(\frac{\partial x^\mu}{\partial y^\rho}\frac{\partial x^\nu}{\partial y^\sigma}\dots T_{\mu\nu\dots}(x(y))\right)|_{\epsilon=0} \equiv L_X T_{\rho\sigma\dots}, \tag{5.56}$$

The infinitesimal transformations of a matter field is given by the Lie derivative L_X. For example, for a vector field V_μ it is

$$L_X V_\mu = X^\rho \partial_\rho V_\mu + X,^\rho_\mu V_\rho. \tag{5.57}$$

Inserting the corresponding Killing field one finds

$$\begin{array}{rl} \text{transl.} & \mathrm{i}L_a V_\mu = a^\rho P_\rho V_\mu \\ \text{Lorentztrf.} & \mathrm{i}L_\omega V_\mu = \frac{1}{2}\omega^{\rho\sigma} L_{\rho\sigma} V_\mu + \mathrm{i}\omega^\rho{}_\mu V_\rho \\ \text{dilatation} & \mathrm{i}L_\lambda V_\mu = \lambda D V_\mu + \mathrm{i}\lambda V_\mu \\ \text{spez. Trf.} & \mathrm{i}L_c V_\mu = c^\rho K_\rho V_\mu + 2\mathrm{i}(c_\mu x^\rho - x_\mu c^\rho + c\cdot x\, \delta^\rho_\mu) V_\rho, \end{array} \tag{5.58}$$

and similarly for tensor fields of higher rank [25].
The algebra of infinitesimal transformations of the fields is the same as that of the generators of the symmetry since

$$[L_X, L_Y] = L_{[X,Y]}.$$

The transformations

$$x^\mu \longrightarrow x^\mu + X^\mu \quad \text{and} \quad \phi \longrightarrow L_X \phi \tag{5.59}$$

is a symmetry of any diffeomorphism invariant theory if one also maps the metric $\eta_{\mu\nu}$ to $e^\varphi \eta_{\mu\nu}$. For a conformally invariant theory the transformation of the metric can be un-done by a compensating Weyl-transformations [26] such that

$$\delta_X T_{\mu\nu\dots} = \left(L_X - \frac{2\alpha}{d} \partial_\rho X^\rho \right) T_{\mu\nu\dots}. \tag{5.60}$$

The real constant α is the *Weyl-weight* of $T_{\mu\nu\dots}$. It can be determined by the following recipe: If the metric transforms as $g_{\mu\nu} \to e^\varphi g_{\mu\nu}$ then the field must transform as $\phi \to e^{\alpha\varphi}\phi$ in order for the action to be invariant. One finds the following weights for scalar-, spinor- and vector fields in d dimensions:

$$\phi: \quad -2\alpha = \frac{d-2}{2}, \quad \psi: \quad -2\alpha = \frac{d-1}{2}, \quad A_\mu \text{ in d=4}: \quad \alpha = 0. \tag{5.61}$$

In particular, for dilatations

$$T_{\mu\nu\dots} \longrightarrow \lambda(x^\rho \partial_\rho + s - 2\alpha) T_{\mu\nu\dots}, \tag{5.62}$$

where s is the number of covariant minus the number of contravariant indexes of T. The number $\Delta_\phi = s - 2\alpha$ is called *conformal weight* of ϕ.

5.2.4 Noether currents

We assume that the action is invariant under conformal transformations of the fields and calculate the corresponding Noether currents. For the action to be invariant the Lagrangean density must have Weyl-weight $\alpha = -d/2$ and

$$\delta_X \mathcal{L} = (X^\mu \partial_\mu \mathcal{L} + \partial_\mu X^\mu)\mathcal{L} = \partial_\mu V^\mu, \qquad V^\mu = X^\mu \mathcal{L}. \tag{5.63}$$

The Noether current for the conformal symmetry has the form

$$J_X^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left(L_X - \frac{2\alpha}{d} \partial_\rho X^\rho \right) \phi - X^\mu \mathcal{L}$$

Recalling the form of the canonical energy-momentum tensor (5.20) this can be rewritten as

$$J_X^\mu = T^\mu_{\ \nu} X^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \left(L_X - X^\rho \partial_\rho - \frac{2\alpha}{d} \partial_\rho X^\rho \right) \phi. \tag{5.64}$$

In particular for dilatations

$$J_D^\mu = T^\mu_{\ \nu} x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta_\phi \phi,$$

where $\Delta_\phi = s - 2\alpha$ is the conformal weight of ϕ. Note that

$$\Delta_{\text{scalar}} = \frac{d-2}{2}, \quad \Delta_{\text{Dirac}} = \frac{d-1}{2} \quad \text{and} \quad \Delta_{\text{Photon}}(d=4) = 1.$$

The conserved *dilation charge* is

$$Q_D = \int_{x^0} d\boldsymbol{x} \, \left(T^0_{\ \nu} x^\nu + \Delta_\phi \phi \, \pi \right) = \int_{x^0} d\boldsymbol{x} \, \left(x^0 \mathcal{H} + \pi (\boldsymbol{x} \nabla \phi) + \Delta_\phi \phi \, \pi \right).$$

As expected, this charge generated the infinitesimal dilatations,

$$\{\phi, Q_D\} = x^\mu \partial_\mu \phi + \Delta_\phi \phi.$$

For example, for a free neutral scalar field with Lagrangean $\frac{1}{2}(\partial\phi)^2$ we find

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L} \quad \text{and} \quad J_D^\nu = T^\mu_{\ \nu} x^\nu + \tfrac{1}{2}(d-2)\, \phi \, \partial^\mu \phi.$$

This current is conserved on shell, as it must be.

5.3 Implementing symmetries in quantised theories

In a quantum field theory for the field ϕ we assign to the classical field the operator $\hat{\phi}(x)$ acting on a suitable Hilbert space $\mathcal{H}$. In accordance with *Wigner's theorem* we assume that the symmetries are represented by unitary representation on $\mathcal{H}$[1]. For the Poincaré group this means

$$U(a,A)\hat{\phi}(x)U^{-1}(a,A) = S\,\hat{\phi}\left((\Lambda^{-1}(x-a)\right) \tag{5.65}$$

with unitary $U(a,A)$ for A from the spin group which is the quantum analog of the Lorentz group. Λ is the Lorentz transformation belonging to A and $S(A)$ is the representation according to which the field transforms. For translations we have

$$U(a) = \exp{(ia^\mu \hat{P}_\mu)}, \tag{5.66}$$

such the infinitesimal form of (5.65) reads

$$[\hat{P}_\mu, \hat{\phi}] = P_\mu \hat{\phi} \quad \text{with} \quad P_\mu = i\partial_\mu. \tag{5.67}$$

For spin transformations we write

$$U(A) = \exp{(\tfrac{i}{2}\omega^{\mu\nu}\hat{J}_{\mu\nu})} \tag{5.68}$$

such that

$$[\hat{J}_{\nu\nu}, \hat{\phi}] = J_{\mu\nu}\hat{\phi} \quad \text{with} \quad J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}. \tag{5.69}$$

Since $(a,A) \to U(a,A)$ is a representation of the quantum mechanical Poincaré group, the infinitesimal generators $(\hat{P}_\mu, \hat{J}_{\mu\nu})$ form a Poincaré algebra.
Let us assume, that the vacuum is left invariant by Poincaré transformations,

$$U(a,A)|0\rangle = |0\rangle. \tag{5.70}$$

It follows at once that a Greenfunction is invariant under a simultaneous shift of its arguments. For example, the 2-point function

$$\begin{aligned} \langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|e^{-i(xP)}\phi(0)e^{i(x-y)P}\phi(0)e^{iyP}|0\rangle \\ &= \langle 0|\phi(x-y)\phi(0)|0\rangle = S(x-y)\,, \end{aligned} \tag{5.71}$$

is just a function of the vector connecting it two arguments. This expresses the homogeneity of space and time. We have skipped the hats. The Greenfunctions also transform covariant under Lorentz transformations. For example,

$$S(x) = \langle 0|\phi(x)\phi(0)|0\rangle = \langle 0|U^{-1}(A)S\phi(\Lambda^{-1}x)\,S\phi(0)U(A)|0\rangle = \langle 0|S\phi(\Lambda^{-1}x)\,S\phi(0)|0\rangle \tag{5.72}$$

For the 2-point function of a spin-1 field this means

$$S_{\mu\nu}(x) \equiv \langle 0|A_\mu(x)A_\nu(0)|0\rangle = \Lambda_\mu^{\ \rho}\Lambda_\nu^{\ \sigma}S_{\rho\sigma}\left(\Lambda^{-1}x\right). \tag{5.73}$$

This condition heavily constrains the Lorentz structure of the propagator in momentum space.

[1]antilinear and antiunitary implementations would be the only other possibility.

How does one (formally) construct the infinitesimal generators in the quantum theory? For that we recall that Poincaré brackets should be replaced by $-i$ times the commutator,

$$\{\phi,\pi\}=\delta\longrightarrow[\hat{\phi},\hat{\pi}]=\mathrm{i}\delta\quad\text{and}\quad\{\phi,Q_X\}=\delta_X\phi\longrightarrow[\hat{\phi},\hat{Q}_X]=\mathrm{i}\delta_X\hat{\phi}.$$

The last relation states, that $\hat{Q}_X$ generates the infinitesimal symmetries,

$$\delta_X\hat{\phi}=i[\hat{Q}_X,\hat{\phi}],\tag{5.74}$$

and thus is the infinitesimal form of the transformation

$$\hat{\phi}\longrightarrow U\hat{\phi}U^{-1},\qquad U=\exp\left(\mathrm{i}\hat{Q}\right),\tag{5.75}$$

in (5.65). We conclude, that the symmetry generators $\hat{Q}$ in quantum field theory are (formally) constructed by replacing the classical field and momentum density in the expression for the classical Noether charges by the field operator and the momentum field operator.

Kapitel 6

The Birth of Supersymmetry

40 years ago the idea came up that perhaps the approximate $SU(3)$ symmetry of strong interaction is part of a larger $SU(6)$ symmetry. Under this bigger symmetry mesons (or baryons) with different spins belong to one multiplet. There were various attempts to generalise the $SU(6)$ symmetry of the non-relativistic quark model to a fully relativistic quantum field theory. These attempts failed, and several authors proved no-go theorems showing that in fact this is impossible. Well-known is the *Coleman-Mandula theorem* which we shall sketch now.

6.1 Coleman Mandula Theorem

In order to better understand the celebrated Coleman-Mandula theorem we consider the theory for one free real scalar field and one free vector field of equal mass m,

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi + \partial_\mu A^\rho\partial^\mu A_\rho - m^2\phi^2 - m^2 A^\rho A_\rho) \tag{6.1}$$

with linear Euler-Lagrange equations

$$\Box\phi + m^2\phi = 0 \quad \text{and} \quad \Box A^\rho + m^2 A^\rho = 0. \tag{6.2}$$

There is an infinite number of conserved currents. For example, the tensor current

$$J_{\mu\nu} = \phi\overleftrightarrow{\partial}_\mu A_\nu = \phi\partial_\mu A_\nu - \partial_\mu\phi A_\nu \tag{6.3}$$

is conserved for solutions of the field equations,

$$\partial^\mu J_{\mu\nu} = \phi\Box A_\mu - \Box\phi A_\mu = -(m^2 - m^2)\phi A_\nu = 0.$$

Since the current is conserved for solutions of the equations of motion only one says that it is conserved *on shell*. This tensor is one out of an infinity of conserved currents of the form

$$J_{\mu_1\dots\mu_n} = \phi\overleftrightarrow{\partial}_{\mu_1}\dots\overleftrightarrow{\partial}_{\mu_n}\phi \quad (n\in 2\mathbb{N}) \quad \text{and} \quad J_{\mu_1\dots\mu_n\nu} = \phi\overleftrightarrow{\partial}_{\mu_1}\dots\overleftrightarrow{\partial}_{\mu_n}A_\nu \quad (n\in\mathbb{N}). \tag{6.4}$$

The corresponding conserved charges obtained from these covariantly conserved currents are tensorial charges of higher rank.
One may ask whether these conservation laws can be extended to the interacting case. In $d > 2$ dimensions the answer to this question is no [16, 17]. In a relativistic quantum theory with a discrete spectrum of massive (but no massless) one-particle states, and with some nontrivial scattering amplitudes, the only conserved tensorial charges that are not Lorentz scalars, are the energy-momentum vector P_μ and the angular momentum tensor $M_{\mu\nu}$ which span the Poincaré algebra. All other conserved charges must be Lorentz scalars. The total algebra is thus *always* of the form

$$iL + \mathcal{G}, \qquad [iL, \mathcal{G}] = 0. \tag{6.5}$$

In the massless case iL can be extended to the conformal algebra $so(2, d)$. Let us formulate the theorem more precisely.

Lemma 3 (The Coleman-Mandula-Theorem [16]) *Let G be a connected symmetry-group of the S-matrix, i.e. a group whose generators commute with the S-matrix, and let the following five assumptions hold:*

- *G contains a subgroup locally isomorphic to the Poincaré group.*
- *All particle types correspond to positive-energy representations of the Poincaré group. For any finite mass m there is only a finite number of particles with mass less than m.*
- *Elastic-scattering amplitudes are analytic functions of center-of-mass energy squared s and invariant momentum transfer squared t, in some neighbourhood of the physical region, except at normal thresholds.*
- *Let $|p_i\rangle$, $i = 1, 2$ be any two 1-particle states and let $|p_1 p_2\rangle$ be the 2-particle state constructed from these. Then $S|p_1 p_2\rangle \neq |p_1 p_2\rangle$ except, perhaps, for certain isolated values of s.*
- *The generators of G, considered as integral operators in momentum space, have distributions for their kernels.*

Then G is locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.

A unitary operator U on the Hilbert space is said to be a symmetry transformation of the S-matrix if:

- U transforms 1-particle states into 1-particle states.
- U acts on many-particle states as if they were tensor products of one-particle states.
- U commutes with the scattering matrix S.

The theorem implies that the most general symmetry algebra of the S-matrix contains the energy momentum operator P_μ, the Lorentz generators $M_{\mu\nu}$ and a finite number of Lorentz scalar operators B_l, that is:

$$[P_\mu, B_l] = 0 \quad , \quad [M_{\mu\nu}, B_l] = 0, \tag{6.6}$$

where the B_l constitute a Lie-algebra with structure constants ${c_{lm}}^k$:

$$[B_l, B_m] = i{c_{lm}}^k B_k. \tag{6.7}$$

It follows at once, that the Casimir-operators of the Poincaré algebra, P^2 and W^2 commute not only with the Poincaré group but also with all internal symmetry generators,

$$[B_l, P^2] = 0 \quad \text{and} \quad [B_l, W^2] = 0. \tag{6.8}$$

The first set of commutators implies that all members of an irreducible multiplet of the internal symmetry group have the same mass. This is known as *O'Raifeartaigh's theorem* [34]. The second set of commutators says that all members have the same spin. For massless states with discrete helicities we have

$$W_\mu = \lambda P_\mu, \qquad \lambda \in \{0, \tfrac{1}{2}, 1, \dots\} \tag{6.9}$$

and no generator B_l can change the helicity since $[B_l, \lambda] = 0$.
To see the arguments leading to the Coleman-Mandula theorem consider a forbidden tensorial charge $\mathcal{Q}_{\mu\nu}$ which for simplicity we shall assume to be traceless, $\mathcal{Q}^\mu_\mu = 0$. Assume that a scalar particle of mass m, carrying the charge $\mathcal{Q}_{\mu\nu}$, appears in the theory and let $|p\rangle$ be a corresponding one-particle state with mass $m^2 = p^2$. Then

$$\langle p|\mathcal{Q}_{\mu\nu}|p\rangle = \left(p_\mu p_\nu - \frac{p^2}{d}\eta_{\mu\nu}\right) C, \quad C \neq 0. \tag{6.10}$$

We consider a $2 \to 2$ scattering process. The incoming particles with momenta p_1, p_2 scatter and then go out with final momenta p_1' and p_2'. The conservation law of $\mathcal{Q}$ applied between asymptotic incoming and outgoing states requires

$$C\left(p_{1\mu}p_{1\nu} + p_{2\mu}p_{2\nu} - \frac{1}{d}\eta_{\mu\nu}(m^2+m^2)\right) = C\left(p'_{1\mu}p'_{1\nu} + p'_{2\mu}p'_{2\nu} - \frac{1}{d}\eta_{\mu\nu}(m^2+m^2)\right).$$

If $C \neq 0$, these equations imply that the scattering must proceed either in the forward or backward direction whereas in all other directions there is no scattering. This conflicts the analyticity properties of scattering amplitudes in more then 2 dimensions and thus $C = 0$. No interacting theory can carry the charge $\mathcal{Q}_{\mu\nu}$. Similar arguments can be used for amplitudes of non-identical particles to prove this 'no-go' theorem.
The Coleman-Mandula theorem shows the impossibility of nontrivial symmetries that connect particles of different spins.

6.2 The Wess-Zumino-Model

We study the simplest supersymmetric model in four spacetime dimensions. It has been constructed by WESS and ZUMINO [29] when they extended the 2-dimensional supersymmetric string-model of GERVAIS and SAKITA[1] [30] to four dimensions. The model contains a supermultiplet with

- a single Majorana field ψ
- a pair of real scalar and pseudo-scalar bosonic fields A and B
- a pair of real scalar and pseudoscalar bosonic auxiliary fields $\mathcal{F}$ and $\mathcal{G}$.

With (4.79) we have the hermiticity properties

$$\begin{aligned} \bar{\varepsilon}\psi,\ \bar{\varepsilon}\gamma^{\mu\nu}\psi,\ \bar{\varepsilon}\gamma_5\gamma^\mu\psi &: \text{ hermitian} \\ \bar{\varepsilon}\gamma_5\psi,\ \bar{\varepsilon}\gamma^\mu\psi &: \text{ antihermitian.} \end{aligned} \tag{6.11}$$

Bilinears without γ_5 are tensor fields and those with γ_5 are pseudo-tensor fields.

6.2.1 The free Model in the on-shell formulation

The hermitean Lagrangean should give rise to a translational and Lorentz invariant action. The action is dimensionless in units for which $\hbar = 1$ and thus $\mathcal{L}$ has dimension

$$[\mathcal{L}] = L^{-4} \quad \text{or} \quad [\mathcal{L}] = m^4. \tag{6.12}$$

In a supersymmetric model we expect an equal number of bosonic and fermionic states and bosons and fermions of equal mass. For example, a spin 1/2-fermion which is its own antiparticle has 2 polarisation states and should be accompanied by two neutral scalar particles. Without interaction the Lagrangean of this model takes the form

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu A)^2 - \tfrac{1}{2}m^2A^2 + \tfrac{1}{2}(\partial_\mu B)^2 - \tfrac{1}{2}m^2B^2 + \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}m\bar{\psi}\psi = \mathcal{L}_0 + \mathcal{L}_m. \tag{6.13}$$

where A and B are real scalar fields and ψ a Majorana spinor. Later it will turn out that B is a real pseudo-scalar field. The unfamiliar factor $\frac{1}{2}$ multiplying the Dirac-term arises because ψ is an uncharged Majorana spinor. The dimensions of the various fields are

$$[A] = [B] = L^{-1} \quad \text{and} \quad [\psi] = L^{-3/2}.$$

The field equations are the Klein-Gordon and Dirac equation

$$(\Box + m^2)A = 0, \quad (\Box + m^2)B = 0 \quad \text{and} \quad (\mathrm{i}\partial\!\!\!/ - m)\psi = 0. \tag{6.14}$$

[1] which was based on earlier work of RAMOND [31] and NEVEU and SCHWARTZ [32]

Besides the well-known space-time symmetries the action $S=\int d^4x\,\mathcal{L}$ admits the following supersymmetry transformations

$$\begin{aligned}\delta_\varepsilon A &= \bar\varepsilon\psi\\ \delta_\varepsilon B &= \mathrm{i}\bar\varepsilon\gamma_5\psi\\ \delta_\varepsilon\psi &= -(\mathrm{i}\partial\!\!\!/ + m)(A+\mathrm{i}\gamma_5 B)\varepsilon,\end{aligned}\tag{6.15}$$

where ε is an arbitrary constant infinitesimal Majorana fermion c-number parameter with dimensions $[\varepsilon]=L^{1/2}$. These transformations map bosons into fermions and vice versa. Note that $\bar\varepsilon\psi$ is a hermitean scalar and $\mathrm{i}\bar\varepsilon\gamma_5\psi$ a hermitean pseudo-scalar in accordance with A and B being hermitean (pseudo)-scalars. With $\gamma^0\gamma_\mu=\gamma_\mu^\dagger\gamma^0$ and $\gamma_5^\dagger=\gamma_5$ the variation of the Dirac conjugate spinor reads

$$\delta_\varepsilon\bar\psi=(\delta_\varepsilon\psi)^\dagger\gamma^0=\bar\varepsilon\left(\mathrm{i}\partial\!\!\!/(A-\mathrm{i}\gamma_5B)-m(A+\mathrm{i}\gamma_5B)\right).\tag{6.16}$$

We shall now prove in several steps, that the Lagrangean density is invariant up to a total divergence which means, that the action is invariant. To prove this property requires some lengthy calculations. For newcomers to supersymmetry this proof is a very useful exercise. To prove the invariance of (6.13) we need

$$\begin{aligned}\tfrac12\delta_\varepsilon(\partial_\mu A\partial^\mu A\ +\ \partial_\mu B\partial^\mu B) &= \bar\varepsilon\partial^\mu(A+\mathrm{i}\gamma_5B)\partial_\mu\psi\\ \tfrac12\delta_\varepsilon\left(A^2+B^2\right) &= \bar\varepsilon(A+\mathrm{i}\gamma_5B)\psi\\ \delta_\varepsilon(\bar\psi\partial\!\!\!/\psi) &= \delta_\varepsilon\bar\psi\gamma^\mu\partial_\mu\psi+\bar\psi\gamma^\mu\partial_\mu\delta_\varepsilon\psi=-\partial_\mu\left(\delta_\varepsilon\bar\psi\gamma^\mu\psi\right) &(6.17)\\ &= -\bar\varepsilon\partial_\mu\left(\mathrm{i}\partial\!\!\!/(A-\mathrm{i}\gamma_5B)\gamma^\mu\psi-m(A+\mathrm{i}\gamma_5B)\gamma^\mu\psi\right) &(6.18)\\ \tfrac12\delta_\varepsilon(\bar\psi\psi) &= \tfrac12(\delta_\varepsilon\bar\psi\psi+\bar\psi\delta_\varepsilon\psi)=\delta_\varepsilon\bar\psi\psi=\bar\varepsilon\left(\mathrm{i}\partial\!\!\!/(A-\mathrm{i}\gamma_5B)\psi-m(A+\mathrm{i}\gamma_5B)\psi\right)\end{aligned}$$

where we used that for Majorana spinors $\chi\gamma^\mu\psi=-\bar\psi\gamma^\mu\chi$ and $\bar\chi\psi=\bar\psi\chi$ hold true.

6.2.2 The interacting model in the off-shell formulation

The Lagrangean should be hermitean and should give rise to a translational and Lorentz invariant action. The action is dimensionless in units with $\hbar=1$ and thus $\mathcal{L}$ must have dimension

$$[\mathcal{L}]=L^{-4}\quad\text{or}\quad[\mathcal{L}]=m^4.\tag{6.19}$$

The Lagrangean density is

$$\begin{aligned}\mathcal{L} &= \tfrac12\partial_\mu A\partial^\mu A+\tfrac12\partial_\mu B\partial^\mu B+\tfrac{i}{2}\bar\psi\partial\!\!\!/\psi+\tfrac12\left(\mathcal{F}^2+\mathcal{G}^2\right)\\ &\quad+m\Big(\mathcal{F}A+\mathcal{G}B-\tfrac12\bar\psi\psi\Big)\\ &\quad+g\Big(\mathcal{F}(A^2-B^2)+2\mathcal{G}AB-\bar\psi(A-\mathrm{i}\gamma_5B)\psi\Big) &(6.20)\\ &= \mathcal{L}_0+\mathcal{L}_m+\mathcal{L}_g.\end{aligned}$$

Parity invariance of $\mathcal{L}$ demands that A, which couples to the scalar $\bar\psi\psi$, is a scalar field and B, which couples to the pseudo-scalar $\bar\psi\gamma_5\psi$, is a pseudo-scalar field. The fields $\mathcal{F}$ and $\mathcal{G}$ enter only algebraically and play the role as Lagrangean multiplier fields. The terms $A\mathcal{F}$ and $B\mathcal{G}$ are parity even if $\mathcal{F}$ is a scalar and $\mathcal{G}$ a pseudo-scalar field. The dimensions of the various fields are

$$[A] = [B] = L^{-1}, \quad [\psi] = L^{-3/2} \quad \text{and} \quad [\mathcal{F}] = [\mathcal{G}] = L^{-2}. \tag{6.21}$$

The field equations are

$$\begin{aligned} 0 &= -\Box A + m\mathcal{F} + 2g\mathcal{F}A + 2g\mathcal{G}B - g\bar\psi\psi \\ 0 &= -\Box B + m\mathcal{G} - 2g\mathcal{F}B + 2g\mathcal{G}A + ig\bar\psi\gamma_5\psi \\ 0 &= \mathcal{F} + mA + g(A^2 - B^2) \\ 0 &= \mathcal{G} + mB + 2gAB \\ 0 &= \mathrm{i}\partial\!\!\!/\psi - m\psi - 2g(A - \mathrm{i}\gamma_5 B)\psi \end{aligned} \tag{6.22}$$

To discover the Fermi-Bose symmetries

$$\delta(\text{Boson}) = \text{Fermion} \quad \text{and} \quad \delta(\text{Fermion}) = \text{Boson}$$

we make the following infinitesimal supersymmetry transformations:

$$\begin{aligned} \delta_\varepsilon A &= \bar\varepsilon\psi, \qquad \delta_\varepsilon B = \mathrm{i}\bar\varepsilon\gamma_5\psi \\ \delta_\varepsilon\psi &= -\mathrm{i}\partial\!\!\!/(A + \mathrm{i}\gamma_5 B)\varepsilon + (\mathcal{F} + \mathrm{i}\gamma_5\mathcal{G})\varepsilon \\ \delta_\varepsilon\mathcal{F} &= -\mathrm{i}\bar\varepsilon\partial\!\!\!/\psi, \qquad \delta_\varepsilon\mathcal{G} = \bar\varepsilon\gamma_5\partial\!\!\!/\psi, \end{aligned} \tag{6.23}$$

where ε is an arbitrary constant infinitesimal Majorana fermion c-number parameter with dimensions $[\varepsilon] = L^{1/2}$. These transformations transform bosons into fermions and vice versa and do not contain the mass or coupling constants of the model.
Note that for example $\bar\varepsilon\psi$ is a hermitean scalar and $\mathrm{i}\bar\varepsilon\gamma_5\psi$ a hermitean pseudo-scalar, as required, since A and B are hermitean (pseudo)-scalars. The variation of the Dirac conjugate spinor is

$$\delta_\varepsilon\bar\psi = \delta_\varepsilon\psi^\dagger\gamma^0 = \mathrm{i}\bar\varepsilon\partial\!\!\!/(A - \mathrm{i}\gamma_5 B) + \bar\varepsilon(\mathcal{F} + \mathrm{i}\gamma_5\mathcal{G}). \tag{6.24}$$

The corresponding action is invariant under the supersymmetry transformation. We shall now prove in several steps, that the Lagrangean density is invariant up to a total divergence.
To prove the invariance of the first line in (6.20) we need

$$\begin{aligned} \tfrac12\delta_\varepsilon(\partial^\mu A\partial_\mu A \;+\; \partial^\mu B\partial_\mu B) &= \bar\varepsilon\partial^\mu(A + \mathrm{i}\gamma_5 B)\partial_\mu\psi \\ \tfrac12\delta_\varepsilon(\mathcal{F}^2 + \mathcal{G}^2) &= -\mathrm{i}\bar\varepsilon(\mathcal{F} + \mathrm{i}\gamma_5\mathcal{G})\partial\!\!\!/\psi \\ \delta_\varepsilon(\bar\psi\partial\!\!\!/\psi) &= \mathrm{i}\bar\varepsilon\partial\!\!\!/(A-\mathrm{i}\gamma_5 B)\partial\!\!\!/\psi + \bar\varepsilon(\mathcal{F}+\mathrm{i}\gamma_5\mathcal{G})\partial\!\!\!/\psi \\ &\quad -\mathrm{i}\bar\varepsilon\Box(A+\mathrm{i}\gamma_5 B)\psi - \bar\varepsilon\partial\!\!\!/(\mathcal{F}-\mathrm{i}\gamma_5\mathcal{G})\psi, \end{aligned}$$

where we used

$$\bar\varepsilon\psi=\bar\psi\varepsilon,\quad \bar\varepsilon\gamma_5\psi=\bar\psi\gamma_5\varepsilon,\quad \bar\varepsilon\gamma^\mu\psi=-\bar\psi\gamma^\mu\varepsilon\quad\text{and}\quad \bar\varepsilon\gamma_5\gamma^\mu\psi=\bar\psi\gamma_5\gamma^\mu\varepsilon. \tag{6.25}$$

The (weighted) sum of these three terms yields the variation of $\mathcal{L}_0$:

$$\delta_\varepsilon\mathcal{L}_0=\partial_\mu(\bar\varepsilon V_0^\mu),\qquad V_0^\mu=\tfrac12\gamma^\mu\{\partial\!\!\!/(A+\mathrm{i}\gamma_5B)-i(\mathcal{F}-\mathrm{i}\gamma_5\mathcal{G})\}\psi. \tag{6.26}$$

This already shows, that the massless non-interacting model is invariant under susy-transformations. Actually this simple subsector is invariant not only under susy but under superconformal transformations. The superconformal group is the generalisation of the conformal group.
For the variation of the mass-term we need

$$\begin{aligned}\delta_\varepsilon(\mathcal{F}A+\mathcal{G}B)&=\bar\varepsilon(\mathcal{F}+\mathrm{i}\gamma_5\mathcal{G})\psi-\mathrm{i}\bar\varepsilon(A+\mathrm{i}\gamma_5B)\partial\!\!\!/\psi\\ \delta_\varepsilon(\bar\psi\psi)&=2\bar\varepsilon(\mathcal{F}+\mathrm{i}\gamma_5\mathcal{G})\psi+2\mathrm{i}\bar\varepsilon\partial\!\!\!/(A-\mathrm{i}\gamma_5B)\psi\end{aligned}$$

from which follows that

$$\delta_\varepsilon\mathcal{L}_m=\partial_\mu(\bar\varepsilon V_m^\mu),\qquad V_m^\mu=-im(A+\mathrm{i}\gamma_5B)\gamma^\mu\psi. \tag{6.27}$$

This shows, that the massive non-interacting model is invariant under susy-transformations.
Finally, for the variation of the interaction term we need

$$\begin{aligned}&\delta_\varepsilon(\mathcal{F}A^2-\mathcal{F}B^2)=\bar\varepsilon\{2\mathcal{F}(A-\mathrm{i}\gamma_5B)-i(A^2-B^2)\partial\!\!\!/\}\psi\\ &\delta_\varepsilon(2AB\mathcal{G})=2\bar\varepsilon(AB\gamma_5\partial\!\!\!/+B\mathcal{G}+iA\mathcal{G}\gamma_5)\psi\\ &\delta_\varepsilon(A\bar\psi\psi)=(\bar\varepsilon\psi)(\bar\psi\psi)+2A\bar\varepsilon\{\mathcal{F}+\mathrm{i}\gamma_5\mathcal{G}+\mathrm{i}\partial\!\!\!/(A-\mathrm{i}\gamma_5B)\}\psi\\ &\delta_\varepsilon(\bar\psi\gamma_5\psi)=2\bar\varepsilon\{(\mathcal{F}+\mathrm{i}\gamma_5\mathcal{G})+\mathrm{i}\partial\!\!\!/(A-\mathrm{i}\gamma_5B)\}\gamma_5\psi\\ &\delta_\varepsilon(B\bar\psi\gamma_5\psi)=i(\bar\varepsilon\gamma_5\psi)(\bar\psi\gamma_5\psi)+2B\bar\varepsilon\{(\mathcal{F}+\mathrm{i}\gamma_5\mathcal{G})+\mathrm{i}\partial\!\!\!/(A-\mathrm{i}\gamma_5B)\}\gamma_5\psi.\end{aligned}$$

Using $(\bar\psi\psi)(\bar\varepsilon\psi)+(\bar\psi\gamma_5\psi)(\bar\varepsilon\gamma_5\psi)=0$ which follows from (4.87), we see that the variation of the interaction term is also a total divergence,

$$\delta_\varepsilon\mathcal{L}_g=\partial_\mu(\bar\varepsilon V_g^\mu),\qquad V_g^\mu=-ig(A+\mathrm{i}\gamma_5B)^2\gamma^\mu\psi. \tag{6.28}$$

Hence, the Lagrangean density transforms under supersymmetry transformations into a total divergence,

$$\begin{aligned}\delta_\varepsilon\mathcal{L}&=\partial_\mu(\bar\varepsilon V^\mu)\\ V^\mu&=\gamma^\mu\{\tfrac12\partial\!\!\!/(A+\mathrm{i}\gamma_5B)-\tfrac{i}{2}(\mathcal{F}-\mathrm{i}\gamma_5\mathcal{G})-im(A-\mathrm{i}\gamma_5B)-ig(A-\mathrm{i}\gamma_5B)^2\}\psi.\end{aligned} \tag{6.29}$$

This shows, that action (6.13) is invariant under the supersymmetry transformations (6.23).
The commutator of two successive transformations of a symmetry group must itself be a symmetry transformation. This way we can identify the algebra of group generators. Let us see what the commutator of two susy transformations looks like. For example,

$$\begin{aligned}[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]A&=\delta_{\varepsilon_1}(\bar\varepsilon_2\psi)-\delta_{\varepsilon_2}(\bar\varepsilon_1\psi)=-\mathrm{i}\bar\varepsilon_2\partial\!\!\!/(A+\mathrm{i}\gamma_5B)\varepsilon_1+\bar\varepsilon_2(\mathcal{F}+\mathrm{i}\gamma_5\mathcal{G})\varepsilon_1-(1\leftrightarrow2)\\ &=-2\mathrm{i}\bar\varepsilon_2\gamma^\mu\varepsilon_1\partial_\mu A.\end{aligned} \tag{6.30}$$

Similarly one finds

$$\begin{aligned}[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]B &= -2\mathrm{i}\bar\varepsilon_2\gamma^\mu\varepsilon_1\partial_\mu B,\\ [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\mathcal{F} &= -2\mathrm{i}\bar\varepsilon_2\gamma^\mu\varepsilon_1\partial_\mu \mathcal{F},\\ [\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\mathcal{G} &= -2\mathrm{i}\bar\varepsilon_2\gamma^\mu\varepsilon_1\partial_\mu \mathcal{G}.\end{aligned}$$

Computing the commutator on the spin-$\frac{1}{2}$ field is more involved. First,

$$\begin{aligned}[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi &= \partial_\mu\{-i(\bar\varepsilon_1\psi)\gamma^\mu + i(\bar\varepsilon_1\gamma_5\psi)\gamma^\mu\gamma_5 - i(\bar\varepsilon_1\gamma^\mu\psi) + i(\bar\varepsilon_1\gamma_5\gamma^\mu\psi)\gamma_5\}\varepsilon_2 - \{1\leftrightarrow 2\}\\ &= \{-\mathrm{i}\gamma^\mu M + \mathrm{i}\gamma^\mu\gamma_5 M\gamma_5 - iM\gamma^\mu + \mathrm{i}\gamma_5 M\gamma_5\gamma^\mu\}\partial_\mu\psi, \qquad (6.31)\end{aligned}$$

where we have introduced

$$M = \varepsilon_2\bar\varepsilon_1 - \varepsilon_1\bar\varepsilon_2 = -\tfrac{1}{2}\gamma_\rho(\bar\varepsilon_1\gamma^\rho\varepsilon_2) + \gamma_{\rho\sigma}(\bar\varepsilon_1\gamma^{\rho\sigma}\varepsilon_2) \qquad (6.32)$$

such that

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}]\psi = -2\mathrm{i}(\bar\varepsilon_2\gamma^\mu\varepsilon_1)\partial_\mu\psi. \qquad (6.33)$$

We see that on all fields the commutator of the supersymmetry transformations yield a translation,

$$[\delta_{\varepsilon_1},\delta_{\varepsilon_2}] = -2\mathrm{i}(\bar\varepsilon_2\gamma^\mu\varepsilon_1)\partial_\mu. \qquad (6.34)$$

Is is not difficult to see, that the supersymmetry transformations commute with the translations,

$$[\delta_\varepsilon,\delta_a] = 0, \qquad \delta_a = a^\mu\partial_\mu.$$

Let us finally calculate the commutator of Lorentz transformations and supersymmetry transformations. For example, for the A field

$$[\delta_\varepsilon,\delta_\omega]A = \delta_\varepsilon\{\tfrac{i}{2}(\omega,L)A\} - \delta_\omega\bar\varepsilon\psi = \tfrac{i}{2}\{(\omega,L)\bar\varepsilon\psi - \bar\varepsilon(\omega,J)\psi\} = -\tfrac{i}{2}\omega_{\mu\nu}\bar\varepsilon\,\Sigma^{\mu\nu}\psi,$$

where $J = L + \Sigma$. Similarly one finds

$$\begin{aligned}[\delta_\varepsilon,\delta_\omega]B &= \tfrac{1}{2}\omega_{\mu\nu}\bar\varepsilon\gamma_5\Sigma^{\mu\nu}\psi\\ [\delta_\varepsilon,\delta_\omega]\mathcal{F} &= -\tfrac{1}{2}\omega_{\mu\nu}\bar\varepsilon\partial\!\!\!/\Sigma^{\mu\nu}\psi\\ [\delta_\varepsilon,\delta_\omega]\mathcal{G} &= -\tfrac{i}{2}\omega_{\mu\nu}\bar\varepsilon\gamma_5\partial\!\!\!/\Sigma^{\mu\nu}\psi\\ [\delta_\varepsilon,\delta_\omega]\psi &= \tfrac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\delta_\varepsilon\psi.\end{aligned}$$

The generator $\mathcal{Q}_\alpha$ of supersymmetriy transformations is a four component Majorana spinor, which we define by the requirement

$$\delta_\varepsilon A = \bar\varepsilon\mathcal{Q}A, \quad \delta_\varepsilon B = \bar\varepsilon\mathcal{Q}B, \ldots.$$

Consistency with (6.34) requires

$$\begin{aligned}[][\delta_{\varepsilon_1},\delta_{\varepsilon_2}] &= [\bar\varepsilon_1\mathcal{Q},\bar\varepsilon_2\mathcal{Q}] = [\bar{\mathcal{Q}}\varepsilon_1,\bar\varepsilon_2\mathcal{Q}] = (\bar{\mathcal{Q}}^\beta\varepsilon_{1\beta}\bar\varepsilon_2^\alpha\mathcal{Q}_\alpha - \bar\varepsilon_2^\alpha\mathcal{Q}_\alpha\bar{\mathcal{Q}}^\beta\varepsilon_{1\beta})\\ &= \varepsilon_{1\beta}\bar\varepsilon_2^\alpha(\bar{\mathcal{Q}}^\beta\mathcal{Q}_\alpha + \mathcal{Q}_\alpha\bar{\mathcal{Q}}^\beta) = -2i(\bar\varepsilon_2\gamma^\mu\varepsilon_1)\partial_\mu = 2\mathrm{i}\varepsilon_{1\beta}\bar\varepsilon_2^\alpha(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu. \end{aligned} \tag{6.35}$$

Comparing the last with the third to last expression yields

$$\{\mathcal{Q}_\alpha,\bar{\mathcal{Q}}^\beta\} = 2i(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu = 2(\gamma^\mu)_\alpha^{\ \beta}P_\mu. \tag{6.36}$$

Since $\bar{\mathcal{Q}}^\beta = -\mathcal{Q}_\gamma(\mathcal{C}^{-1})^{\gamma\beta}$ and $\mathcal{C}^T = -\mathcal{C}$ we may rewrite this commutation relation as

$$\{\mathcal{Q}_\alpha,\mathcal{Q}_\beta\} = -2(\gamma^\mu\mathcal{C})_{\alpha\beta}P_\mu. \tag{6.37}$$

Also, since the susy transformations commute with the translations, we have

$$[\mathcal{Q}_\alpha, P_\mu] = 0.$$

Finally, since

$$[\delta_\varepsilon,\delta_w]A = \tfrac{i}{2}\bar\varepsilon^\alpha\omega^{\mu\nu}[\mathcal{Q}_\alpha,J_{\mu\nu}]A = -\tfrac{i}{2}\omega^{\mu\nu}\bar\varepsilon^\alpha(\Sigma_{\mu\nu}\psi)_\alpha$$

we conclude, that

$$[\mathcal{Q}_\alpha,J_{\mu\nu}]A = -(\Sigma_{\mu\nu}\mathcal{Q})_\alpha A.$$

The same holds true for the other fields, such that

$$[J_{\mu\nu},\mathcal{Q}_\alpha] = (\Sigma_{\mu\nu}\mathcal{Q})_\alpha. \tag{6.38}$$

As expected from a Majorana spinor, the supercharges transform as spin-$\frac{1}{2}$ fields.

6.2.3 On-shell formulation

The Lagrangean and equations of motion in their present form are not very illuminating. Notice, however, that the fields $\mathcal{F}$ and $\mathcal{G}$ are just Lagrangean multiplier fields. These auxiliary (dummy) fields satisfy the algebraic equations of motion

$$0 = \frac{\partial\mathcal{L}}{\partial\mathcal{F}} = \mathcal{F} + mA + g(A^2 - B^2) \quad \text{and} \quad 0 = \frac{\partial\mathcal{L}}{\partial\mathcal{G}} = \mathcal{G} + mB + 2gAB$$

see eq. (6.22), and can be eliminated from the Lagrangean and the equations of motion. With

$$\begin{aligned}&\tfrac{1}{2}(\mathcal{F}^2+\mathcal{G}^2) + m(\mathcal{F}A+\mathcal{G}B) + g\{\mathcal{F}(A^2-B^2) + 2\mathcal{G}AB\}\\ &= -\tfrac{1}{2}m^2(A^2+B^2) - mgA(A^2+B^2) - \tfrac{1}{2}g^2(A^2+B^2)^2\end{aligned}$$

the 'on-shell' Lagrangean density reads

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{1}{2}\partial_\mu B\partial^\mu B - \tfrac{1}{2}m^2(A^2+B^2) + \tfrac{i}{2}\bar\psi\partial\!\!\!/\psi - \tfrac{1}{2}m\bar\psi\psi\\ &\quad -mgA(A^2+B^2) - \tfrac{1}{2}g^2(A^2+B^2)^2 - g\bar\psi(A - \mathrm{i}\gamma_5 B)\psi.\end{aligned} \tag{6.39}$$

The field equations are

$$\begin{aligned}(\Box + m^2)A &= -mg(3A^2 + B^2) - 2g^2 A(A^2 + B^2) - g\bar{\psi}\psi \\ (\Box + m^2)B &= -2mgAB - 2g^2 B(A^2 + B^2) + ig\bar{\psi}\gamma_5\psi \\ (\mathrm{i}\not{\partial} - m)\psi &= 2g(A - \mathrm{i}\gamma_5 B)\psi.\end{aligned} \tag{6.40}$$

The following features of $\mathcal{L}$ are characteristic for supersymmetric theories:

- The scalars and the fermions have *equal mass*.
- There are quartic and cubic couplings between the scalars and a Yukawa-interaction between fermions and scalars.
- There are only few parameters: For the Wess-Zumino model m and g.

The relation between boson and fermion masses and couplings is a common feature of SUSY and it is stable under renormalisation. This has been found to be so, since there exists a Pauli-Villars regularisation which preserves supersymmetry for the model [33] Indeed, the Wess-Zumino Model has also some remarkable renormalisation properties. Despite the presence of scalar fields, there is no renormalisation of the mass and coupling constant. There is only a wave-function renormalisation. The divergences arising from boson loops are cancelled by those from fermion loops which have the opposite sign. These powerful *non-renormalisation theorems* make SUSY particularly attractive.
The on-shell transformations read

$$\begin{aligned}\delta_\varepsilon A &= \bar{\varepsilon}\psi, \\ \delta_\varepsilon B &= \mathrm{i}\bar{\varepsilon}\gamma_5\psi, \\ \delta_\varepsilon \psi &= -\{\mathrm{i}\not{\partial} + m + g(A + \mathrm{i}\gamma_5 B)\}\{A + \mathrm{i}\gamma_5 B\}\varepsilon,\end{aligned} \tag{6.41}$$

so that the Dirac-conjugated spinor transforms as

$$\delta_\varepsilon\bar{\psi} = \bar{\varepsilon}\{\mathrm{i}\not{\partial}(A - \mathrm{i}\gamma_5 B) - (m + g(A + \mathrm{i}\gamma_5 B))(A + \mathrm{i}\gamma_5 B)\}.$$

Note that these transformations have become *nonlinear* and *model-dependend* (they depend on the parameters in $\mathcal{L}$), and there is no part of the Lagrangean which separately transforms as a density under them. Of course, the action is invariant under these transformation. For example, for $m = g = 0$ we have

$$\delta_\varepsilon\mathcal{L} = \partial_\mu\bar{\varepsilon}V^\mu, \qquad V^\mu = \tfrac{1}{2}\gamma^\mu\not{\partial}(A + \mathrm{i}\gamma_5 B)\psi.$$

This is just the result for the off-shell formulation in which one replaces the auxiliary fields by their equation of motion. Now we shall see, that the susy algebra closes only on-shell. It closes off-shell on the fields A and B, but not on the fermion field,

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\psi = -\{\mathrm{i}\not{\partial} + m + 2g(A + \mathrm{i}\gamma_5 B)\}\{M - \gamma_5 M\gamma_5\}\psi,$$

where $M = \varepsilon_2\bar{\varepsilon}_1 - \varepsilon_1\bar{\varepsilon}_2$ has already been introduced in (6.32). Since

$$M - \gamma_5 M \gamma_5 = -\gamma_\rho(\bar{\varepsilon}_1\gamma^\rho\varepsilon_2) \quad \text{and} \quad \partial\!\!\!/\gamma_\rho = -\gamma_\rho\partial\!\!\!/ - 2\partial_\rho$$

we can rewrite the commutator of 2 susy-transformations on the Fermi-field as

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\psi = -2(\bar{\varepsilon}_2\gamma^\rho\varepsilon_1)\partial_\rho\psi + \gamma_\rho(\bar{\varepsilon}_2\gamma^\rho\varepsilon_1)\{\mathrm{i}\partial\!\!\!/ - m - 2g(A - \mathrm{i}\gamma_5 B)\}\psi. \qquad (6.42)$$

Only if we impose the field equation for the spinor field, then the last term vanishes and

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\psi = -2(\bar{\varepsilon}_2\gamma^\rho\varepsilon_1)\partial_\rho\psi.$$

We see, that after the elimination of the auxiliary fields the susy algebra closes only if the equations of motion hold. This could be disastrous for quantum corrections: there the fields must be taken off-shell, away from their classical paths through configuration space. However, if there is (as for the WZ-model) some off-shell version, we expect no problems. But what happens if the theory were intrinsically only on-shell supersymmetric (as $N = 4$ and $N = 8$ theories may well be) or if there exist several off-shell versions (as there are for $N = 1$ supergravity), is unclear.

Counting degrees of freedom: A heuristic understanding for the necessity of auxiliary fields in the off-shell formulation is based on a counting of degrees of freedom. In a supersymmetric theory the fermionic degrees of freedom must match the bosonic ones. In 4 dimensions a real field $A(x)$ describes one neutral scalar particle and the Dirac field $\psi(x)$ has 8 real components, but describes only 4 states of a spin-$\frac{1}{2}$ particle and its anti-particle. Correspondingly a Marjoran field describes the 2 states of a spin-$\frac{1}{2}$ particle which is its own anti-particle. Thus, in going from fields to states we loose some dimensions of our representation space, but differently so for different spins. Supersymmetric models for which the number of fermionic degrees freedom is the number of bosonic degrees of freedeom somehow take care of this, and they do so by means of auxiliary fields whose off-shell degrees of freedom disappear completely on-shell. For the Wess-Zumino model

$$\text{off-shell: } (A, B, \mathcal{F}, \mathcal{G}) \longleftrightarrow \psi_C = \psi \quad , \quad \text{on-shell: } (A, B) \longleftrightarrow \psi_C = \psi.$$

6.2.4 Noether current and supercharge

We proceed as we did in the last chapter. For space-times symmetries the Lagrangean density is invariant up to a total divergence and the Noether current

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_X\phi - V^\mu$$

acquires an additional term $-V^\mu$. Here we are mainly interested in the conserved charges and not so much in the currents. Hence we do not attempt to improve the current. For example, the conserved energy momentum current

$$T^\mu_\nu = \partial^\mu A\partial_\nu A + \partial^\mu B\partial_\nu B + \tfrac{i}{2}\bar{\psi}\gamma^\mu\partial_\nu\psi - \delta^\mu_\nu\mathcal{L}$$

is not symmetric and not traceless for vanishing mass. But it leads to the conserved *energy-momentum*

$$P_\nu = \int_{x^0} d\boldsymbol{x}(\pi_A\partial_\nu A + \pi_B\partial_\nu B + \mathrm{i}\psi^\dagger\partial_\nu\psi - \delta^0{}_\nu\mathcal{L}).$$

To construct the supercurrent, i.e. the Noether current associated to supersymmetry, we must determine

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\varepsilon\phi.$$

Because $\bar\psi\not\partial\psi = -\partial_\mu\bar\psi\gamma^\mu\psi$ we obtain

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta_\varepsilon\phi &= \partial^\mu A\bar\varepsilon\psi + \mathrm{i}\partial^\mu B\bar\varepsilon\gamma_5\psi - \tfrac{i}{2}\delta_\varepsilon\bar\psi\gamma^\mu\psi \\ &= \bar\varepsilon\{2\partial^\mu(A+\mathrm{i}\gamma_5 B) - \tfrac{1}{2}\gamma^\mu\not\partial(A+\mathrm{i}\gamma_5 B) - \tfrac{i}{2}(\mathcal{F}-\mathrm{i}\gamma_5\mathcal{G})\}\psi.\end{aligned}$$

Now we must subtract V^μ in (6.29) to obtain the conserved current

$$\begin{aligned}\bar\varepsilon J^\mu &= \bar\varepsilon\{\not\partial(A-\mathrm{i}\gamma_5 B)\gamma^\mu\psi + im\gamma^\mu(A-\mathrm{i}\gamma_5 B)\psi + ig\gamma^\mu(A-\mathrm{i}\gamma_5 B)^2\}\psi \\ &= -\mathrm{i}\delta_\varepsilon\bar\psi\gamma^\mu\psi = \mathrm{i}\bar\psi\gamma^\mu\delta_\varepsilon\psi. \qquad (6.43)\end{aligned}$$

In these expressions for the conserved super-current we must replace the velocities by the momenta. Note, that the auxiliary fields do not appear in the expression for the Noether current.

To arrive at the *symplectic structure* we first recall the fundamental equal-time commutators for the Bose-fields

$$[A(x),\pi_A(y)] = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad , \quad [B(x),\pi_B(y)] = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y}) \qquad (6.44)$$

and equal-time anticommutators for the Majorana fields,

$$\{\psi_\alpha(x),\psi^{\beta\dagger}(y)\} = \delta_\alpha^{\,\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) \quad \text{or} \quad \{\psi_\alpha(x),\bar\psi^\beta(y)\} = (\gamma^0)_\alpha^{\,\beta}\delta(\boldsymbol{x}-\boldsymbol{y}). \qquad (6.45)$$

For a Majorana spinor, the last anti-commutator implies

$$\{\psi_\alpha(x),\psi_\beta(y)\} = -(\gamma^0\mathcal{C})_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}), \qquad (\gamma^0\mathcal{C})^T = \gamma^0\mathcal{C}. \qquad (6.46)$$

The Hamiltonian splits into three terms,

$$\begin{aligned}H &= H_0 + H_m + H_g, \quad \text{where} \quad H_m + H_g = -\int_{x^0} d\boldsymbol{x}\,(\mathcal{L}_m + \mathcal{L}_g) \quad \text{and} \\ H_0 &= \tfrac{1}{2}\int_{x^0} d\boldsymbol{x}\left(\pi_A^2 + (\nabla A)^2 + \pi_B^2 + (\nabla B)^2 - \mathrm{i}\psi^\dagger\alpha^i\partial_i\psi - (\mathcal{F}^2+\mathcal{G}^2)\right). \qquad (6.47)\end{aligned}$$

We used the (from the Dirac theory known) hermitian matrices $\alpha^i = \gamma^0\gamma^i$. Using

$$[AB,C] = A\{B,C\} - \{A,C\}B$$

in the time-evolution equation

$$\dot{O} = i[H, O]$$

one finds the Hamiltonian *equation of motion* for the fields and their conjugate momenta. For example, for the free massless model

$$\begin{aligned}
\dot{A}(x) &= i[H_0, A(x)] = \pi_A \quad , \quad \dot{\pi}_A = i[H_0, \pi_A] \\
\dot{\psi}_\alpha(x) &= i[H_0, \psi_\alpha(x)] = \tfrac{1}{2}(\alpha^i)_\beta^{\ \gamma} \int d\boldsymbol{y} \left(\psi^{\dagger\beta}\{\partial_i\psi_\gamma, \psi_\alpha(x)\} - \{\psi^{\dagger\beta}, \psi_\alpha(x)\}\partial_i\psi_\gamma\right) \\
&= -\tfrac{1}{2}(\alpha^i\partial_i\psi)_\alpha - \tfrac{1}{2}(\alpha^i)_\beta^{\ \gamma} \int d\boldsymbol{y}\partial_i\psi^{\dagger\beta}\{\psi_\gamma, \psi_\alpha(x)\} = -(\alpha^i\partial_i\psi)_\alpha,
\end{aligned}$$

which are just the Hamiltonian field equations of the free massless Wess-Zumino model. The *supercharge* of the interacting model has the form

$$\bar{\varepsilon}\mathcal{Q} = \bar{\varepsilon}\int_{x^0} d\boldsymbol{x} \left(\pi_A + \mathrm{i}\gamma_5\pi_B - \alpha^i\partial_i(A + \mathrm{i}\gamma_5 B) + im(A + \mathrm{i}\gamma_5 B)\gamma^0 + ig(A + \mathrm{i}\gamma_5 B)^2\gamma^0\right)\psi,$$

and generates the following transformations

$$\begin{aligned}
i[\bar{\varepsilon}\mathcal{Q}, A] &= \bar{\varepsilon}\psi, \\
i[\bar{\varepsilon}\mathcal{Q}, B] &= \mathrm{i}\bar{\varepsilon}\gamma_5\psi \\
i[\bar{\varepsilon}\mathcal{Q}, \psi_\beta] &= -\mathrm{i}\left(\gamma^0(\pi_A + \mathrm{i}\pi_B\gamma_5) + \gamma^i\partial_i(A + \mathrm{i}\gamma_5 B) + m(A + \mathrm{i}\gamma_5 B) + g(A + \mathrm{i}\gamma_5 B)^2\right)\varepsilon,
\end{aligned}$$

which are just the on-shell susy-transformations. We used the identities

$$\begin{gathered}
\gamma_5\mathcal{C} = \mathcal{C}\gamma_5^T, \quad (\bar{\varepsilon}\gamma^0\mathcal{C})^T = \gamma^0\varepsilon, \quad (\bar{\varepsilon}\gamma_5\gamma^0\mathcal{C})^T = \gamma^0\gamma_5\varepsilon \\
(\bar{\varepsilon}\alpha^i\gamma^0\mathcal{C})^T = -\gamma^i\varepsilon \quad \text{and} \quad (\bar{\varepsilon}\alpha^{\mathrm{i}}\gamma_5\gamma^0\mathcal{C})^T = \gamma_5\gamma^i\varepsilon.
\end{gathered}$$

For the infinitesimal transformation of the momenta one finds

$$i[\bar{\varepsilon}\mathcal{Q}, \pi_A] = \bar{\varepsilon}\dot{\psi} \quad \text{and} \quad i[\bar{\varepsilon}\mathcal{Q}, \pi_B] = \mathrm{i}\bar{\varepsilon}\gamma_5\dot{\psi},$$

where we used the abbreviation

$$\dot{\psi} = -(\alpha^i\partial_i + \mathrm{i}\gamma^0(m + 2g(A - \mathrm{i}\gamma_5 B))\psi.$$

The Hamiltonian equations for the fields and their conjugate momentum fields are equivalent to the field equations. The spinorial *supercharges* of the interacting model are

$$\begin{aligned}
\mathcal{Q} &= \int \{\pi_A + \mathrm{i}\gamma_5\pi_B - \alpha^i\partial_i(A + \mathrm{i}\gamma_5 B) + \mathrm{i}(m + g[A + \mathrm{i}\gamma_5 B])(A + \mathrm{i}\gamma_5 B)\gamma^0\}\psi \\
\bar{\mathcal{Q}} &= \int \bar{\psi}\{\pi_A + \mathrm{i}\gamma_5\pi_B + \alpha^i\partial_i(A + \mathrm{i}\gamma_5 B) - \mathrm{i}\gamma^0(m + g[A + \mathrm{i}\gamma_5 B])(A + \mathrm{i}\gamma_5 B)\}. \quad (6.48)
\end{aligned}$$

To finally calculate the anticommutator of the supercharges we use

$$\{B_1F_1, F_2\} = B_1\{F_1, F_2\} - [B_1, F_2]F_1, \quad \{F_1, B_2F_2\} = B_2\{F_1, F_2\} + [F_1, B_2]F_2 \quad (6.49)$$

so that

$$\{B_1F_1, B_2F_2\} = B_1B_2\{F_1, F_2\} + B_1[F_1, B_2]F_2 - [B_1, B_2]F_2F_1 - B_2[B_1, F_2]F_1 \quad (6.50)$$

holds true. The B_i are bosonic and the F_i are fermionic operators. Since $\mathcal{Q}, \bar{\mathcal{Q}}$ have the form

$$\mathcal{Q}_\alpha = \int d\boldsymbol{x} B_\alpha^{\ \gamma}(x)\psi_\gamma(x) \quad \text{and} \quad \bar{\mathcal{Q}}^\beta = \int d\boldsymbol{y} \bar{\psi}^\delta(y) C_\delta^{\ \beta}(y),$$

where the bosonic operator $B_\alpha^{\ \gamma}$ commutes with ψ we obtain

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}^\beta\} = \int d\boldsymbol{x}d\boldsymbol{y} \left(B_\alpha^{\ \gamma}(x)\,\{\psi_\gamma(x), \bar{\psi}^\delta(y)\}\, C_\delta^{\ \beta}(y) + \bar{\psi}^\delta(y)\,[C_\delta^{\ \beta}(y), B_\alpha^{\ \gamma}(x)]\,\psi_\gamma(x)\right).$$

A rather involved calculations shows, that

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}^\beta\} = 2(\gamma^\mu)_\alpha^{\ \beta} P_\mu,$$

as expected. Actually, one gets an additional term

$$-\tfrac{i}{2}\gamma^0[\gamma^i, \gamma^j]\gamma_5 \int (\partial_i A \partial_j B - \partial_j A \partial_i B),$$

but the integrand is a total divergence and hence the integral vanishes. One gets the off-shell Hamiltonian P_0 with eliminated auxiliary fields. Also, one automatically obtains a hermitian expression for the momentum operator. For example, the bosonic contribution to the momentum is calculated to be

$$P_i = \tfrac{1}{2}(\pi_A \partial_i A + \partial_i A \pi_A + \pi_B \partial_i B + \partial_i B \pi_B).$$

6.2.5 The superpotential

To generalise the results it is convenient to work with Weyl spinors. The following calculations can be done in any representations. For notational simplicity I prefer to choose the chiral representation introduced in section (4.5). In the van der Waerden notation with dotted and un-dotted indexes a Dirac spinor has the form

$$\psi = \begin{pmatrix} \psi_A \\ \bar{\chi}^{\dot{A}} \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger \gamma^0 = (\,\bar{\psi}_{\dot{A}}, \chi^A\,)\,\gamma^0 = (\,\chi^A, \bar{\psi}_{\dot{A}}\,), \quad \text{where} \quad \bar{\psi}_{\dot{A}} = (\psi_A)^\dagger \dots.$$

A Majorana spinor ψ can be formed entirely from a lefthanded Weyl spinor,

$$\psi = \psi_L + (\psi_L)_C, \quad \psi = P_L\psi, \quad P_L = \tfrac{1}{2}(1 - \gamma_5). \quad (6.51)$$

In the chiral representation representation introduced in the last chapter we have

$$\gamma_5 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \varepsilon \\ -\varepsilon & 0 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2, \quad \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad (6.52)$$

where

$$\sigma_\mu = (\sigma_0, -\sigma_i), \quad \tilde{\sigma}_\mu = (\sigma_0, \sigma_i) \quad \text{so that} \quad \varepsilon\sigma_\mu\varepsilon = -\tilde{\sigma}^T_\mu. \tag{6.53}$$

Left and righthanded spinors are simply

$$\psi_L = \begin{pmatrix} \psi_A \\ 0 \end{pmatrix} \quad \text{and} \quad \psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{A}} \end{pmatrix}, \tag{6.54}$$

and a Majorana spinor has the form

$$\psi = \psi_L + (\psi_L)_C = \begin{pmatrix} \psi_A \\ -(\varepsilon\psi^*)^{\dot{A}} \end{pmatrix} \equiv \begin{pmatrix} \chi_A \\ \bar{\chi}^{\dot{A}} \end{pmatrix} = \psi_C \quad \Longrightarrow \quad \bar{\psi} = (\psi^A, \bar{\psi}_{\dot{A}}). \tag{6.55}$$

The raising and lowering of the indexes are performed with the ε-symbols

$$\psi^A = \varepsilon^{AB}\,\psi_B, \quad \psi_A = \psi^B\epsilon_{BA} \quad \text{and} \quad \bar{\psi}^{\dot{A}} = \bar{\psi}_{\dot{B}}\,\varepsilon^{\dot{B}\dot{A}}, \quad \bar{\psi}_{\dot{A}} = \epsilon_{\dot{A}\dot{B}}\bar{\psi}^{\dot{B}}$$

where we introduced

$$\begin{aligned} (\varepsilon^{AB}) = (\varepsilon_{AB}) &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ (\varepsilon^{\dot{A}\dot{B}}) = (\varepsilon_{\dot{A}\dot{B}}) &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

It follows that

$$\varepsilon^{AB}\varepsilon_{BC} = -\delta^A_C, \quad \varepsilon_{\dot{A}\dot{B}}\varepsilon^{\dot{B}\dot{C}} = -\delta^{\dot{A}}_{\dot{C}}.$$

For fermionic variables we have

$$\psi^A\chi_A = \varepsilon^{AB}\psi_B\chi_A = \varepsilon^{BA}\chi_A\psi_B = \chi^A\psi_A \equiv \psi\chi, \quad \bar{\psi}\bar{\chi} = \bar{\psi}_{\dot{A}}\bar{\chi}^{\dot{A}} = \bar{\chi}_{\dot{A}}\bar{\psi}^{\dot{A}}.$$

Note, however, that

$$\psi^A\chi_A = \varepsilon^{AB}\psi_B\chi^C\varepsilon_{CA} = -\delta^B_C\psi_B\chi^C = -\psi_B\chi^B, \quad \bar{\psi}_{\dot{A}}\bar{\chi}^{\dot{A}} = -\bar{\psi}^{\dot{A}}\bar{\chi}_{\dot{A}}.$$

The mass term reads

$$\bar{\psi}\psi = \psi^A\psi_A + \bar{\psi}_{\dot{A}}\bar{\psi}^{\dot{A}} = \psi\psi + \bar{\psi}\bar{\psi}, \qquad \bar{\psi}\bar{\psi} = (\psi\psi)^\dagger, \tag{6.56}$$

and

$$\bar{\psi}\gamma_5\psi = -\psi\psi + \bar{\psi}\bar{\psi}.$$

The vector current can be written as

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_{\dot{A}}(\tilde{\sigma}^\mu)^{\dot{A}A}\psi_A + \psi^A(\sigma^\mu)_{A\dot{A}}\bar{\psi}^{\dot{A}} = \bar{\psi}\tilde{\sigma}^\mu\psi + \psi\sigma^\mu\bar{\psi} = \bar{\psi}\tilde{\sigma}^\mu\psi + \text{ h.c.},$$

so that the 'kinetic term' for Majorana fermions is

$$\bar{\psi}\partial\!\!\!/\psi = \bar{\psi}\tilde{\sigma}^\mu\partial_\mu\psi + \psi\sigma^\mu\partial_\mu\bar{\psi}. \tag{6.57}$$

Now we are ready to rewrite the susy-Lagrangean (6.13) in terms of a single lefthanded field ψ, and complex fields ϕ and F for its scalar partners

$$\phi = \frac{1}{\sqrt{2}}(A + iB) \quad \text{and} \quad F = \frac{1}{\sqrt{2}}(\mathcal{F} - \mathrm{i}\mathcal{G}). \tag{6.58}$$

With

$$\begin{aligned}
\partial_\mu\phi^\dagger\partial^\mu\phi &= \tfrac{1}{2}\{\partial_\mu A\partial^\mu A + \partial_\mu B\partial^\mu B\} \quad , \quad F^\dagger F = \tfrac{1}{2}(\mathcal{F}^2 + \mathcal{G}^2)\\
\phi F + (\phi F)^\dagger &= A\mathcal{F} + B\mathcal{G} \quad , \quad \sqrt{2}\{\phi^2 F + (\phi^2 F)^\dagger\} = \mathcal{F}(A^2 - B^2) + 2AB\mathcal{G}\\
\sqrt{2}\{\phi\psi\psi + \phi^\dagger\bar{\psi}\bar{\psi}\} &= \bar{\psi}(A - \mathrm{i}\gamma_5 B)\psi.
\end{aligned}$$

we can rewrite (6.13) as

$$\begin{aligned}
\mathcal{L} &= \partial_\mu\phi^\dagger\partial^\mu\phi + F^\dagger F + \tfrac{1}{2}\{\mathrm{i}\bar{\psi}\tilde{\sigma}^\mu\partial_\mu\psi + 2m\phi F - m\psi^2 + \text{ h.c.}\}\\
&\qquad + \sqrt{2}g\{\phi^2 F - \phi\psi\psi + \text{ h.c.}\}.
\end{aligned} \tag{6.59}$$

Then, using the equation of motion for F, which gives

$$F^\dagger = -m\phi - \sqrt{2}g\phi^2$$

we can eliminate the auxiliary field so that the Lagrangean density becomes

$$\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - |m\phi + \sqrt{2}g\phi^2|^2 + \{\tfrac{i}{2}\bar{\psi}\tilde{\sigma}^\mu\partial_\mu\psi - \tfrac{1}{2}\bar{\psi}\psi - \sqrt{2}g\phi^2 F + \text{ h.c.}\}. \tag{6.60}$$

It is useful to re-express these Lagrangean in terms of an analytic function

$$W(\phi) = \tfrac{1}{2}m\phi^2 + \tfrac{1}{3}\sqrt{2}g\phi^3, \tag{6.61}$$

known as *superpotential*, as

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_K + F^\dagger F + \Big\{F\frac{\partial W}{\partial\phi} + \tfrac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi + \text{ h.c.}\Big\}\\
\mathcal{L} &= \mathcal{L}_K - |\frac{\partial W}{\partial\phi}|^2 + \tfrac{1}{2}\Big\{\frac{\partial^2 W}{\partial\phi^2}\psi\psi + \text{ h.c.}\Big\}.
\end{aligned} \tag{6.62}$$

For a normalisable theory W can be, at most, a cubic function of ϕ. The superpotential is the only free function in the susy Lagrangean and determines both the potential for the scalar fields and the masses and couplings of the fermions and bosons.

In general there may be several chiral multiplets. For example, if the lefthanded spinor fields ψ_i belongs to a representation of $SU(N)$, we will have supermultiplets

$$(\phi_i, \chi_i),$$

where in the fundamental representation $i = 1, 2, \dots, N$. Now (6.62) is readily generalised to a density that is invariant under the additional symmetry and contains the new supermultiplets. It is

$$\mathcal{L}_{\text{chiral}} = \sum_i |\partial_\mu \phi_i|^2 + \Big\{ \tfrac{i}{2} \sum_i \bar\psi_i \tilde\sigma^\mu \partial_\mu \psi_i - \sum_i |\frac{\partial W}{\partial \phi_i}|^2 + \tfrac{1}{2} \sum_{ij} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_i \psi_j + \text{ h.c.} \Big\}. \quad (6.63)$$

Supersymmetry transformations: we write the Majorana spinor α in terms of its lefthanded part,

$$\alpha = \begin{pmatrix} \alpha_A \\ \bar\varepsilon^{\dot A} \end{pmatrix}, \quad \bar\varepsilon = (\alpha^A, \bar\varepsilon_{\dot A}).$$

Then it follows at once that

$$\begin{aligned} \bar\varepsilon\psi = \alpha\chi + \bar\varepsilon\bar\chi \quad &, \quad \bar\varepsilon\gamma_5\psi = -\alpha\psi + \bar\varepsilon\bar\psi \\ \bar\varepsilon\gamma^\mu\psi = \bar\varepsilon\tilde\sigma^\mu\psi + \alpha\sigma^\mu\bar\psi \quad &, \quad \bar\varepsilon\gamma_5\gamma^\mu\psi = \bar\varepsilon\tilde\sigma^\mu\psi - \alpha\sigma^\mu\bar\psi, \end{aligned}$$

where we used, that the σ_μ are hermitian. The susy-transformations take now the form

$$\delta_\varepsilon\phi = \sqrt{2}\alpha\psi, \quad \delta_\varepsilon F = -\mathrm{i}\sqrt{2}\bar\varepsilon\tilde\sigma^\mu\partial_\mu\psi, \quad \delta_\varepsilon\psi = \sqrt{2}(F\alpha - \mathrm{i}\sigma^\mu\partial_\mu\phi\,\bar\varepsilon).$$

6.2.6 Supersymmetry algebra in Weyl-basis

We may recast the supersymmetry algebra for the Majorana spinor charges $\mathcal{Q}_\alpha$, namely

$$\begin{aligned} \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = -2(\gamma^\mu\mathcal{C})_{\alpha\beta}P_\mu \quad &\text{or} \quad \{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}^\beta\} = 2(\gamma^\mu)_\alpha^{\ \beta}P_\mu \\ [\mathcal{Q}_\alpha, P_\mu] = 0 \qquad &\text{and} \qquad [J_{\mu\nu}, \mathcal{Q}_\alpha] = (\Sigma_{\mu\nu}\mathcal{Q})_\alpha, \end{aligned} \quad (6.64)$$

plus the Poincaré algebra, in terms of the lefthanded charges. For that we expand (6.35) in the lefthanded supercharges

$$\begin{aligned} [\bar{\mathcal{Q}}\alpha, \bar\beta\mathcal{Q}] &= \alpha_A\beta^B\{\mathcal{Q}^A, \mathcal{Q}_B\} + \bar\varepsilon^{\dot A}\bar\beta_{\dot B}\{\bar{\mathcal{Q}}_{\dot A}, \bar{\mathcal{Q}}^d b\} + \alpha_A\bar\beta_{\dot B}\{\mathcal{Q}^A, \bar{\mathcal{Q}}^{\dot B}\} + \bar\varepsilon^{\dot A}\beta^B\{\bar{\mathcal{Q}}_{\dot A}, \mathcal{Q}_B\} \\ &= -2\{\beta^B(\sigma^\mu)_{B\dot A}\bar\varepsilon^{\dot A} + \bar\beta_{\dot B}(\tilde\sigma^\mu)^{\dot B A}\alpha_A\}P_\mu. \end{aligned}$$

Comparing the two expressions yields

$$\{\mathcal{Q}_A, \bar{\mathcal{Q}}_{\dot B}\} = 2(\sigma^\mu)_{A\dot B}P_\mu \quad \text{or} \quad \{\bar{\mathcal{Q}}^{\dot A}, \mathcal{Q}^B\} = 2(\tilde\sigma^\mu)^{\dot A B}P_\mu \quad (6.65)$$

and

$$\{\mathcal{Q}_A, \mathcal{Q}_B\} = \{\bar{\mathcal{Q}}^{\dot A}, \bar{\mathcal{Q}}^{\dot B}\} = 0.$$

Analogously, when we express the Majorana spinors in

$$[J_{\mu\nu}, \bar\varepsilon\mathcal{Q}] = \frac{\bar\varepsilon}{2}\begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde\sigma_{\mu\nu} \end{pmatrix}\mathcal{Q}, \qquad \sigma_{\mu\nu} = \tfrac{1}{2i}(\sigma_\mu\tilde\sigma_\nu - \sigma_\nu\tilde\sigma_\mu), \quad \tilde\sigma_{\mu\nu} = \tfrac{1}{2i}(\tilde\sigma_\mu\sigma_\nu - \tilde\sigma_\nu\sigma_\mu),$$

in terms of their lefthanded components, we obtain

$$[J_{\mu\nu}, \mathcal{Q}_A] = \tfrac{1}{2}(\sigma_{\mu\nu})_A{}^B \mathcal{Q}_B \quad \text{or} \quad [J_{\mu\nu}, \bar{\mathcal{Q}}^{\dot{A}}] = \tfrac{1}{2}(\tilde{\sigma}_{\mu\nu})^{\dot{A}}{}_{\dot{B}} \bar{\mathcal{Q}}^{\dot{B}}, \tag{6.66}$$

Hence, an alternative form of the susy algebra (6.64) reads

$$\begin{aligned} \{\mathcal{Q}_A, \bar{\mathcal{Q}}_{\dot{B}}\} &= 2(\sigma^\mu)_{A\dot{B}} P_\mu \quad , \quad \{\mathcal{Q}_A, \mathcal{Q}_B\} = 0 \\ [\mathcal{Q}_A, P_\mu] &= 0 \qquad \text{and} \quad [J_{\mu\nu}, \mathcal{Q}_A] = \tfrac{1}{2}(\sigma_{\mu\nu})_A{}^B \mathcal{Q}_B, \end{aligned} \tag{6.67}$$

plus the Poincaré algebra.

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Supersymmetry algebras

Supersymmetric theories avoid the restriction of the COLEMAN-MANDULA theorem by relaxing one condition: they generalize the notion of a Lie algebra and include in the defining relation anticommutators in addition to the usual commutators. We shall develop the form of the susy algebra from first principles, following HAAG, LOPUSZANSKI AND SOHNIUS [35]. Under the conditions imposed in the COLEMAN-MANDULA theorem the supersymmetric structure is almost uniquely fixed by the requirement of Lorentz invariance. The supermultiplett structure of one-particle states will later be deduced from the susy algebra.
In 4 dimensions the generalization of the Poincaré algebra to a superalgebra is obtained in its simplest version by the following procedure: one adds to the Poincaré algebra a Majorana spinor charge, with components $\mathcal{Q}_\alpha, \alpha = 1, \dots, 4$, with the properties

$$\{\mathcal{Q}_\alpha, \bar{\mathcal{Q}}^\beta\} = 2(\gamma^\mu)_\alpha^{\ \beta} P_\mu, \qquad [\mathcal{Q}_\alpha, P_\mu] = 0 \quad \text{and} \quad [M_{\mu\nu}, \mathcal{Q}_\alpha] = (\Sigma_{\mu\nu}\mathcal{Q})_\alpha. \tag{7.1}$$

Here P_μ and $M_{\mu\nu}$ are the generators of translations and homogeneous Lorentz transformations of space-time. Since P_μ has the dimensions L^{-1}, the supersymmetry generators $\mathcal{Q}_\alpha$ must have dimension $L^{-1/2}$. The $M_{\mu\nu}$ are dimensionless. We have seen that the $\mathcal{Q}_\alpha$ map fermions into bosons and bosons into fermions. The relation $[M_{\mu\nu}, \mathcal{Q}_\alpha] = (\Sigma_{\mu\nu}\mathcal{Q})_\alpha$ expresses the fact, that the $\mathcal{Q}_\alpha$ transform as a spinor under Lorentz transformations. In the last section we have explicitly realized this algebraic structure on a Majorana spinor field and spin-0 fields. Now we shall investigate, how unique this algebraic structure is.

7.1 Graded Lie algebras

Supersymmetry is expressed in terms of symmetry generators t_a that form a graded Lie algebra or Lie superalgebra. We have to deal with two types of elements: Bose and FERMI. Thus we use a graded vector space such that each of its vectors has a grade defined mod 2. Hence a $\mathbb{Z}_2$-graded algebra consists of a vector space $\mathcal{S}$ which is the set-theoretic union[1]

[1] Superselection rules forbid the linear combination of Bose with Fermi elements [39], and hence $\mathcal{S}$ is the set-theoretic union and not the direct sum of $\mathcal{S}_0$ and $\mathcal{S}_1$.

of two subspaces,

$$\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1, \tag{7.2}$$

and is equipped with a binary operation which respects this grading. The dimension of $\mathcal{S}_0$ is N_0 and that of $\mathcal{S}_1$ is N_1. We assign to an element $t_a \in \mathcal{S}$ a grade η_a. The elements in $\mathcal{S}_0$ have grade 0 and those in $\mathcal{S}_1$ have grade 1. $\mathcal{S}_0$ consists of even elements and $\mathcal{S}_1$ of odd ones. The bilinear product

$$[.,.\} : \mathcal{S} \times \mathcal{S} \longrightarrow \mathcal{S}, \qquad [t_a, t_b\} = iC_{ab}{}^c t_c, \tag{7.3}$$

has the following properties

$$\begin{array}{rl} \text{grading:} & [\mathcal{S}_0, \mathcal{S}_0\} \subset \mathcal{S}_0, \quad [\mathcal{S}_0, \mathcal{S}_1\} \subset \mathcal{S}_1, \quad [\mathcal{S}_1, \mathcal{S}_1\} \subset \mathcal{S}_0 \\ \text{super-ACR:} & [t_a, t_b\} = -(-)^{\eta_a\eta_b}[t_b, t_a\} \\ \text{super-Jacobi:} & (-)^{\eta_a\eta_c}[t_a, [t_b, t_c\}\} + (-)^{\eta_c\eta_b}[t_c, [t_a, t_b\}\} + (-)^{\eta_b\eta_a}[t_b, [t_c, t_a\}\} = 0. \end{array} \tag{7.4}$$

The even elements in $\mathcal{S}_0$ correspond to bosonic generators, the odd elements in $\mathcal{S}_1$ correspond to fermionic generators. Hence, the product of two bosonic or two fermionic operators is bosonic, and the product of a fermionic with a bosonic operator is fermionic, so that

$$C_{ab}{}^c = 0, \quad \text{unless} \quad \eta_c = (\eta_a + \eta_b) \bmod 2. \tag{7.5}$$

According to the symmetry properties of the product in (7.4) the structure constants in (7.3) must satisfy the conditions

$$C_{ba}{}^c = -(-1)^{\eta_a\eta_b} C_{ab}{}^c. \tag{7.6}$$

Let us finally see, what are the consequences of the super Jacobi identity for the structure constants $C_{ab}{}^c$ in (7.3):

$$\begin{aligned} 0 &= (-)^{\eta_a\eta_c}[t_a, [t_b, t_c\}\} + (-)^{\eta_c\eta_b}[t_c, [t_a, t_b\}\} + (-)^{\eta_b\eta_a}[t_b, [t_c, t_a\}\} \\ &= i(-)^{\eta_a\eta_c} C_{bc}{}^d [t_a, t_d\} + i(-)^{\eta_c\eta_b} C_{ab}{}^d [t_c, t_d\} + i(-)^{\eta_b\eta_a} C_{ca}{}^d [t_b, t_d\} \\ &= -\Big((-)^{\eta_a\eta_c} C_{bc}{}^d C_{ad}{}^e + (-)^{\eta_c\eta_b} C_{ab}{}^d C_{cd}{}^e + (-)^{\eta_b\eta_a} C_{ca}{}^d C_{bd}{}^e\Big)\, t_e\,. \end{aligned}$$

This means that the structure constants obey the quadratic relations

$$(-)^{\eta_a\eta_c} C_{bc}{}^d C_{ad}{}^e + (-)^{\eta_c\eta_b} C_{ab}{}^d C_{cd}{}^e + (-)^{\eta_b\eta_a} C_{ca}{}^d C_{bd}{}^e = 0\,. \tag{7.7}$$

Introducing the adjoint action

$$\mathrm{ad}_{t_a} : t_b \longrightarrow \mathrm{ad}_{t_a}(t_b) = [t_a, t_b\} \tag{7.8}$$

the super-Jacobi identity can be rewritten as

$$(\mathrm{ad}_{t_a} \circ \mathrm{ad}_{t_b})(t_c) - (-)^{\eta_a\eta_b}(\mathrm{ad}_{t_b} \circ \mathrm{ad}_{t_a})(t_c) = \mathrm{ad}_{[t_a, t_b\}}(t_c).$$

Since this holds for any generator t_c we conclude, that

$$[\text{ad}_{t_a}, \text{ad}_{t_b}\} \equiv \text{ad}_{t_a} \circ \text{ad}_{t_b} - (-)^{\eta_a \eta_b} \text{ad}_{t_b} \circ \text{ad}_{t_a} = \text{ad}_{[t_a, t_b\}} = iC_{ab}{}^c \text{ad}_{t_c}. \tag{7.9}$$

In applications the t_a are realized as linear operators on a Hilbert space and the product of operators $t_a t_b$ is defined. In this case

$$[t_a, t_b\} = t_a t_b - (-)^{\eta_a \eta_b} t_b t_a. \tag{7.10}$$

The super-Jacobi identity is then automatically fulfilled. If one operator is bosonic we get the commutator, if both operators are fermionic we get the anticommutator. If the t_a are hermitian, then the structure constants satisfy a reality conditions

$$-i\bar{C}_{ab}{}^c t_c = [t_a, t_b\}^\dagger = t_b t_a - (-)^{\eta_a \eta_b} t_a t_b = iC_{ba}{}^c t_c$$

which is equivalent to

$$\bar{C}_{ab}{}^c = -C_{ba}{}^c. \tag{7.11}$$

Note that $\mathcal{S}$ is not a Lie algebra, since the product of two elements in $\mathcal{S}_1$ is symmetric and not antisymmetric,

$$[t_a, t_b\} = [t_b, t_a\} \quad \text{if} \quad t_a, t_b \in \mathcal{S}_1. \tag{7.12}$$

In the literature one writes $[.,.]$ instead of $[.,.\}$ if the product is antisymmetric, i.e. if at least one argument is even and $\{.,,\}$ instead of $[,.,\}$ if the product is symmetric, i.e. if both arguments are odd. The subspace $\mathcal{S}_0$ spans an ordinary Lie algebra. The subspace $\mathcal{S}_1$ is not even a subalgebra, since $\mathcal{S}_1$ is not closed under $[.,.\}$.

7.1.1 From a Lie algebra to a super-Lie algebra

We construct a $\mathbb{Z}_2$ grading of a Lie algebra which is generated by N_0 generators T_i with commutation relations

$$[T_i, T_j] = iC_{ij}{}^k T_k, \tag{7.13}$$

with Lie algebra structure constants $C_{ij}{}^k$. We denote the generators of $\mathcal{S}_1$ by $\mathcal{Q}_\alpha$. Since $[\mathcal{S}_0, \mathcal{S}_1] \subset \mathcal{S}_1$ and $\{\mathcal{S}_1, \mathcal{S}_1\} \subset \mathcal{S}_0$ we have, in addition to (7.13), the relations

$$[T_i, \mathcal{Q}_\alpha] = iC_{i\alpha}{}^\beta \mathcal{Q}_\beta \quad \text{and} \quad \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = iC_{\alpha\beta}^{\;i} T_i. \tag{7.14}$$

Because of the symmetry properties (7.6) the non-vanishing structure constants obey

$$C_{ij}{}^k = -C_{ji}{}^k, \qquad C_{i\alpha}{}^\beta = -C_{\alpha i}{}^\beta \quad \text{and} \quad C_{\alpha\beta}{}^i = C_{\beta\alpha}{}^i. \tag{7.15}$$

If the generators T_i and $\mathcal{Q}_\alpha$ are *hermitian*, then the reality conditions (7.11) and symmetry properties imply:

$$\bar{C}_{ij}{}^k = C_{ij}{}^k, \qquad \bar{C}_{i\alpha}{}^\beta = C_{i\alpha}{}^\beta \quad \text{and} \quad \bar{C}_{\alpha\beta}{}^i = -C_{\alpha\beta}{}^i. \tag{7.16}$$

Let us study the consequences of the super Jacobi identities for two bosonic generators and one fermionic generator. We just must set $a = i, b = j$ and $c = \alpha$ in (7.7) in which case all $(-)^{\eta_a\eta_b} = 1$ and find

$$0 = C_{j\alpha}{}^{\beta} C_{i\beta}{}^{\gamma} + C_{ij}{}^{k} C_{\alpha k}{}^{\gamma} + C_{\alpha i}{}^{\beta} C_{j\beta}{}^{\gamma}.$$

Now we introduce the N_1-dimensional matrices C_i with $i = 1, \dots N_0$, with (real) matrix elements $(C_i)_\alpha^\beta$ and using $C_{\alpha i}{}^{\beta} = -C_{i\alpha}{}^{\beta}$ we end up with

$$[C_i, C_j] = -C_{ij}{}^{k} C_k. \tag{7.17}$$

This proves the

Lemma 4 *The N_0 matrices C_i, whose matrix elements are the structure constants $C_{i\alpha}{}^{\beta}$, form a N_1-dimensional representation of the bosonic Lie algebra $\mathcal{S}_0$.*

Now we investigate the super Jacobi identity for one bosonic and two fermionic generators. Hence we set $a = i$ and $b = \alpha, c = \beta$ in (7.7), so that $\eta_a = 0$ and $\eta_b = \eta_c = 1$ and find

$$C_{ij}{}^{k} C_{\alpha\beta}{}^{j} = C_{i\alpha}{}^{\gamma} C_{\beta\gamma}{}^{k} + C_{i\beta}{}^{\gamma} C_{\alpha\gamma}{}^{k} = C_{i\alpha}{}^{\gamma} C_{\gamma\beta}{}^{k} + C_{i\beta}{}^{\gamma} C_{\gamma\alpha}{}^{k}.$$

If we introduce the N_0 symmetric N_1-dimensional matrices

$$C^i \quad \text{with matrix elements} \quad (C^i)_{\alpha\beta} = C_{\alpha\beta}{}^{i}$$

then these relations can be written as

$$C_{ij}{}^{k} C^j = C_i C^k + (C_i C^k)^T. \tag{7.18}$$

Finally we study the consequences of the super Jacobi identity for three fermionic operators. Hence we set $a = \alpha, b = \beta$ and $c = \gamma$ in (7.7). This results in

$$0 = C_{\beta\gamma}{}^{i} C_{\alpha i}{}^{\delta} + C_{\alpha\beta}{}^{i} C_{\gamma i}{}^{\delta} + C_{\gamma\alpha}{}^{i} C_{\beta i}{}^{\delta} = (C^i)_{\alpha\beta} (C_i)_\gamma{}^{\delta} + \text{cycl.}(\alpha, \beta, \gamma). \tag{7.19}$$

Let us now study some graduations of simple and physically relevant Lie algebras.

7.1.2 The grading of $SU(2)$

Here we study graded Lie algebras with $\mathcal{S}_0 =$su(2). The Lie-subalgebra $\mathcal{S}_0$ is generated by the 3 hermitian matrices

$$T_i = \tfrac{1}{2}\sigma_i, \quad \text{such that} \quad C_{ij}{}^{k} = \epsilon_{ijk}. \tag{7.20}$$

The condition (7.17) for the matrices C_i with matrix elements $C_{i\alpha}{}^{\beta}$ reads

$$[C_i, C_j] = -\epsilon_{ijk} C_k, \tag{7.21}$$

and hence the real matrices C_i must form a N_1 dimensional representation of $su(2)$. For the 2-dimensional representations we have

$$(C_i)_\alpha^{\ \beta} = \tfrac{i}{2}(\sigma_i)_\alpha^{\ \beta}, \quad \text{so that} \quad [T_i, \mathcal{Q}_\alpha] = -\tfrac{1}{2}(\sigma_i \mathcal{Q})_\alpha.$$

Now we proceed to the constraints (7.18) for the symmetric matrices C^k. For the two-dimensional representation we set $C^k = \sigma_k \sigma_2$. Now one can prove, that (7.18) holds true,

$$C_i C^k + (C_i C^k)^T = \tfrac{i}{2}(\sigma_i \sigma_k \sigma_2 - \sigma_2 \sigma_k^T \sigma_i^T) = \tfrac{i}{2}[\sigma_i, \sigma_k]\sigma_2 = -\epsilon_{ikj} C^j = C_{ij}^{\ \ k} C^j,$$

where we used the identity $\sigma_2 \sigma_i^T = -\sigma_i \sigma_2$. With this solution we find

$$\{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = i(\sigma_i \sigma_2)_{\alpha\beta} T_i.$$

Now one can show, that for our choice for C_i and C^k the third set of constraints in (7.19) are automatically fulfilled. Hence we have obtained the following grading of $su(2)$:

$$[T_i, T_j] = i\epsilon_{ijk} T_k, \quad [T_i, \mathcal{Q}_\alpha] = -\tfrac{1}{2}(\sigma_i \mathcal{Q})_\alpha \quad \text{and} \quad \{\mathcal{Q}_\alpha, \mathcal{Q}_\beta\} = i(\sigma_i \sigma_2)_{\alpha\beta} T_i. \tag{7.22}$$

One can write down an explicit matrix representation of the graded $SU(2)$ on a 3-dimensional graded vector space $V(1|2)$ over $\mathbb{C}$. We represent a vector in V as a column vector with $1+2$ entries. The Bose element is in the top entry whereas the Fermi elements are in the bottom 2 entries. Then the matrix representation of the generators $\{T_i, \mathcal{Q}_\alpha\}$ takes the form

$$T_i = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\sigma_i \end{pmatrix}, \quad \mathcal{Q}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Q}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{7.23}$$

The superalgebra is then represented by the following matrices

$$M = \begin{pmatrix} 0 & B \\ \varepsilon B^T & D \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{7.24}$$

where $D = D^\dagger$ a 2-dimensional matrix which maps fermions into fermions and B is a 2-dimensional row vector which maps fermions into bosons, so that

$$\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \in \mathcal{S}_0 \quad \text{and} \quad \begin{pmatrix} 0 & B \\ \varepsilon B^T & 0 \end{pmatrix} \in \mathcal{S}_1. \tag{7.25}$$

Using $\varepsilon D = -D^T \epsilon$ the super-anticommutator of 2 elements can be written as

$$[M_1, M_2\} = \begin{pmatrix} 0 & B_3 \\ \varepsilon B_3^T & D_3 \end{pmatrix}, \quad B_3 = B_1 D_2 - B_2 D_1, \quad D_3 = [D_1, D_2] + \varepsilon(B_1^T B_2 + B_2^T B_1).$$

It turns out, that for the 3-dimensional representation

$$(C_i)_\alpha^{\ \beta} = \epsilon_{i\alpha\beta}$$

we cannot satisfy the constraints (7.18,7.19).

7.1.3 Supertrace and super-Killing form

Guided by the grading of $SU(2)$ we consider a mod 2 graded vector space $V(N|M)$ over $\mathbb{C}$ with N Bose and M Fermi dimensions. A vector in V can be represented as a column vector with $M+N$ entries. For Bose (FERMI) elements the top N (bottom M) entries will be non-zero. Now consider the complex linear transformations on $V(N|M)$. The grading of V induces an obvious grading of these linear transformations,

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{7.26}$$

In the matrix representation a Bose linear transformation is block diagonal

$$M \in \mathcal{S}_0 : \quad M = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \tag{7.27}$$

whereas a Fermi transformation is block off-diagonal

$$M \in \mathcal{S}_1 : \quad M = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{7.28}$$

A and D are square matrices, whereas B and C are in general rectangular. In the matrix representation the bracket is the usual commutator in all cases but one, namely when both bracketed transformations are Fermi in which case it is an anticommutator. The so obtained Lie superalgebra is $gl(N|M)$ and it is not simple.
Simplicity of a superalgebra is defined as for the ordinary Lie algebra: a Lie superalgebra $\mathcal{S}$ is simple if any sub-superalgebra $\mathcal{A}$ of $\mathcal{S}$, such that $[\mathcal{A}, \mathcal{S}\} \subset \mathcal{A}$, is trivial, i.e. either $\mathcal{A} = 0$ or $\mathcal{A} = \mathcal{S}$. In ordinary Lie algebras one achieves simplicity by imposing the traceless condition. However, if M_1 and M_2 are traceless, then the bracket

$$[M_1, M_2\} = \begin{pmatrix} [A_1, A_2] + B_1C_2 + B_2C_1 & A_1B_2 - A_2B_1 + B_1D_2 - B_2D_1 \\ C_1A_2 - C_2A_1 + D_1C_2 - D_2C_1 & [D_1, D_2] + C_1B_2 + C_2B_1 \end{pmatrix} \tag{7.29}$$

need not be traceless since, it involves anti-commutators. We therefore need a new concept of trace, called *supertrace*, such that it vanishes for the bracketing of two matrices. It is easily checked that the supertrace, given by

$$\mathrm{S\,Tr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, D, \tag{7.30}$$

has the required property,

$$\mathrm{S\,Tr}\,[M_1, M_2\} = 0. \tag{7.31}$$

Recall, that the invariant Killing form of an ordinary Lie algebra is defined as

$$K(T_i, T_j) = \mathrm{Tr}\,(\mathrm{ad}_{T_i} \circ \mathrm{ad}_{T_j}) = -\sum_{k,l} C_{ik}{}^l C_{jl}{}^k, \quad \text{where} \quad [T_i, T_j] = iC_{ij}{}^k T_k. \tag{7.32}$$

This bilinear-form is non-degenerate for semi-simple Lie groups and positive definite for compact Lie groups. The generalized Killing form on $\mathcal{S}$ is analogously defined as

$$K(t_a, t_b) = \mathrm{S\,Tr}\,(\mathrm{ad}_{t_a} \circ \mathrm{ad}_{t_b}) = -\sum_{c,d} (-)^{\eta_d} C_{ac}{}^{d} C_{bd}{}^{c} \equiv K_{ab}. \tag{7.33}$$

To prove the second equality we introduce an orthogonal basis $\{t_a\}$ in $\mathcal{S}$ for which

$$(t_a, t_b) = \mathrm{S\,Tr}\,(t_a, t_b) = (-)^{\eta_a} \delta_{ab} \tag{7.34}$$

such that indeed

$$\mathrm{S\,Tr}\,(\mathrm{ad}_{t_a} \circ \mathrm{ad}_{t_b}) = \sum_c (t_c, [t_a, [t_b, t_c\}\}) = -C_{bc}{}^{e} C_{ae}{}^{d} (t_c, t_d) = -K_{ab}.$$

It reduces to a multiple of the usual Killing form on the bosonic subspace $\mathcal{S}_0$ and hence is a symmetric bilinear form when acting on the subspace $\mathcal{S}_0$. However, it is antisymmetric when acting on the fermionic subspace $\mathcal{S}_1$,

$$K(\mathcal{Q}_\alpha, \mathcal{Q}_\beta) = K_{\alpha\beta} = -K_{\beta\alpha}. \tag{7.35}$$

This follows from the following symmetry property of the super-Killing form

$$K_{ab} = (-)^{\eta_a \eta_b} K_{ba}, \tag{7.36}$$

which is easily gotten from (7.9), applied to 2 fermionic generators. Finally, we have

$$K(T_i, \mathcal{Q}_\alpha) = 0, \tag{7.37}$$

because the structure constants respect the grading. The generalized Killing form is invariant under bracketing,

$$K([t_c, t_a\}, t_b) + (-)^{\eta_a \eta_b} K(t_a, [t_c, t_b\}) = 0. \tag{7.38}$$

I leave the proof as exercise.
Let us finally calculate the super Killing form for the graded $SU(2)$ algebra with structure constants

$$C_{ij}{}^{k} = \epsilon_{ijk}, \quad C_{i\alpha}{}^{\beta} = -C_{\alpha i}{}^{\beta} = \tfrac{i}{2}(\sigma_i)_\alpha^{\ \beta}, \quad C_{\alpha\beta}{}^{i} = C_{\beta\alpha}{}^{i} = (\sigma_i \sigma_2)_{\alpha\beta}.$$

Inserting into (7.33) yields the following super Killing matrix

$$\begin{aligned} K_{ij} &= -\sum_{kl} C_{ik}{}^{l} C_{jl}{}^{k} + \sum_{\alpha\beta} C_{i\alpha}{}^{\beta} C_{j\beta}{}^{\alpha} = -\sum_{kl} \epsilon_{ikl}\epsilon_{jlk} - \tfrac{1}{4}\sum_{\alpha\beta} (\sigma_i)_\alpha^{\ \beta} (\sigma_j)_\beta^{\ \alpha} = \tfrac{3}{2}\delta_{ij} \\ K_{\alpha\beta} &= -\sum_{\gamma i} C_{\alpha\gamma}{}^{i} C_{\beta i}{}^{\gamma} + (\alpha \leftrightarrow \beta) = \tfrac{i}{2}\sum_{\gamma i} (\sigma_i\sigma_2)_{\alpha\gamma} (\sigma_i)_\beta^{\ \gamma} - (\alpha \leftrightarrow \beta) = -3i(\sigma_2)_{\alpha\beta}. \end{aligned}$$

Writing $(a) = (i, \alpha)$ the matrix (K_{ab}) takes the form

$$(K_{ab}) = \tfrac{3}{2} \begin{pmatrix} \mathbb{1}_3 & 0 \\ 0 & -2\varepsilon \end{pmatrix}, \tag{7.39}$$

where ε has been introduced in (7.24). K_{ab} is non-degenerated as expected for the simple graded $su(2)$ algebra.

7.1.4 Supergroups

An endomorphism $M : V(N|M) \to V(N|M)$ can be represented by a graded $(N+M)$-dimensional matrix with the following structure

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \qquad (M_a{}^b) = \begin{pmatrix} A_i^{\ j} & B_i^{\ \alpha} \\ C_\alpha^{\ i} & D_\alpha^{\ \beta} \end{pmatrix}, \tag{7.40}$$

where $A : V_0 \to V_0$ maps bosons into bosons and $D : V_1 \to V_1$ maps fermions into fermions. A and D are square matrices. The matrices $B : V_1 \to V_0$ maps fermions into bosons and $C : V_0 \to V_1$ maps bosons into fermions. If $N \neq M$ then B and C are not square matrices:

$$B : N \times M \quad \text{and} \quad C : M \times N. \tag{7.41}$$

Since A, D do not change the grade they are even submatrices with degree $\eta = 0$. On the other hand, B, C change the grade and hence they are odd submatrices. In this subsection (7.1.4) we only use *commutators (no anticommutators)* and then we must assume, that the elements of B and C are anticommuting variables and therefore behave as Grassmann parameters:

$$B_{i\alpha} C_{\beta j} = -C_{\beta j} B_{i\alpha}.$$

From this we obtain the rules

$$(BC)^T = -C^T B^T \quad \text{and} \quad \mathrm{Tr}\,(BC) = -\mathrm{Tr}\,(CB). \tag{7.42}$$

Next we return to the *supertrace* which has been defined by

$$\mathrm{S\,Tr}\, M = \mathrm{Tr}\, A - \mathrm{Tr}\, D = \mathrm{Tr}\,(-)^{N_F} M. \tag{7.43}$$

Here N_F is the fermionic number and it is zero on the bosonic subspace and 1 on the fermionic subspace,

$$N_F = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1} \end{pmatrix} \Longrightarrow (-)^{N_F} = \begin{pmatrix} \mathbb{1}_N & 0 \\ 0 & -\mathbb{1}_M \end{pmatrix}. \tag{7.44}$$

On the bosonic subspace the supertrace becomes the ordinary trace.

Lemma 5 *The supertrace is a symmetric bilinear-form on $L(V)$ which is cyclic, i.e.*

$$\mathrm{S\,Tr}\,(M_1 M_2) = \mathrm{S\,Tr}\,(M_2 M_1) \quad \textit{and} \quad \mathrm{S\,Tr}\,(M_1 M_2 M_3) = \mathrm{S\,Tr}\,(M_3 M_1 M_2). \tag{7.45}$$

The symmetry is shown by explicit calculation:

$$\begin{aligned}
\mathrm{S\,Tr}\,(M_1 M_2) &= \mathrm{Tr}\,(A_1 A_2 + B_1 C_2) - \mathrm{Tr}\,(C_1 B_2 + D_1 D_2) \\
&= \mathrm{Tr}\,(A_1 A_2) + \mathrm{Tr}\,(B_1 C_2) - \mathrm{Tr}\,(C_1 B_2) - \mathrm{Tr}\,(D_1 D_2) \\
&= \mathrm{Tr}\,(A_2 A_1) - \mathrm{Tr}\,(C_2 B_1) + \mathrm{Tr}\,(B_2 C_1) - \mathrm{Tr}\,(D_2 D_1) \\
&= \mathrm{Tr}\,(A_2 A_1 + B_2 C_1) - \mathrm{Tr}\,(C_2 B_1 + D_2 D_1) = \mathrm{S\,Tr}\,(M_2 M_1).
\end{aligned}$$

The cyclic property follows then at once. Having defined the supertrace we introduce the superdeterminant:

Definition: The determinant of a supermatrix M is the *superdeterminant* defined by

$$\mathrm{S}\det M = \exp\{\mathrm{S}\,\mathrm{Tr}\,\log M\}. \tag{7.46}$$

On the bosonic subspace the superdeterminant is just the ordinary determinant. The superdeterminant shares many properties with the ordinary determinant. In particular we have the

Lemma 6 *Let M_1, M_2 be two graded matrices. Then we have the product rule*

$$\mathrm{S}\det(M_1 M_2) = \mathrm{S}\det(M_1)\cdot \mathrm{S}\det(M_2). \tag{7.47}$$

Proof: We set $M_i = \exp(P_i)$ and use the BAKER-CAMPBELL-HAUSDORFF formula

$$M_1 M_2 = e^{P_1} e^{P_2} = e^{P_1+P_2+\frac{1}{2}[P_1,P_2]+\dots}$$

and take the logarithm of both sides

$$\log(M_1 M_2) = P_1 + P_2 + \tfrac{1}{2}[P_1, P_2] + \dots .$$

Since the supertrace of the commutator of two supermatrices vanish, we immediately find

$$\mathrm{S}\,\mathrm{Tr}\,\log(M_1 M_2) = \mathrm{S}\,\mathrm{Tr}\,P_1 + \mathrm{S}\,\mathrm{Tr}\,P_2 = \mathrm{S}\,\mathrm{Tr}\,\log(M_1) + \mathrm{S}\,\mathrm{Tr}\,\log(M_2).$$

After exponentiation we find the product rule for the determinant. Now one has the following

Lemma 7 *The superdeterminant can be expressed in terms of ordinary determinants by the following formulae*

$$\mathrm{S}\det M = \frac{\det(A - BD^{-1}C)}{\det D} = \det A \det(D - CA^{-1}B). \tag{7.48}$$

A corresponding formula holds for $\det M$ if all entries of M are c-numbers. Then the determinants in the denominators of these formulae are in the numerator. To prove this lemma we first calculate the logarithm of particular triangular matrices. When exponentiating both sides, it is straightforward to prove, that

$$\begin{aligned} R &= \begin{pmatrix} \mathbb{1} & B \\ 0 & D \end{pmatrix} \quad \Longrightarrow \quad \log R = \begin{pmatrix} 0 & B(D-1)^{-1}\log D \\ 0 & \log D \end{pmatrix} \\ L &= \begin{pmatrix} A & 0 \\ C & \mathbb{1} \end{pmatrix} \quad \Longrightarrow \quad \log L = \begin{pmatrix} \log A & 0 \\ C(A-1)^{-1}\log A & 0 \end{pmatrix}. \end{aligned} \tag{7.49}$$

It follows that

$$\mathrm{S}\,\mathrm{Tr}\,R = -\mathrm{Tr}\,\log D,\quad \mathrm{S}\det R = \frac{1}{\det D} \quad \text{and} \quad \mathrm{S}\,\mathrm{Tr}\,L = \mathrm{Tr}\,\log A,\quad \mathrm{S}\det L = \det A.$$

Now we decompose the supermatrix M as

$$M = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ C & D \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ CA^{-1} & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} A & B \\ 0 & \mathbb{1} \end{pmatrix}.$$

Applying the product rule for superdeterminants to these two decomposition immediately leads to the formulae (7.48). Finally we need the generalization of transposition for supermatrices.

Definition: The supertransposition of a supermatrix is defined by

$$M^{ST} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{ST} = \begin{pmatrix} A^T & C^T \\ -B^T & D^T \end{pmatrix}. \tag{7.50}$$

This definition lead to the ordinary transposition law

$$(M_1 M_2)^{ST} = M_2^{ST} M_1^{ST}. \tag{7.51}$$

This can be checked by explicitly calculating both sides and using (7.42). Also, one has the

Lemma 8 *The superdeterminant is invariant under supertransposition,*

$$\mathrm{S}\det(M^{ST}) = \mathrm{S}\det(M). \tag{7.52}$$

Proof: We use the formula (7.48) for the superdeterminants and find

$$\mathrm{S}\det(M^{ST}) = \frac{\det(A^T + C^T(D^T)^{-1}B^T)}{\det D^T} = \frac{\det(A - BD^{-1}C)^T}{\det D} = \mathrm{S}\det(M).$$

After we have introduced the supertrace, superdeterminant and super-transposition of supermatrices we can now proceed and define the supergroups relevant in physics. For the infinitesimal super-group transformation generated by elements of a super-Lie algebra one prefers to work with bosonic variables only. But then one needs to introduce anti-commutators in addition to commutators. The transition is similar as that from the Noether charge $\bar{\alpha}\mathcal{Q}$ to the supercharges $\mathcal{Q}_\alpha$. Whereas $\bar{\alpha}\mathcal{Q}$ obeys commutation relations the supercharges $\mathcal{Q}_\alpha$ satisfy anti-commutation relations.

7.2 Superalgebras

The most important Lie superalgebras are the ones, which have no nontrivial invariant subalgebra. These simple finite-dimensional superalgebras are fully classified [37]. There are eight infinite families

$$s\ell(N|M), \quad osp(N|M), \quad P(N), \quad Q(N), \quad W(N), \quad S(N), \quad \tilde{S}(N), \quad H(N), \tag{7.53}$$

a continuum of 17-dimensional exceptional superalgebras and one exceptional superalgebra each in 31 and 40 dimensions,

$$D(2|1;\alpha), \quad G(3) \quad \text{and} \quad F(3). \tag{7.54}$$

In passing we note, that a semi-simple superalgebra is in general not the direct sum of simple superalgebras. A good review on superalgebras is [38].
The superalgebras of main interest in physics are the special linear, orthosymplectic and unitary ones. We shall now describe the relevant superalgebras in detail.

7.2.1 The special linear superalgebras $s\ell(N|M)$.

The supergroup $GL(N|M)$ consists of matrices of the form (7.40) with non-zero superdeterminant. This supergroup is not simple. But the sub-supergroup $S\ell(N|M)$ of matrices with superdeterminant one,

$$S\ell(N|M) = \{U \in GL(N|M)|\, \mathrm{S}\det U = 1\} = \{e^{iM} \in GL(N|M)|\, \mathrm{S}\,\mathrm{Tr}\, M = 0\}. \quad (7.55)$$

is simple. For $N \neq M$ the super-traceless $(N+M)\times(N+M)$ matrices form a simple $((N+M)^2-1)$-dimensional subalgebra $s\ell(N|M)$. For $m = n$ the unit matrix is super-traceless and generates a one-dimensional center of $s\ell(N|N)$. To achieve simplicity we must divide out this center and this leads to the superalgebras $ps\ell(N|N)$ with has dimension $4N^2-2$.
Instead of using anticommuting as well as commuting entries, it is possible to use only commuting entries. One defines the supertrace of M as in (7.30) or (7.43) but now A, B, C and D contain only commuting c-numbers. The generators of $s\ell(N|M)$ are still defined by $\mathrm{S}\,\mathrm{Tr}\,(M) = 0$, but the composition rule is modified. Instead of a commutator one has an anticommutator for the bracket of B and C matrices with themselves and with each other. As earlier we denote this composition rule by $[M_1, M_2\}$.
The dimensions of the bosonic and fermionic subspaces for gl, sl and psl superalgebras are listed in the following table

superalgebra	$\dim(\mathcal{S})$	N_0=$\dim(\mathcal{S}_0)$	N_1=$\dim(\mathcal{S}_1)$
$g\ell(N\|M)$	$(N+M)^2$	N^2+M^2	$2NM$
$s\ell(N\|M)$	$(N+M)^2-1$	N^2+M^2-1	$2NM$
$ps\ell(N)$	$(2N)^2-2$	$2N^2-2$	$2N^2$

7.2.2 The orthosymplectic superalgebras $osp(N|M)$

The orthosymplectic supergroup $OSp(N|M)$ can be defined by considering the bilinear form

$$x^i\eta_{ij}y^j + \theta^\alpha\Omega_{\alpha\beta}\zeta^\beta = x^T\eta\, y + \theta^T\Omega\,\zeta \quad (7.56)$$

with commuting $x^i\,(i = 1, \dots, N)$ and y^j and anticommuting $\theta^\alpha\,(\alpha = 1, \dots, M)$ and ζ^β. The η_{ij} and $\Omega_{\alpha\beta}$ are symmetric real and antisymmetric real metrics, respectively. Let (x, θ) and (y, ζ) transform linearly under a supermatrix U. Thus

$$\begin{pmatrix} x \\ \theta \end{pmatrix} \longrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} x \\ \theta \end{pmatrix} = U\begin{pmatrix} x \\ \theta \end{pmatrix}, \qquad \begin{pmatrix} x \\ \theta \end{pmatrix}^T = (x^T, \theta^T) \longrightarrow (x^T, \theta^T)U^{ST}. \quad (7.57)$$

The supergroups $OSp(N|M)$ leave the line 'line element' (7.56) invariant which means

$$U^{ST}\begin{pmatrix}\eta & 0\\ 0 & \Omega\end{pmatrix}U=\begin{pmatrix}\eta & 0\\ 0 & \Omega\end{pmatrix}. \tag{7.58}$$

Writing $U=\mathbb{1}+M$ this implies the following condition for the infinitesimal transformations

$$M^{ST}\begin{pmatrix}\eta & 0\\ 0 & \Omega\end{pmatrix}+\begin{pmatrix}\eta & 0\\ 0 & \Omega\end{pmatrix}M=0. \tag{7.59}$$

One may verify that the set of matrices satisfying the condition (7.59) generate a closed algebraic system: if M_1 and M_2 satisfy this condition, so does the ordinary commutator $[M_1,M_2]$. Recall, that the sub-blocks B and C are anticommuting, while A and D are commuting. Again we make the transition to commuting entries. One defines the transposition on U as in (7.50) but now A,B,C and D contain only commuting c-numbers. The generators of $OSp(N|M)$ are still defined by (7.59), but the composition rule is modified. Instead of a commutator one has an anticommutator for the bracket of B and C matrices with themselves and with each other. As earlier we denote this composition rule by $[M_1,M_2\}$. The explicit form of (7.59) reads

$$A^T\eta+\eta A=0,\quad D^T\Omega+\Omega D=0,\quad C^T\Omega+\eta B=0\quad\text{and}\quad B^T\eta-\Omega C=0. \tag{7.60}$$

The solution of the last two relations are

$$B=-\eta^{-1}C^T\Omega.$$

Using these relations for the submatrices A_i,B_i,C_i and D_i of $M_i, i=1,2$, one can explicitly check that $[M_1,M_2\}$ in (7.29) satisfies the condition (7.59). Let us recall that $Sp(M,\mathbb{R})$ is generated by real matrices D with $D^T\Omega+\Omega D=0$ with antisymmetric and real Ω. Thus, D generates $Sp(M,\mathbb{R})$, A generates $SO(N,\mathbb{R})$. The dimension of the bosonic sector is thus $\frac{1}{2}N(N-1)+\frac{1}{2}M(M+1)$, and that of the fermionic sector NM. The total dimension of the superalgebra is therefore

$$\dim(osp(N|M))=\frac{1}{2}[(N+M)^2+M-N]. \tag{7.61}$$

7.2.3 The unitary superalgebras $su(N|M)$

We consider the line element in superspace

$$(ds)^2=z^\dagger\eta z+\theta^\dagger\Omega\theta. \tag{7.62}$$

and linear transformations

$$\begin{pmatrix}z\\ \theta\end{pmatrix}\longrightarrow\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}z\\ \theta\end{pmatrix}=U\begin{pmatrix}z\\ \theta\end{pmatrix},\quad (z^\dagger,\theta^\dagger)\longrightarrow(z^\dagger,\theta^\dagger)\begin{pmatrix}A^\dagger & C^\dagger\\ -B^\dagger & D^\dagger\end{pmatrix}=(z^\dagger,\theta^\dagger)U^\dagger \tag{7.63}$$

which leave the line element invariant. Writing $U = \mathbb{1} + M$ we see, that the infinitesimal symmetry transformation obey

$$M^\dagger \begin{pmatrix} \eta & 0 \\ 0 & \Omega \end{pmatrix} + \begin{pmatrix} \eta & 0 \\ 0 & \Omega \end{pmatrix} M = 0, \quad \text{where} \quad M^\dagger = \begin{pmatrix} A^\dagger & C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix}. \tag{7.64}$$

Again we take commuting entries, but now complex, and define $u(N|M)$ by those ordinary complex matrices M which satisfy (7.64). The explicit form of this condition reads

$$A^\dagger \eta + \eta A = 0, \quad D^\dagger \Omega + \Omega D = 0, \quad C^\dagger \Omega + \eta B = 0 \quad \text{and} \quad B^\dagger \eta - \Omega C = 0. \tag{7.65}$$

The solution of the last two relations are

$$B = -\eta^{-1} C^\dagger \Omega, \qquad \eta^\dagger = \eta, \quad \Omega^\dagger = -\Omega. \tag{7.66}$$

The superalgebra $su(N|M)$ is obtained by retaining only the generators with vanishing supertrace. Since $\mathrm{S\,Tr}[M_1, M_2\} = 0$ always, these matrix form also a closed algebraic system called $su(N|M)$.

7.2.4 Further superalgebras

The elements of $s\ell(N+1|N+1)$ defined by matrices of the form (now we work with commuting entries)

$$M = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \quad \text{with} \quad \mathrm{Tr}\, A = 0,\ B = B^T,\ C = -C^T \tag{7.67}$$

form a sub-superalgebra denoted by $P(N)$. The dimension of this superalgebra is $2(N+1)^2 - 1$.
$Q(N)$ is the $2(N+1)^2 - 1$-dimensional sub-superalgebra of $s\ell(N+1|N+1)$ defined by

$$M = \begin{pmatrix} A & B \\ B & A \end{pmatrix} \quad \text{with} \quad \mathrm{S\,Tr}\, M = 0,\ \mathrm{Tr}\, B = 0. \tag{7.68}$$

Just as for ordinary Lie algebras, there exist superalgebras in some particular dimensions. They are called *exceptional*. I refer to the literature for the definition and discussion of the exceptional superalgebras $D(2|1;\alpha), G(3)$ and $F(4)$. The dimensions of these superalgebras are

$$\dim(D(2|1;\alpha)) = 9 + 8 = 17, \quad \dim(G(3)) = 17 + 14 = 31, \quad \dim(F(4)) = 30 + 10 = 40 \tag{7.69}$$

Besides there are the superalgebras of Cartan-*type*. These are $W(N), S(N), \tilde{S}(N)$ and $H(N)$.
Note, that in any spacetime dimensions the ordinary Poincaré algebra is not even semi-simple. So one may ask why one puts so much attention on simple Lie superalgebras. But similarly as for the bosonic Poincaré algebra one can obtain the super Poincaré algebra as Wigner-Inönü contraction from the simple anti-deSitter algebra.

7.3 Superalgebras containing spacetime algebras

The superalgebras of interest contain spacetime algebras. There are four spacetime algebras of interest for supergravity and supersymmetry. These are:

- The Poincaré *group* which leave the line element $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ invariant. The Poincaré algebra has been discussed in section (3.1).
- The *conformal group*, the group of transformations of x^μ which leave $ds^2 = 0$ invariant. In contains, in addition to the Poincaré transformations the dilatations and special conformal transformations. The corresponding algebra has been investigated in section (5.2).
- The anti-deSitter group $SO(2, d-1)$. This is the group of symmetries of a maximally symmetric spacetime with constant negative curvature. It admits $\frac{1}{2}d(d+1)$ Killing fields which generate a $so(2, d-1)$ algebra. This group and its algebra has been discussed in section (3.4).
- The deSitter group $SO(1, d)$. This is the group of a maximally symmetric spacetime, whose constant curvature is positive. The $\frac{1}{2}d(d+1)$ Killing fields generate a $so(1, d)$ algebra.

The deSitter and Anti-deSitter spacetimes also admit conformal Killing vector fields, and in both cases the conformal group is $SO(2, d)$, as it is in Minkowski spacetime. At this point it might be useful to recapitulate the real forms of bosonic algebras of 'classical type'. That is done in table (7.1) (taken from the review of VAN PROEYEN). The conventions which has been used for groups is that $Sp(2N) = Sp(2N, \mathbb{R})$ (always even entry) and $USp(2N, 2M) = U(N, M, \mathbb{H})$. $S\ell(N)$ is $S\ell(N, \mathbb{R})$. Furthermore, $SU^*(2N) = S\ell(N, \mathbb{H})$ and $SO^*(2N) = O(N, \mathbb{H})$. Here it maybe useful to recall the CARTAN classification of the classical groups:

groups	A_N	B_N	C_N	D_N
complexified group	$S\ell(N+1, \mathbb{C})$	$Spin(2N+1, \mathbb{C})$	$Sp(2N, \mathbb{C})$	$Spin(2N, \mathbb{C})$
compact form	$SU(N+1, \mathbb{C})$	$Spin(2N+1, \mathbb{R})$	$U(N, \mathbb{H})$	$Spin(2N, \mathbb{R})$

In these bosonic algebras there are isomorphisms which will be important later. To discuss these, it is of use to recall the Dynkin diagrams of the classical groups. These are depicted in figure (7.1).

The isomorphisms one needs in supersymmetry and supergravity are:

$$\begin{aligned}
B_1 \sim A_1 : &\quad SO(3) \simeq SU(2), \quad SO(2,1) \simeq SU(1,1) \simeq Sp(2,\mathbb{R}) \simeq Sl(2) \\
D_2 \sim A_1 \times A_1 : &\quad SO(4) \simeq SU(2) \times SU(2), \quad SO(2,2) \simeq SU(1,1) \times SU(1,1) \\
B_2 \sim C_2 : &\quad SO(5) \simeq USp(4), \quad SO(4,1) \simeq USp(2,2), \quad SO(2,3) \simeq Sp(4,\mathbb{R}) \\
D_3 \sim A_3 : &\quad SO(6) \simeq SU(4), \quad SO(1,5) \simeq SU^*(4) \\
&\quad SO(2,4) \simeq SU(2,2), \quad SO(3,3) \simeq S\ell(4)
\end{aligned}$$

Compact	Real Form	Maximal compact subalg.
$SU(N)$	$SU(p,N-p)$	$SU(p)\times SU(N-p)\times U(1)$
$SU(N)$	$S\ell(N)$	$SO(N)$
$SU(2N)$	$SU^*(2N)$	$USp(2N)$
$SO(N)$	$SO(p,N-p)$	$SO(p)\times SO(q)$
$SO(2N)$	$SO^*(2N)$	$U(N)$
$USp(2N)$	$Sp(2N)$	$U(N)$
$USp(2N)$	$USp(2p,2N-2p)$	$USp(2p)\times USp(2N-2p)$
$G_{2,-14}$	$G_{2,2}$	$SU(2)\times SU(2)$
$F_{4,-52}$	$F_{4,-20}$	$SO(9)$
$F_{4,-52}$	$F_{4,4}$	$USp(6)\times SU(2)$
$E_{6,-78}$	$E_{6,-26}$	$F_{4,-52}$
$E_{6,-78}$	$E_{6,-14}$	$SO(10)\times SO(2)$
$E_{6,-78}$	$E_{6,2}$	$SU(6)\times SU(2)$
$E_{6,-78}$	$E_{6,6}$	$USp(8)$
$E_{7,-133}$	$E_{7,-25}$	$E_{6,-78}\times SO(2)$
$E_{7,-133}$	$E_{7,-5}$	$SO(12)\times SU(2)$
$E_{7,-133}$	$E_{7,7}$	$SU(8)$
$E_{8,-248}$	$E_{8,-24}$	$E_{7,-133}\times SU(2)$
$E_{8,-248}$	$E_{8,8}$	$SO(16)$

Tabelle 7.1: Real forms of simple bosonic Lie algebras.

Note that the equality sign is not correct for the groups. For the algebras there are these isomorphisms, but it is rather the covering group of the orthogonal groups which are listed at the right hand sides.

It is remarkable that for the Poincaré, conformal, deSitter and anti-deSitter algebras super-extensions exist. I will not discuss the deSitter superalgebras, because the corresponding supergravity theories (which do exist) contain ghosts [36]. The anti-deSitter algebras are free from ghosts. The superalgebras which correspond to supersymmetric theories can now be found by looking up which superalgebra contains a given spacetime group. According to the spin-statistic theorem the odd generators (the columns of C and rows of B) must transform under the spacetime group as spinors. This means that the spacetime generators must be in the spinor representations of the $SO(p,q)$ groups. The supergroups containing the AdS and conformal algebras as spacetime symmetries in $2,4$ and 5 dimensions are

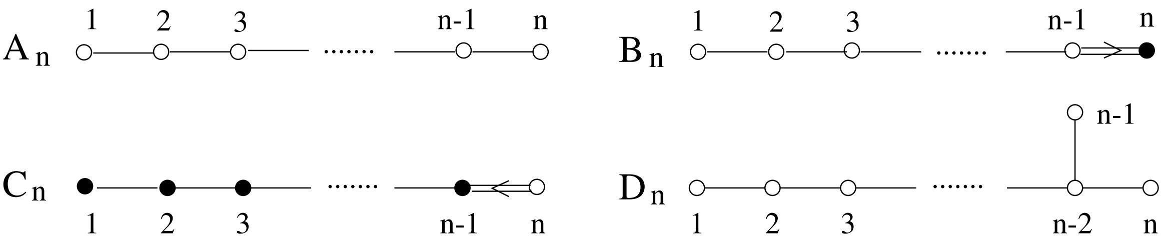


Abbildung 7.1: *Dynkin diagrams of the A_N, B_N, C_N and D_N series. $\bullet$ are short roots.*

dimensions	spacetime-group	supergroup	remarks
$d=2$	adS conformal	$OSp(N\|2)$ $OSp(N\|2)\times OSp(N\|2)$	
$d=4$	adS conformal	$OSp(N\|4)$ $SU(N\|2,2)$	$Sp(4)\simeq SO(3,2)$ is adS $SU(2,2)\simeq SO(4,2)$
$d=5$	adS conformal	$SU(N\|2,2)$ F_4?	complex spinors

In the appendix to this chapter I give a list of superalgebras of 'classical type' (taken from van Proeyen). A superalgebra is of classical type if the representation of $\mathcal{S}_0$, according to which the fermionic generators transform, is completely reducible. The non-classical superalgebras are the Cartan-type superalgebras $W(N), S(N), \tilde{S}(N)$ and $H(N)$.

7.4 Physically relevant examples of superalgebras

Let us see in more detail, how to construct supergroups and superalgebras explicitly. We shall investigate the important anti-deSitter and Poincaré superalgebras in detail.

7.4.1 Anti-deSitter algebras in $2,4$ and 5 dimensions

The Poincaré algebra is not a simple algebra, but a Wigner-Inönü contraction of the simple anti-deSitter algebra $SO(2,d-1)$. The anti-deSitter algebra is

$$[M_{mn},M_{pq}]=i(\eta_{mp}M_{nq}+\eta_{nq}M_{mp}-\eta_{mq}M_{np}-\eta_{np}M_{mq}),\qquad m,n=0,\dots d,\quad (7.70)$$

where $\eta_{mn}=\mathrm{diag}(+-\dots-+)$. Setting

$$M_{d\mu}=-M_{\mu d}\equiv RP_\mu,\quad \text{where}\quad \mu=0,\dots d-1 \qquad (7.71)$$

and denoting the remaining components of M_{mn} by $M_{\mu\nu}$, this algebra takes the form

$$\begin{aligned}
[M_{\mu\nu},M_{\rho\sigma}] &= i(\eta_{\mu\rho}M_{\nu\sigma}+\eta_{\nu\sigma}M_{\mu\rho}-\eta_{\mu\sigma}M_{\nu\rho}-\eta_{\nu\rho}M_{\mu\sigma})\\
[P_\rho,M_{\mu\nu}] &= -i(\eta_{\rho\mu}P_\nu-\eta_{\rho\nu}P_\mu)\\
[P_\mu,P_\nu] &= \frac{i}{R^2}M_{\mu\nu}.
\end{aligned}\qquad (7.72)$$

The Wigner-Inönü contraction is the statement that the $SO(2,d-1)$ algebra reduces to the Poincaré algebra when we take the limit $R\to\infty$ with constant P_μ in the limit. The same applies to the corresponding superalgebras as we shall see later.
For the following calculations it is useful to recall the formula

$$\left(\mathcal{C}^{-1}\gamma^{(n)}\right)^T=-\varepsilon(-)^{[n/2]}(-\eta)^n\mathcal{C}^{-1}\gamma^{(n)}. \tag{7.73}$$

We also need the formulae (4.75) and (4.83). But now ψ and χ are commuting objects, so that

$$\bar\psi\gamma^{(n)}\chi=-\epsilon(-)^{n(n+1)/2}\eta^t\,\bar\chi\gamma^{(n)}\psi \tag{7.74}$$

and

$$\Delta\psi\bar\chi=\sum_{p=0}^{D}\frac{(-)^{p(p-1)/2}}{p\,!}\gamma_{\mu_1\dots\mu_p}\,(\bar\chi\gamma^{\mu_1\dots\mu_p}\psi). \tag{7.75}$$

Remember, that $\Delta=2^{[d/2]}$ and $D=d$ for even d and $D=(d-1)/2$ for odd dimensions. If we choose Majorana spinors, then we take $B=\mathcal{C}A^T=\mathbb{1}$ such that Majorana spinors are real. For an antisymmetric charge conjugation matrix this implies $\mathcal{C}^{-1}=-A$ and $\bar\psi=-\psi^T\mathcal{C}^{-1}$.

Two spacetime dimensions

First we consider the anti-deSitter spacetime in 2 spacetime dimensions. The corresponding supergroup is $OSp(N,2)$. According to (7.55) the generators of $sp(2,\mathbb{R})$ satisfy

$$D^T\Omega+\Omega D=-(\Omega D)^T+\Omega D=0,\qquad \Omega^T=-\Omega, \tag{7.76}$$

i.e. the real matrices ΩD are symmetric. Let us relate $sp(2,\mathbb{R})$ to the isomorphic $so(2,1)$. Consulting our tables in section 3 we see that there exists a Majorana representation of $SO(2,1)$. We take $\mathcal{C}=\gamma_0\gamma_2$ such that Majorana spinors are real. Since $\epsilon=\eta=1$ we have

$$\mathcal{C}^T=-\mathcal{C},\quad (\mathcal{C}^{-1}\gamma_m)^T=\mathcal{C}^{-1}\gamma_m,\quad (\mathcal{C}^{-1}\gamma_{mn})^T=\mathcal{C}^{-1}\gamma_{mn},\quad \gamma_{mn}=\tfrac12[\gamma_m,\gamma_n], \tag{7.77}$$

where the indices m,n take the values $0,1$ and 2. It follows in particular, that

$$\gamma_0^T=\gamma_0,\quad \gamma_1^T=-\gamma_1\quad\text{and}\quad \gamma_2^T=\gamma_2.$$

From the known hermiticity properties of the gamma-matrices we conclude, that the γ_m and hence the charge conjugation matrix $\mathcal{C}$ are all real. The real $\frac12\gamma_{mn}$ satisfy the $SO(2,1)$ algebra (7.70) with i on the right hand side replaced by -1,

$$[\tfrac12\gamma_{mn},\tfrac12\gamma_{pq}]=-\tfrac12(\eta_{mp}\gamma_{nq}+\eta_{nq}\gamma_{mp}-\eta_{mq}\gamma_{np}-\eta_{np}\gamma_{mq}),\quad \eta=\mathrm{diag}(1,-1,1). \tag{7.78}$$

The following matrices form a Majorana representation of $so(2,1)$:

$$\gamma_0=\sigma_1,\quad \gamma_1=-i\sigma_2,\quad \gamma_2=\sigma_3\Longrightarrow\{\gamma_m,\gamma_n\}=2\eta_{mn}\mathbb{1}_2,\quad \eta_{mn}=\mathrm{diag}(1,-1,1)$$

In this representation the generators of $SO(2,1)$ have the form

$$\{\gamma_{01},\gamma_{02},\gamma_{12}\}=\{\sigma_3,-i\sigma_2,\sigma_1\},$$

and the charge conjugation matrix reads

$$\mathcal{C}=\gamma_0\gamma_2=-i\sigma_2.$$

Now we compare (7.76) with (7.77) we conclude that we may choose $\Omega=\mathcal{C}^{-1}$ in which case D is a linear combination of the $SO(2,1)$ generators with real coefficients,

$$D=\tfrac{1}{2}\omega^{mn}\gamma_{mn},\qquad \Omega=\mathcal{C}^{-1} \tag{7.79}$$

Now we choose $\eta=\mathbb{1}$ in (7.55). Then A is real and antisymmetric, that is in the Lie algebra of $SO(N)$. Hence

$$A=\alpha^{ij}\Omega_{ij},\quad (\Omega_{ij})_{kl}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk},\quad \text{e.g.}\quad \Omega_{12}=\begin{pmatrix}i\sigma_2&0\\0&0\end{pmatrix}. \tag{7.80}$$

The generators Ω_{ij} fulfill the $so(N)$ algebra, that is the algebra (7.70) with i on the right hand side replaced by -1 and η replaced by δ. Now we take as generators of the bosonic sector of the super Lie algebra

$$M_{mn}=\begin{pmatrix}0&0\\0&\frac{1}{2}\gamma_{mn}\end{pmatrix}\quad\text{and}\quad T_{ij}=\begin{pmatrix}\Omega_{ij}&0\\0&0\end{pmatrix}. \tag{7.81}$$

Clearly, the M_{mn} generate the $so(2,1)$ subalgebra and the T_{ij} the $so(N)$ subalgebra of $\mathcal{S}_0$. The generators of $so(2,1)$ commute with those of $so(N)$ such that $\mathcal{S}_0=so(2,1)\oplus so(N)$. Now we turn to the fermionic generators. According to (7.55) we have $B=-C^T\mathcal{C}^{-1}$ so that these generators have the form

$$M=\begin{pmatrix}0&-C^T\mathcal{C}^{-1}\\C&0\end{pmatrix}\in\mathcal{S}_1. \tag{7.82}$$

Here C is a general rectangular $2\times N$ matrix and therefore a linear combination of $2N$ basic generators with all but one entries equals zero. We choose as basis elements the matrices

$$e_\alpha\otimes e_i^T, \tag{7.83}$$

where the 2-dimensional column-vector e_α has entries $(e_\alpha)_\beta=\delta_{\alpha\beta}$ and the N-dimensional column vector e_i has entries $(e_i)_j=\delta_{ij}$. and correspondingly the following generators of the fermionic subspace

$$\mathcal{Q}^i_\alpha\;=\;\begin{pmatrix}0&-e^T_\alpha\mathcal{C}^{-1}\otimes e_i\\e_\alpha\otimes e_i^T&0\end{pmatrix}\equiv\begin{pmatrix}0&\bar e_\alpha\otimes e_i\\e_\alpha\otimes e_i^T&0\end{pmatrix}. \tag{7.84}$$

We used, that $-e^T_\alpha\mathcal{C}^{-1}=\bar e_\alpha$. The commutators of the bosonic with these fermionic generators are

$$[M_{mn},\mathcal{Q}^i_\alpha]=\tfrac{1}{2}(\gamma_{mn}\mathcal{Q}^i)_\alpha,\quad [T_{ij},\mathcal{Q}^i_\alpha]=-(\mathcal{Q}_\alpha\Omega)^i. \tag{7.85}$$

This shows, that the real $\mathcal{Q}^1,\dots,\mathcal{Q}^m$ transform according to to the spin $\frac{1}{2}$-representation of the $so(2,1)$. These Majorana charges do not mix under spin rotations. On the other hand, the $so(N)$ transforms the $\mathcal{Q}^i$ without affecting the spinor index α. They are rotated into each other with the N-dimensional defining representation. The anticommutator of two $\mathcal{Q}$ is easily found to be

$$\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}=\begin{pmatrix}-\mathcal{C}^{-1}_{\alpha\beta}(e_ie_j^T-e_je_i^T) & 0\\ 0 & \delta^{ij}(e_\alpha\bar e_\beta+e_\beta\bar e_\alpha\end{pmatrix}. \tag{7.86}$$

Now it is easy to calculate the anticommutator of two fermionic generators for $i\neq j$. One finds

$$\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}=-(\mathcal{C}^{-1})_{\alpha\beta}T_{ij}. \tag{7.87}$$

To calculate the anticommutator for $i=j$ we use (7.75) which implies for $so(1,2)$ the Fierz identity

$$2e_\alpha\bar e_\beta=\bar e_\alpha e_\beta\mathbb{1}+\gamma_m(\bar e_\alpha\gamma^m e_\beta)\Longrightarrow(e_\alpha\bar e_\beta+e_\beta\bar e_\alpha)=-(\mathcal{C}^{-1}\gamma^m)_{\alpha\beta}\gamma_m.$$

Now we use that $\gamma^0\gamma^1\gamma^2=\pm\mathbb{1}$ which implies

$$\gamma_m=\pm\epsilon_{mpq}\gamma^{pq}\ (\epsilon_{012}=1)\quad\text{so that}\quad(-\mathcal{C}^{-1}\gamma^m)_{\alpha\beta}\gamma_m=\tfrac{1}{2}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}\gamma_{mn}.$$

This implies the following anticommutator for $i=j$:

$$\{\mathcal{Q}^i_\alpha,\mathcal{Q}^i_\beta\}=(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}M_{mn}. \tag{7.88}$$

For contracting the super-AdS algebra to the super-Poincaré algebra, we set

$$M_{20}=RP_0\quad\text{and}\quad M_{21}=RP_1$$

so that the anti-deSitter algebra reads

$$[M_{01},P_0]=iP_1,\quad[M_{01},P_1]=-iP_0,\quad[P_0,P_1]=iR^{-2}M_{01}. \tag{7.89}$$

For $R\to\infty$ it contracts to the Poincaré algebra, as discussed above. Now, in addition, we replace $\mathcal{Q}_\alpha$ by $\mathcal{Q}_\alpha/\sqrt{R}$ in which case

$$\begin{aligned}\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}&=2\delta^{ij}\{(\mathcal{C}^{-1}\gamma^2\gamma^\mu)_{\alpha\beta}P_\mu+\tfrac{1}{R}(\mathcal{C}^{-1}\gamma^{01})_{\alpha\beta}M_{01}\}-\tfrac{1}{R}(\mathcal{C}^{-1})_{\alpha\beta}T_{ij}\\ [P_\mu,\mathcal{Q}^i_\alpha]&=\tfrac{1}{2R}(\gamma_{2\mu}\mathcal{Q}^i)_\alpha,\quad[M_{01},\mathcal{Q}^i_\alpha]=\tfrac{1}{2}(\gamma_{01}\mathcal{Q}^i)_\alpha,\quad[T_{ij},\mathcal{Q}^k_\alpha]=-(\mathcal{Q}_\alpha\Omega_{ij})^k\end{aligned} \tag{7.90}$$

Now we perform the limit $R\to\infty$ in which case (7.89) contracts to the Poincaré algebra as expected. Using that $\mathcal{C}^{-1}\gamma^2=-\gamma^0=-\mathcal{C}_2$ is, up to a sign, the charge conjugation matrix in 2-dimensional Minkowski space (with $\eta=\epsilon=-1$), the anticommutation relations in (7.90) simplify to

$$\begin{aligned}\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}&=-2\delta^{ij}(\mathcal{C}_2\gamma^\mu)_{\alpha\beta}P_\mu,\quad[P_\mu,\mathcal{Q}^i_\alpha]=0\\ [M_{01},\mathcal{Q}^i_\alpha]&=\tfrac{1}{2}(\gamma_{01}\mathcal{Q}^i)_\alpha,\quad[T_{ij},\mathcal{Q}^k_\alpha]=-(\mathcal{Q}_\alpha\Omega_{ij})^k.\end{aligned} \tag{7.91}$$

We have found a superalgebra extending the Poincaré algebra consisting of N Majorana charges $\mathcal{Q}^i$. The $\mathcal{Q}^i$ are transformed into each other by the internal $SO(N)$ transformation. The $so(N)$ rotate the Majorana supercharges $\mathcal{Q}^i$. This automorphism group is called *R-symmetry.* In accordance with the Coleman-Mandula theorem the R-symmetry commutes with the Poincaré transformations.

Four spacetime dimensions

Again we must solve $D^T\Omega+\Omega D=0$ with a real 4-dimensional, nondegenerate and antisymmetric matrix Ω. This condition is equivalent to

$$(\Omega D)^T=\Omega D \Longrightarrow D\in sp(4). \tag{7.92}$$

To relate $Sp(4)$ to $SO(2,3)$ we recall, that $SO(2,3)$ admits a Majorana representation with $\epsilon=1$ and $\eta=-1$. From (7.73) it follows, that

$$\mathcal{C}=-\mathcal{C}^T,\quad (\mathcal{C}^{-1}\gamma^m)^T=-\mathcal{C}^{-1}\gamma^m \quad\text{and}\quad (\mathcal{C}^{-1}\gamma_{mn})^T=\mathcal{C}^{-1}\gamma_{mn},\qquad m=0,\dots,4. \tag{7.93}$$

The γ_{mn} satisfy the $SO(2,4)$-algebra (7.78) with $\eta_{mn}=$diag$(1,-1,-1,-1,1)$. As charge conjugation matrix we take

$$\mathcal{C}=\gamma_0\gamma_4 \tag{7.94}$$

such that Majorana spinors are real. Then

$$\gamma_0^T=-\gamma_0,\quad \gamma_4^T=-\gamma_4$$

which implies, that the γ^m are imaginary. For example, we may choose the following explicit Majorana representation

$$\gamma^0=\sigma_0\otimes\sigma_2,\quad \gamma^1=i\sigma_0\otimes\sigma_3,\quad \gamma^2=i\sigma_1\otimes\sigma_1,\quad \gamma^3=i\sigma_3\otimes\sigma_1\quad\text{and}\quad \gamma^4=\sigma_2\otimes\sigma_1$$

in which case $\mathcal{C}=-i\sigma_2\otimes\sigma_3$. Comparing (7.92) with (7.93) we see, that we can choose

$$\Omega=\mathcal{C}^{-1}\quad\text{and}\quad D=\tfrac{1}{2}\omega^{mn}\gamma_{mn},\qquad \omega^{mn}\in\mathbb{R}.$$

This identification yields the isomorphism between $sp(4)$ and $so(2,3)$. Choosing $\eta=\mathbb{1}$ in the other constraint in (7.55) we conclude that $A\in so(N)$ and hence

$$A=\alpha^{ij}\Omega_{ij},\quad (\Omega_{ij})_{kl}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}. \tag{7.95}$$

As in two dimensions we take the following basis of the bosonic subalgebra

$$M_{mn}=\begin{pmatrix}0&0\\0&\frac{1}{2}\gamma_{mn}\end{pmatrix}\quad\text{and}\quad T_{ij}=\begin{pmatrix}\Omega_{ij}&0\\0&0\end{pmatrix}. \tag{7.96}$$

The M_{mn} satisfy the $so(2,3)$ commutation relations whereas the T_{ij} generate $so(N)$. These two subalgebras commute, in accordance with the Coleman-Mandula theorem. As basis for the fermionic subsector we again choose

$$\mathcal{Q}^i_\alpha=\begin{pmatrix}0&-e_\alpha^T\mathcal{C}^{-1}\otimes e_i\\ e_\alpha\otimes e_i^T&0\end{pmatrix}\equiv\begin{pmatrix}0&\bar e_\alpha\otimes e_i\\ e_\alpha\otimes e_i^T&0\end{pmatrix}, \tag{7.97}$$

where now e_α is the 4-dimensional column vector with matrix elements $(e_\alpha)_\beta=\delta_{\alpha\beta}$ and we used $\bar e_\alpha=-e_\alpha^T\mathcal{C}^{-1}$. The commutator of the bosonic with the fermionic generators are

$$[M_{mn},\mathcal{Q}^i_\alpha]=\tfrac{1}{2}(\gamma_{mn}\mathcal{Q}^i)_\alpha,\quad [T_{ij},\mathcal{Q}^k_\alpha]=-(\mathcal{Q}_\alpha\Omega)^k. \tag{7.98}$$

As expected, the 4-dimensional real column vectors $\mathcal{Q}^1,\dots,\mathcal{Q}^N$ transform as Majorana spinors under the spacetime group $SO(2,3)$ and the N-dimensional real row vectors $\mathcal{Q}_1,\dots\mathcal{Q}_4$ transform as vectors under the group $SO(N)$. As in the 2-dimensional case the R-symmetry group is $SO(N)$. To calculate the anticommutator of two fermionic charges we use

$$4(e_\alpha\bar{e}_\beta+e_\beta\bar{e}_\alpha)=\gamma_{mn}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}$$

which follows from (7.75) and the (anti)symmetry properties of the matrices $\mathcal{C}^{-1}\gamma^{(n)}$. Therefore the anticommutator of two supercharges is given by

$$\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}=\tfrac{1}{2}\delta^{ij}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}M_{mn}-(\mathcal{C}^{-1})_{\alpha\beta}T_{ij}. \tag{7.99}$$

Now we perform the $4+1$-split and rescale the supercharges and $M_{d\mu}$ and end up with the algebra

$$\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}=(\mathcal{C}^{-1}\gamma^4\gamma^\mu)_{\alpha\beta}\delta^{ij}P_\mu+\tfrac{1}{2R}(\mathcal{C}^{-1}\gamma^{\mu\nu})_{\alpha\beta}\delta^{ij}M_{\mu\nu}-\tfrac{1}{R}(\mathcal{C}^{-1})_{\alpha\beta}T_{ij}. \tag{7.100}$$

Since $\mathcal{C}^{-1}\gamma^4=-\gamma^0=\mathcal{C}_4$ is the antisymmetric charge conjugation matrix in 4-dimensional Minkowski spacetime we obtain in the limit $R\to\infty$ the Poincaré algebra plus the following (anti)commutators:

$$\begin{aligned}\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}&=(\mathcal{C}_4\gamma^\mu)_{\alpha\beta}\delta^{ij}P_\mu,\qquad[P_\mu,\mathcal{Q}^i_\alpha]=0,\qquad[T_{ij},\mathcal{Q}^k_\alpha]=-(\mathcal{Q}_\alpha\Omega_{ij})^k\\ [M_{\mu\nu},\mathcal{Q}^i_\alpha]&=\tfrac{1}{2}(\gamma_{\mu\nu}\mathcal{Q}^i)_\alpha,\qquad[M_{\mu\nu},T_{ij}]=[P_\mu,T_{ij}]=0.\end{aligned} \tag{7.101}$$

Together with the Poincaré algebra we have derived the (anti)commutation for supersymmetric theories with N-Majorana supercharges. Later we shall restrict the values of N by physical arguments. For the simplest $N=1$ theories there are no additional generators besides $M_{\mu\nu},P_\mu$ and $\mathcal{Q}_\alpha$ and we rediscover the supersymmetry algebra of the corresponding Noether charges in the Wess-Zumino model.

Five spacetime dimensions: AdS_5

I discuss this case since it plays an important role in recent developments in field theory. The so-called AdS/CFT-correspondence is base on the relation between supersymmetric theories on AdS_5 and conformal field theories on the boundary of AdS_5 which can be identified with Minkowski spacetime. The symmetry algebra of AdS_5 is the same as the conformal algebra in Minkowski spacetime, namely $so(2,4)$.
There is a Majorana representation of $SO(2,4)$ and it has $(\epsilon,\eta)=(1,-1)$. Then the 8-dimensional γ-matrices and charge conjugation have the following symmetry properties,

$$(\mathcal{C}^{-1}\gamma^{n)})^T=-(-)^{[n/2]}\mathcal{C}^{-1}\gamma^n,\qquad n=0,\dots,6. \tag{7.102}$$

It follows

$$\mathcal{C}^{-1}\gamma_{mn},\quad\mathcal{C}^{-1}\gamma_{mnp}\quad\text{and}\quad\mathcal{C}^{-1}\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5\equiv\mathcal{C}^{-1}\gamma_* \tag{7.103}$$

are symmetric[2] and the others are antisymmetric. Note that $\gamma_*^2 = -\mathbb{1}$ and $\gamma_*^T = -\gamma_*$. Choosing $CA^T = \mathbb{1}$, such that Majorana spinors are real, we find

$$C = \gamma_0\gamma_5 = -\mathcal{C}^T = -\mathcal{C}^{-1} = -\gamma_5\gamma_0, \quad \gamma_0^T = \mathcal{C}^{-1}\gamma_0\mathcal{C} = -\gamma_0. \tag{7.104}$$

Hence, γ_0 and γ_5 are antisymmetric, whereas the others γ_i are symmetric. All γ_m are imaginary in this representation.

We want to use the isomorphism $SO(2,4) \sim SU(2,2)$ and construct the superalgebra $su(N|2,2)$ which contains the anti-deSitter algebra as subalgebra of the bosonic algebra $su(N) \times su(2,2) \times u(1)$. If we want to use the 8-dimensional real Majorana representation then we should use the 8-dimensional real forms of $su(2,2)$ and $su(N)$. For that one notes, that a unitary matrix $U = \Re U + i\Im U \in U(N)$ can be represented as

$$R = \begin{pmatrix} \Re U & -\Im U \\ \Im U & \Re U \end{pmatrix} \in SO(2N).$$

To see that we note that $UU^\dagger = \mathbb{1}$ implies

$$\Re U(\Re U)^T + \Im U(\Im U)^T = \mathbb{1}, \quad \Re U(\Im U)^T - \Im U(\Re U)^T = 0$$

so that R is orthogonal, $R^T R = \mathbb{1}$. Correspondingly the Lie algebra $u(N)$ can be represented by matrices

$$A = \begin{pmatrix} a & -s \\ s & a \end{pmatrix} = \sigma_0 \otimes a - i\sigma_2 \otimes s, \quad \text{where} \quad a^T = -a, \quad s^T = s.$$

An equivalent way to characterize the real form of the Lie algebra of $u(N)$ is

$$u(N) \sim \{A \in \mathrm{Mat}_{2N} \,|\, A^T = -A,\ A\Omega = \Omega A\}, \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & -\mathbb{1}_N \\ \mathbb{1}_N & 0 \end{pmatrix}. \tag{7.105}$$

Note that Ω itself is element of the subalgebra and commutes with all elements. Hence we may decompose $s(N) = su(N) \times U(1)$ as follows

$$su(N) = \{A \in u(N) \,|\, \mathrm{Tr}\,\Omega A = 0\}, \qquad u(1) = \{A = \alpha\Omega\}. \tag{7.106}$$

In other words, for the generators of $su(N)$ the symmetric matrix s is traceless. With similar arguments one shows that there exists a 8-dimensional real form $u(2,2)$. In this representation the elements have the form

$$u(2,2) = \{D \in Mat_8 | (\mathcal{C}^{-1}D)^T = \mathcal{C}^{-1}D,\ \gamma_* D = D\gamma_*\}. \tag{7.107}$$

Instead of repeating the arguments leading to the real form of $u(N)$ we show, that the infinitesimal generators satisfy the $so(2,4) \times so(2) \sim u(2,2)$ commutation relation. According to (7.103) the matrix D must be a linear combination of

$$\gamma^{mn}, \quad i\gamma^{mnp} \quad \text{and} \quad \gamma_*.$$

[2]For convenience we have chosen a choice for γ_* which differs from the choice we made in section 3.

But the product of three gamma-matrices does not commute with γ_* and hence D must be a linear combination of the real matrices γ^{mn} and γ_*. Since γ_* commutes with the γ^{mn} and the latter generate $so(2,4)$ this proves (7.107). Now we are ready to define our bosonic generators. For that we define the following traceless matrices

$$a_{ij} = e_i e_j^T - e_j e_i^T \quad \text{and} \quad s_{ij} = e_i e_j^T + e_j e_i^T - \frac{2}{N}\delta_{ij}\,\mathbb{1}_N \tag{7.108}$$

and the associated generators of $su(N)$:

$$A_{ij} = \begin{pmatrix} \sigma_0 \otimes a_{ij} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad S_{ij} = \begin{pmatrix} -i\sigma_2 \otimes s_{ij} & 0 \\ 0 & 0 \end{pmatrix}. \tag{7.109}$$

As generators of $su(2,2)$ we take

$$M_{mn} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\gamma_{mn} \end{pmatrix}. \tag{7.110}$$

As further generator which commutes with $su(N)$ and with $su(2,2)$ we take

$$S = \frac{1}{N}\begin{pmatrix} \Omega & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 0 & 0 \\ 0 & \gamma_* \end{pmatrix}. \tag{7.111}$$

In the complex form this would correspond to the $u(1)$ generator with vanishing supertrace. The real bosonic generators

$$\{A_{ij}, S_{ij}, S, M_{mn}\}$$

generate the sub Lie algebra $su(N) \times u(1) \times su(2,2)$. Now we come the the fermionic generators. Guided by our earlier experience or by rewriting the complex into the real form one is lead to the following form for these generators:

$$\mathcal{Q}^i_\alpha = \begin{pmatrix} 0 & 0 & -e_\alpha^T \mathcal{C}^{-1} \otimes e_i \\ 0 & 0 & -(\gamma_* e_\alpha)^T \mathcal{C}^{-1} \otimes e_i \\ e_\alpha \otimes e_i^T & \gamma_* e_\alpha \otimes e_i^T & 0 \end{pmatrix}. \tag{7.112}$$

The e_i are the usual basis column vectors in $\mathbb{R}^N$ and the e_α the basis column vectors in $\mathbb{R}^8$. When calculating the anticommutator of $\mathcal{Q}^i_\alpha$ with $\mathcal{Q}^j_\beta$ one encounters the following expressions: The $u(N)$ part in the anticommutator is

$$-(\mathcal{C}^{-1})_{\alpha\beta}\sigma_0 \otimes a_{ij} - i(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}\sigma_2 \otimes (s_{ij} + \tfrac{2}{N}\delta_{ij}\mathbb{1}),$$

whereas the $u(2,2)$ part reads

$$-\delta_{ij}(e_\alpha e_\beta^T + e_\beta e_\alpha^T)\mathcal{C}^{-1} + \delta_{ij}\gamma_*(e_\alpha e_\beta^T + e_\beta e_\alpha^T)\gamma_*\mathcal{C}^{-1}.$$

Now one uses the following Fierz identity for the $so(2,4)$ Clifford algebra:

$$(e_\alpha e_\beta^T + e_\beta e_\alpha^T)\mathcal{C}^{-1} = -\gamma_{mn}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta} - \tfrac{1}{3}\gamma_{mnp}(\mathcal{C}^{-1}\gamma^{mnp})_{\alpha\beta} - 2\gamma_*(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta} \tag{7.113}$$

to prove, that the $U(2,2)$ part becomes

$$\tfrac{1}{4}\delta_{ij}\{\gamma_{mn}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}+2\gamma_*(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}\}.$$

Collecting our results we end up with the following anticommutator

$$\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}=-(\mathcal{C}^{-1})_{\alpha\beta}A_{ij}+(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}S_{ij}+2(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}\delta_{ij}S+\tfrac{1}{2}\delta_{ij}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}M_{mn} \tag{7.114}$$

The commutator of the bosonic charges with the supercharges is easily computed to be

$$\begin{aligned}[A_{jk},\mathcal{Q}^i_\alpha]&=(a_{jk})_{pi}\mathcal{Q}^p_\alpha \quad , \quad [S_{jk},\mathcal{Q}^i_\alpha]=-(s_{jk})_{pi}(\gamma_*)_{\sigma\alpha}\mathcal{Q}^p_\sigma\\ [S,\mathcal{Q}^i_\alpha]&=(\tfrac{1}{4}-\tfrac{1}{N})(\gamma_*)_{\beta\alpha}\mathcal{Q}^i_\beta \quad , \quad [M_{mn},\mathcal{Q}^i_\alpha]=\tfrac{1}{2}(\gamma_{mn})_{\beta\alpha}\mathcal{Q}^i_\beta.\end{aligned} \tag{7.115}$$

Note that for $N=4$ the $U(1)$ charge S commutes with all other generators. But the anticommutator of two supercharges still contains S. Hence, for $N=4$ the anti-deSitter superalgebra becomes non-simple. The non-extended $N=1$-superalgebra simplifies considerably since in this case the R-symmetry group is just a $U(1)$. The super-CR read

$$\{\mathcal{Q}_\alpha,\mathcal{Q}_\beta\}=2(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}S+\tfrac{1}{2}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}M_{mn}\quad\text{and}\quad[S,\mathcal{Q}_\alpha]=\tfrac{3}{4}(\gamma_*)_{\alpha\beta}\mathcal{Q}_\beta. \tag{7.116}$$

This makes clear, that the $U(1)$-transformation is just a chiral rotation of the supercharges. The lefthanded charges have the opposite transformation rule as the righthanded ones.

7.4.2 Appendix: AdS_5 in chiral basis

The elements of the superalgebras $SU(N|2,2)$ have the form

$$M=\begin{pmatrix}A & -C^\dagger\Omega\\ C & D\end{pmatrix}\quad\text{with}\quad A^\dagger=-A,\quad(\Omega D)^\dagger=\Omega D,\quad\text{where}\quad\Omega^\dagger=-\Omega. \tag{7.117}$$

We see, that $A\in u(N)$. In addition, since the columns of C transform according to the 4-dimensional complex representation of $su(2,2)\sim so(2,4)$ this can only be a chiral spinor representation of $SO(2,4)$. To continue we note that

$$(\mathcal{A}\gamma^{(n)})^\dagger=-(-)^{n(n+1)/2}\mathcal{A}\gamma^{(n)},\qquad\mathcal{A}=\gamma_0\gamma_5=-\mathcal{A}^\dagger. \tag{7.118}$$

Since we want to a chiral representation we can only use those matrices which preserve chirality. Out of these we have the following hermitian and antihermitian matrices

$$\text{hermitian:}\quad\mathcal{A}\gamma^{mn},\ i\mathcal{A}\gamma_*\quad,\quad\text{antihermitian:}\quad\mathcal{A},\ \mathcal{A}\gamma^{mnpq}. \tag{7.119}$$

Thus we may take

$$\Omega=P_L\mathcal{A}\equiv\mathcal{A},\quad\text{where}\quad P_L=\tfrac{1}{2}(1-\gamma_*),\quad\gamma_*=i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$$

and

$$D=\tfrac{1}{2}\omega^{mn}\sigma_{mn}+\tfrac{i}{4}\omega\,\mathbb{1}_4,\qquad\sigma_{mn}=P_L\gamma_{mn}$$

since $\mathcal{A}D$ is then a general hermitian matrix. Note that now we have chosen a hermitian γ_*, contrary to our choice in the Majorana basis. Also we have used that on the chiral sectors γ_* is $\pm\mathbb{1}$. Next we observe that σ_{mn} is traceless:

$$\mathrm{Tr}\,\gamma_m\gamma_n = 8\eta_{mn},\quad \mathrm{Tr}\,\gamma_*\gamma_m\gamma_n = \mathrm{Tr}\,\gamma_*\gamma_m\gamma_n\gamma_0\gamma_0 = -\mathrm{Tr}\,\gamma_0\gamma_*\gamma_m\gamma_n\gamma_0 = -\mathrm{Tr}\,\gamma_*\gamma_m\gamma_n$$

from which immediately follows that $\mathrm{Tr}\,\sigma_{mn} = \mathrm{Tr}\,P_L\gamma_{mn} = 0$. As a consequence

$$\mathrm{Tr}\,D = i.$$

In passing we note, that

$$\mathcal{A}^2 = -\mathbb{1},\ \mathrm{Tr}\mathcal{A} = 0,\ \mathcal{A}^\dagger = -\mathcal{A},\quad \text{so that}\quad \mathcal{A}\sim \mathrm{diag}\,(i,i,-i,-i)$$

which shows, that $D\in u(2,2)$. As bosonic generators of $su(N|2,2)$ we may take the following supertraceless supermatrices

$$M_{mn} = \begin{pmatrix} 0 & 0\\ 0 & \frac{1}{2}\sigma_{mn}\end{pmatrix},\quad X = iS,\quad X_{ij} = \begin{pmatrix} s_{ij} & 0\\ 0 & 0\end{pmatrix},\quad Y_{ij} = \begin{pmatrix} a_{ij} & 0\\ 0 & 0\end{pmatrix},\tag{7.120}$$

where the a_{ij}, s_{ij} have been introduce in the previous subsection. As earlier e_i denotes the N-dimensional column vector with $(e_i)_j = \delta_{ij}$. This shows that the bosonic subalgebra of $su(N|2,2)$ is $su(N)\times su(2|2)\times u(1)$. In particular we have

$$[X, M_{mn}] = [X, X_{ij}] = [X, Y_{ij}] = 0.$$

Note that for $N=4$ the $U(1)$ factor is generated by the identity and hence the superalgebra is not simple in this case. To get a simple superalgebra for $N=4$ we must not include X in the list of generators and then the bosonic subalgebra of $su(4|2,2)$ is $su(4)\times su(2,2)$. Since $\psi^\dagger\mathcal{A} = \bar\psi$ we may take as fermionic generators

$$\mathcal{Q}^i_\alpha = \begin{pmatrix} 0 & -\bar e_\alpha\otimes e_i\\ e_\alpha\otimes e_i^T & 0\end{pmatrix}\quad\text{and}\quad \tilde{\mathcal{Q}}^i_\alpha = \begin{pmatrix} 0 & i\bar e_\alpha\otimes e_i\\ ie_\alpha\otimes e_i^T & 0\end{pmatrix}.\tag{7.121}$$

The 4-dimensional column vector e_α has entries $(e_\alpha)_\beta = \delta_{\alpha\beta}$. The bosonic generators M_{mn} satisfy the $so(2,4)$ commutation relations. In addition

$$\begin{aligned}
[M_{mn}, \mathcal{Q}^i_\alpha] &= \tfrac12(\Re\sigma_{mn})_{\beta\alpha}\mathcal{Q}^i_\beta + \tfrac12(\Im\sigma_{mn})_{\beta\alpha}\tilde{\mathcal{Q}}^i_\beta\\
[M_{mn}, \tilde{\mathcal{Q}}^i_\alpha] &= \tfrac12(\Re\sigma_{mn})_{\beta\alpha}\tilde{\mathcal{Q}}^i_\beta - \tfrac12(\Im\sigma_{mn})_{\beta\alpha}\mathcal{Q}^i_\beta\\
[X_{jk}, \mathcal{Q}^i_\alpha] &= i(s_{jk})_{pi}\tilde{\mathcal{Q}}^p_\alpha\quad,\quad [Y_{jk}, \mathcal{Q}^i_\alpha] = \mathcal{Q}^p_\alpha(a_{jk})_{pi}\\
[X_{jk}, \tilde{\mathcal{Q}}^i_\alpha] &= -i(s_{jk})_{pi}\mathcal{Q}^p_\alpha\quad,\quad [Y_{jk}, \tilde{\mathcal{Q}}^i_\alpha] = \tilde{\mathcal{Q}}^p_\alpha(y_{jk})_{pi}.\\
[X, \mathcal{Q}^i_\alpha] &= (\tfrac14 - \tfrac1N)\tilde{\mathcal{Q}}^i_\alpha\quad\text{and}\quad [X, \tilde{\mathcal{Q}}^i_\alpha] = -(\tfrac14 - \tfrac1N)\mathcal{Q}^i_\alpha.
\end{aligned}$$

The supercharges $\mathcal{Q}^i,\quad i=1,\dots,N$ transform as left handed Weyl-spinors under the generators M_{mn}. The $\mathcal{Q}_\alpha, \alpha = 1\dots,4$ transform under the defining N-dimensional representation of $su(N)$. The anticommutator of the supercharges with $i\neq j$ yields

$$\begin{aligned}
\{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} &= -(\Re\mathcal{A})_{\alpha\beta}Y_{ij} - (\Im\mathcal{A})_{\alpha\beta}(X_{ij} + 2X\delta_{ij}) - \delta_{ij}\begin{pmatrix} 0 & 0\\ 0 & e_\alpha\bar e_\beta + e_\beta\bar e_\alpha - \frac{i}{2}\mathbb{1}(\Im\mathcal{A})_{\alpha\beta}\end{pmatrix}\\
\{\mathcal{Q}^i_\alpha, \tilde{\mathcal{Q}}^j_\beta\} &= -(\Re\mathcal{A})_{\alpha\beta}(X_{ij} + 2X\delta_{ij}) + (\Im\mathcal{A})_{\alpha\beta}Y_{ij} + i\delta_{ij}\begin{pmatrix} 0 & 0\\ 0 & e_\alpha\bar e_\beta - e_\beta\bar e_\alpha + \frac12\mathbb{1}(\Re\mathcal{A})_{\alpha\beta}\end{pmatrix}.
\end{aligned}$$

where we have decomposed

$$\mathcal{A} = \Re\mathcal{A} + i\Im\mathcal{A}, \qquad (\Re\mathcal{A})_{\alpha\beta} = -(\Re\mathcal{A})_{\beta\alpha}, \quad (\Im\mathcal{A})_{\alpha\beta} = (\Im\mathcal{A})_{\beta\alpha}.$$

Now we use the Fierz identities

$$\begin{aligned} 4(e_\alpha\bar{e}_\beta + e_\beta\bar{e}_\alpha) &= -\tfrac{1}{2}\sigma_{mn}\{(\bar{e}_\beta\sigma^{mn}e_\alpha) + (\bar{e}_\alpha\sigma^{mn}e_\beta)\} + 2\,\mathbb{1}_4(\Im\mathcal{A})_{\alpha\beta} \\ 4i(e_\alpha\bar{e}_\beta - e_\beta\bar{e}_\alpha) &= -\tfrac{i}{2}\sigma_{mn}\{(\bar{e}_\beta\sigma^{mn}e_\alpha) - (\bar{e}_\alpha\sigma^{mn}e_\beta)\} - 2i\,\mathbb{1}_4(\Re\mathcal{A})_{\alpha\beta} \end{aligned}$$

and find

$$\begin{aligned} \{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} &= -(\Re\mathcal{A})_{\alpha\beta}Y_{ij} - (\Im\mathcal{A})_{\alpha\beta}(X_{ij} + 2X\delta_{ij}) + \tfrac{1}{4}\delta_{ij}\{(\mathcal{A}\sigma^{mn})_{\alpha\beta} + (\mathcal{A}\sigma^{mn})_{\beta\alpha}\}M_{mn} \\ \{\mathcal{Q}^i_\alpha, \tilde{\mathcal{Q}}^j_\beta\} &= -(\Re\mathcal{A})_{\alpha\beta}(X_{ij} + 2X\delta_{ij}) + (\Im\mathcal{A})_{\alpha\beta}Y_{ij} + \tfrac{i}{4}\delta_{ij}\{(\mathcal{A}\sigma^{mn})_{\alpha\beta} - (\mathcal{A}\sigma^{mn})_{\beta\alpha}\}M_{mn}. \end{aligned}$$

The $\tilde{\mathcal{Q}}$ fulfill the same anticommutation rules as the $\mathcal{Q}$. In addition, one has

$$\{\mathcal{Q}^i_\alpha, \tilde{\mathcal{Q}}^j_\beta\} = -\{\tilde{\mathcal{Q}}^i_\alpha, \mathcal{Q}^j_\beta\}.$$

Again we see, that for $N = 4$ the $U(1)$ generator commutes with all other generators and the superalgebra ceases to be simple.
I leave it to an exercise to prove the equivalence of the superalgebras in the Majorana and the Weyl basis. In any case, once again we see that it is advantageous to the real basis, if possible.

7.5 Appendix: superalgebras of classical type

Tabelle 7.2: Lie superalgebras of classical type.

Name	Range	Bosonic algebra	Defining repres.	Number of generators
$SU(N\vert M)$	$N \geq 2$ $N \neq M$ $N = M$	$SU(N) \oplus SU(M)$ $\oplus U(1)$ no $U(1)$	$(N, \bar{M}) \oplus$ $(\bar{N}, M)$	$N^2 + M^2 - 1$, $2NM$ $2(N^2 - 1), 2N^2$
$S\ell(N\vert M)$ $SU(N-p, p\vert M-q, q)$ $SU^*(2N\vert 2N)$		$S\ell(N) \oplus S\ell(M) \oplus SO(1,1)$ $SU(N-p, p) \oplus SU(M-q, q) \oplus U(1)$ $SU^*(2N) \oplus SU^*(2M) \oplus SO(1,1)$		if $N \neq M$
$OSp(N\vert M)$	$N \geq 1$ $M = 2, 4, ..$	$SO(N) \oplus Sp(M)$	(N, M)	$\frac{1}{2}(N^2 - N +$ $M^2 + M), NM$
$OSp(N-p, p\vert M)$ $OSp(N^*\vert M-q, q)$		$SO(N-p, p) \oplus Sp(M)$ $SO^*(N) \oplus USp(M-q, q)$		M even N, M, q even
$D(2, 1, \alpha)$	$0 < \alpha \leq 1$	$SO(4) \oplus S\ell(2)$	$(2, 2, 2)$	9, 8
$D^p(2, 1, \alpha)$		$SO(4-p, p) \oplus S\ell(2)$		$p = 0, 1, 2$
$F(4)$		$\overline{SO(7)} \oplus S\ell(2)$	$(8, 2)$	21, 16
$F^p(4)$		$SO(7-p, p) \oplus S\ell(2)$		$p = 0, 1, 2, 3$
$G(3)$		$G_2 \oplus S\ell(2)$	$(7, 2)$	14, 14
$G_p(3)$		$G_{2,p} \oplus S\ell(2)$		$p = -14, 2$
$P(N-1)$	$N \geq 3$	$S\ell(N)$	$(N \otimes N)$	$N^2 - 1, N^2$
$Q(N-1)$	$N \geq 3$	$SU(N)$	Adjoint	$N^2 - 1, N^2 - 1$
$Q(N-1)$ $Q((N-1)^*)$ $UQ(p, N-1-p)$		$S\ell(N)$ $SU^*(N)$ $SU(p, N-p)$		

Kapitel 8

Representations of susy algebras

We want to study *representations* of the most relevant spacetime superalgebras we have introduced in the last chapter. Let us start with the algebra in 4 dimensions in the Weyl basis. I recall our conventions: We take the chiral representation

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \tilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \tag{8.1}$$

where we have introduced the matrices

$$\sigma_\mu = (\sigma_0, -\sigma_i), \quad \tilde{\sigma}_\mu = (\sigma_0, \sigma_i).$$

The infinitesimal spinor-rotations take the form

$$\gamma_{\mu\nu} = \begin{pmatrix} \sigma_{\mu\nu} & 0 \\ 0 & \tilde{\sigma}_{\mu\nu} \end{pmatrix} \quad \text{with} \quad \begin{aligned} \sigma_{\mu\nu} &= \tfrac{1}{2}(\sigma_\mu\tilde{\sigma}_\nu - \sigma_\nu\tilde{\sigma}_\mu) \\ \tilde{\sigma}_{\mu\nu} &= \tfrac{1}{2}(\tilde{\sigma}_\mu\sigma_\nu - \tilde{\sigma}_\nu\sigma_\mu). \end{aligned} \tag{8.2}$$

Note that the σ_{0i} are hermitean, whereas the σ_{ij} are antihermitian. A Dirac spinor consists of a lefthanded and a righthanded part,

$$\mathcal{Q} = \begin{pmatrix} \mathcal{Q}_\alpha \\ \bar{\mathcal{Q}}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\mathcal{Q}} = (\mathcal{Q}^\alpha, \bar{\mathcal{Q}}_{\dot{\alpha}}) = \mathcal{Q}^\dagger\gamma^0 \Longrightarrow \bar{\mathcal{Q}}_{\dot{\alpha}} = \mathcal{Q}^\dagger_\alpha, \quad \bar{\mathcal{Q}}^{\dot{\alpha}} = (\mathcal{Q}^\alpha)^\dagger. \tag{8.3}$$

The raising and lowering of the indices are done with ε:

$$\begin{aligned} \mathcal{Q}^\alpha &= \varepsilon^{\alpha\beta}\,\mathcal{Q}_\beta \quad , \quad \mathcal{Q}_\alpha = \varepsilon_{\alpha\beta}\mathcal{Q}^\beta \\ \bar{\mathcal{Q}}^{\dot{\alpha}} &= \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{Q}}_{\dot{\beta}} \quad , \quad \bar{\mathcal{Q}}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\mathcal{Q}}^{\dot{\beta}} \end{aligned}$$

where we introduced

$$(\varepsilon_{\alpha\beta}) = (\varepsilon_{\dot{\alpha}\dot{\beta}}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad (\varepsilon^{\alpha\beta}) = (\varepsilon^{\dot{\alpha}\dot{\beta}}) \quad = \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{8.4}$$

The index structure of the relevant generators are

$$(\sigma_\mu)_{\alpha\dot{\alpha}}, \quad (\tilde{\sigma}_\mu)^{\dot{\alpha}\alpha}, \quad (\sigma_{\mu\nu})_\alpha^{\ \beta}, \quad (\tilde{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}. \tag{8.5}$$

We have discussed that the supercharges should have spin $\frac{1}{2}$ which fixes their commutation relations with the Lorentz boosts. They must commute with the translations such that the extended superalgebra contains the Poincaré algebra plus

$$\begin{aligned}
[J_{\mu\nu}, \mathcal{Q}^i_\alpha] &= \tfrac{1}{2i}(\sigma_{\mu\nu})_\alpha{}^\beta \mathcal{Q}^i_\beta \qquad [J_{\mu\nu}, \bar{\mathcal{Q}}^{\dot\alpha}_i] = \tfrac{1}{2i}(\tilde\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{\mathcal{Q}}^{\dot\beta}_i, \\
\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}^j_{\dot\beta}\} &= 2\delta^{ij}(\sigma^\mu)_{\alpha\dot\beta}P_\mu \qquad [\mathcal{Q}^i_\alpha, P_\mu] = 0.
\end{aligned} \tag{8.6}$$

The missing supercommutators will be discussed soon.

8.1 Hamiltonian and Central charges

The sign in the anticommutator of $\mathcal{Q}$ and $\bar{\mathcal{Q}}$ is determined by the requirement that the energy should be a positive definite operator: we get for each value of the index i

$$\sum_{\alpha=1}^{2}\{\mathcal{Q}^i_\alpha, \mathcal{Q}^{i\dagger}_\alpha\} = 2\mathrm{Tr}\,\sigma^\mu P_\mu = 4P_0 \quad \text{no sum over i.} \tag{8.7}$$

The left hand side is manifestly positive, since each term is positive,

$$\langle\psi|\{\mathcal{Q}^i_\alpha, \mathcal{Q}^{i\dagger}_\alpha\}\psi\rangle = ||\mathcal{Q}^i_\alpha\psi||^2 + ||\mathcal{Q}^{i\dagger}_\alpha\psi||^2 \geq 0,$$

it follows that

• the *spectrum of $H = P_0$ in a theory with supersymmetry contains no negative eigenvalues.*

We denote the state (or family of states) with the lowest energy by $|0\rangle$ and call it vacuum state. The vacuum will have zero energy

$$H|0\rangle = 0 \tag{8.8}$$

if and only if

$$\mathcal{Q}^i_\alpha|0\rangle = 0 \quad \text{and} \quad \mathcal{Q}^{i\dagger}_\alpha|0\rangle = 0 \qquad \forall\, \alpha, i. \tag{8.9}$$

Any state with positive energy cannot be invariant under supersymmetry transformations. It follows in particular that every one-particle state $|1\rangle$ must have super partner states $\mathcal{Q}^i_\alpha|1\rangle$ or $\mathcal{Q}^{i\dagger}_\alpha|1\rangle$. The spin of these partners will differ by $\frac{1}{2}$ from that of $|1\rangle$. Thus

• *each supermultiplett must contain at least one boson and one fermion whose spins differ by $\frac{1}{2}$.*

The translation invariance of $\mathcal{Q}$ implies that $\mathcal{Q}$ does not change energy and momentum

$$P_\mu|p\rangle = p_\mu|p\rangle \Longrightarrow P_\mu\mathcal{Q}^i_\alpha|p\rangle = p_\mu\mathcal{Q}^i_\alpha|p\rangle, \qquad P_\mu\mathcal{Q}^{i\dagger}_\alpha|p\rangle = p_\mu\mathcal{Q}^{i\dagger}_\alpha|p\rangle,$$

and therefore

• *all states in a multiplet of unbroken supersymmetry have the same mass.*

Supersymmetry is spontaneously broken if the ground state will not be invariant under all supersymmetry transformations,

$$\mathcal{Q}^i_\alpha|0\rangle \neq 0 \quad \text{or} \quad \mathcal{Q}^{i\dagger}_\alpha|0\rangle \neq 0 \tag{8.10}$$

for same α and i. We conclude that

• *supersymmetry is spontaneously broken if and only if the energy of the lowest lying state is not exactly zero.*

We come back to supersymmetrie-breaking in the next chapter. The supercharge $\mathcal{Q}^i$ could be a DIRAC, Majorana or Weyl spinor. In certain dimensions (e.g. 2 and 10) it could even be a MAJORANA-WEYL spinor.
As we have seen, in *extended supersymmetry* the $\mathcal{Q}$ may carry some representation of the internal symmetry,

$$[T_r, \mathcal{Q}^i_\alpha] = (t_r)^i{}_j \mathcal{Q}^j_\alpha. \tag{8.11}$$

Since we assume this so-called *R-symmetry* to be compact, the representation matrices t can be chosen Hermitian, $t_r = t_r^\dagger$. Then

$$[T_r, \bar{\mathcal{Q}}^i_{\dot\alpha}] = -\bar{\mathcal{Q}}^j_{\dot\alpha}(t_r)_j{}^i. \tag{8.12}$$

Now let us consider the anticommutator $\{\mathcal{Q},\mathcal{Q}\}$. It must be a linear combination of the bosonic operators in the representation $(0,0)$ and $(1,0)$ of the Lorentz group. The only three-dimensional $(1,0)$ representation in the bosonic sector is the (anti)selfdual part of $J_{\mu\nu}$. Such a term in $\{\mathcal{Q},\mathcal{Q}\}$ would not commute with the 4-momentum, whereas $\{\mathcal{Q},\mathcal{Q}\}$ does. Thus we are left with

$$\{\mathcal{Q}^i_\alpha, \mathcal{Q}^j_\beta\} = 2\varepsilon_{\alpha\beta} Z^{ij}, \tag{8.13}$$

were Z^{ij} commutes with the space-time symmetries and hence is some linear combination of the internal symmetry generators,

$$Z^{ij} = \alpha^{rij} T_r. \tag{8.14}$$

We show that the Z^{ij} commute with the superalgebra. For this reason they are called *central charges*. First we show that the commutator of any T_r with the central charges is a central charge: First we have

$$\{\mathcal{Q}^i_\alpha, [T_r, \mathcal{Q}^j_\beta]\} = (t_r)^j{}_k \{\mathcal{Q}^i_\alpha, \mathcal{Q}^k_\beta\} = 2\varepsilon_{\alpha\beta}(t_r)^j{}_k Z^{ik}.$$

By using the super-Jacobi identity the left hand side can be rewritten as follows,

$$\begin{aligned}\{\mathcal{Q}^i_\alpha, [T_r, \mathcal{Q}^j_\beta]\} &= [T_r, \{\mathcal{Q}^j_\beta, \mathcal{Q}^i_\alpha\}] - \{\mathcal{Q}^j_\beta, [T_r, \mathcal{Q}^i_\alpha]\} \\ &= 2\varepsilon_{\beta\alpha}[T_r, Z^{ji}] - (t_r)^i{}_k \{\mathcal{Q}^j_\beta, \mathcal{Q}^k_\alpha\} = 2\varepsilon_{\beta\alpha}\Big([T_r, Z^{ji}] - (t_r)^i{}_k Z^{jk}\Big).\end{aligned}$$

If not all $\epsilon_{\alpha\beta}$ vanish then

$$[T_r, Z^{ji}] = (t_r)^j{}_k Z^{ki} + (t_r)^i{}_k Z^{jk}. \tag{8.15}$$

With (8.14) we conclude that

$$[Z^{ij}, Z^{kl}] = a^{rij}[T_r, Z^{kl}] = a^{rij}\left((t_r)^k{}_p Z^{pl} + (t_r)^l{}_p Z^{kp}\right). \tag{8.16}$$

The last two equations imply that the Z^{ij} span an *invariant subalgebra* of the internal symmetry algebra. We use once more the super Jacobi identity for $\{\bar{\mathcal{Q}}, [\mathcal{Q}, T]\}$ and with (8.12) conclude

$$\{\bar{\mathcal{Q}}^j_{\dot\beta}, [\mathcal{Q}^i_\alpha, T_r]\} + [T_r, \{\bar{\mathcal{Q}}^j_{\dot\beta}, \mathcal{Q}^i_\alpha\}] = \{\mathcal{Q}^i_\alpha, [T_r, \bar{\mathcal{Q}}^j_{\dot\beta}]\} = -\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}^k_{\dot\beta}\}(t_r)_k{}^j.$$

Now we use (8.13) in the commutator of the supercharges with the central charges and find

$$2\varepsilon_{\beta\gamma}[\bar{\mathcal{Q}}^i_{\dot\alpha}, Z^{jk}] = [\bar{\mathcal{Q}}^i_{\dot\alpha}, \{\mathcal{Q}^j_\beta, \mathcal{Q}^k_\gamma\}] = -[\mathcal{Q}^k_\gamma, \{\bar{\mathcal{Q}}^i_{\dot\alpha}, \mathcal{Q}^j_\beta\}] - [\mathcal{Q}^j_\beta, \{\mathcal{Q}^k_\gamma, \bar{\mathcal{Q}}^i_{\dot\alpha}\}] = 0,$$

where we have used that $\{\mathcal{Q}, \bar{\mathcal{Q}}\} \sim P$ commutes with the supercharges. Hence, the central charges commute with the supercharges $\bar{\mathcal{Q}}$. It follows then, that

$$0 = [\bar{\mathcal{Q}}^i_{\dot\alpha}, Z^{jk}] = -a^{rjk}(t_r)_l{}^i \bar{\mathcal{Q}}^l_{\dot\alpha} \quad \text{or} \quad a^{rjk}(t_r)_l{}^i = 0$$

so that the central charges also commute with the remaining supercharges $\mathcal{Q}$

$$[Z^{jk}, \mathcal{Q}^i_\alpha] = a^{rjk}(t_r)^i{}_l \mathcal{Q}^l_\alpha = 0,$$

and hence among themselves, $[Z^{ij}, Z^{kl}] = 0$. We see that the invariant subalgebra spanned by the Z^{ij} is Abelian. Since the internal symmetry group is assumed to be compact it follows that the Z^{ij} commute with the T_r. Thus

$$[T_r, Z^{ij}] = 0. \tag{8.17}$$

This proves that the Z^{ij} commute with all elements of the superalgebra and hence are central.

8.2 Representations

Is maybe useful to collect the (anti)commutation relations of the $\mathcal{N}$-extended superalgebra in 4 dimensions. It contains the Poincaré algebra

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\rho} - \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\nu\rho}J_{\mu\sigma}) \\
[P_\rho, J_{\mu\nu}] &= -i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \qquad [P_\mu, P_\nu] = 0
\end{aligned} \tag{8.18}$$

as subalgebra. The supercharges commute with the translations,

$$[\mathcal{Q}^i_\alpha, P_\mu] = 0, \tag{8.19}$$

transform under the spin-$\frac{1}{2}$ representation of the Lorentz group

$$[J_{\mu\nu},\mathcal{Q}^i_\alpha]=\tfrac{1}{2i}(\sigma_{\mu\nu})_\alpha{}^\beta\mathcal{Q}^i_\beta\quad,\quad[J_{\mu\nu},\bar{\mathcal{Q}}^{i\dot\alpha}]=\tfrac{1}{2i}(\tilde\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\bar{\mathcal{Q}}^{i\dot\beta}\tag{8.20}$$

and under the R-symmetry group as

$$[T_r,\mathcal{Q}^i_\alpha]=(t_r)^i{}_j\mathcal{Q}^j_\alpha\quad,\quad[T_r,\bar{\mathcal{Q}}^i_{\dot\alpha}]=-\bar{\mathcal{Q}}^j_{\dot\alpha}(t_r)_j{}^i.\tag{8.21}$$

The generators of the R-symmetry fulfill

$$[T_r,T_s]=iC_{rs}{}^tT_t\tag{8.22}$$

and commute with the bosonic generators of the Poincaré algebra,

$$[T_r,J_{\mu\nu}]=[T_r,P_\mu]=0.\tag{8.23}$$

The supercharges fulfill the following anticommutation relations

$$\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}=2\varepsilon_{\alpha\beta}Z^{ij}\quad,\quad\{\bar{\mathcal{Q}}^i_{\dot\alpha},\bar{\mathcal{Q}}^j_{\dot\beta}\}=2\varepsilon_{\dot\alpha\dot\beta}\bar{Z}^{ij}\quad,\quad\{\mathcal{Q}^i_\alpha,\bar{\mathcal{Q}}^j_{\dot\alpha}\}=2\delta^{ij}\sigma^\mu_{\alpha\dot\alpha}P_\mu\,,\tag{8.24}$$

where Z^{ij} is the antisymmetric central charge matrix. It commutes with all other generators of the super-Poincaré algebra.
Let us now discuss the representation theory of $\mathcal{N}$-extended supersymmetry in four dimensions. In general we can have more than one supersymmetry (extended supersymmetry). For each conserved Weyl spinor charge we have one supersymmetry. We will first assume that the *central charges are zero.*

8.2.1 Massive representations without central charges

For vanishing central charges the supercharges anticommute,

$$\{\mathcal{Q}^i_\alpha,\mathcal{Q}^j_\beta\}=\{\bar{\mathcal{Q}}^i_{\dot\alpha},\bar{\mathcal{Q}}^j_{\dot\beta}\}=0.\tag{8.25}$$

For a massive particle we may choose the rest frame in which $P\sim(M,\boldsymbol{0})$. Then the relations (8.24) simplify as follows:

$$\{\mathcal{Q}^i_\alpha,\bar{\mathcal{Q}}^j_{\dot\alpha}\}=2M\delta^{ij}(\sigma_0)_{\alpha\dot\alpha}=2M\delta_{\alpha\dot\alpha}\delta^{ij}.\tag{8.26}$$

$\mathcal{Q}$ is a tensor operator of spin $\frac{1}{2}$, as follows from

$$[\mathcal{Q},\boldsymbol{J}\,]=\tfrac{1}{2}\boldsymbol{\sigma}\mathcal{Q}.\tag{8.27}$$

Therefore, the result of the action of $\mathcal{Q}$ on a state with spin s will be a linear combination of states with spin $s+\frac{1}{2}$ and $s-\frac{1}{2}$,

$$\mathcal{Q}|m\,s\,s_3\rangle=\sum_{\tilde s_3}c^+_{s_3\tilde s_3}|m\,s+\tfrac{1}{2}\,\tilde s_3\rangle+\sum_{\tilde s_3}c^-_{s_3\tilde s_3}|m\,s-\tfrac{1}{2}\,\tilde s_3\rangle.$$

Now we define the following $2\mathcal{N}$ fermionic creation and annihilation operators

$$A^i_\alpha = \frac{1}{\sqrt{2M}}\,\mathcal{Q}^i_\alpha \qquad \bar{A}^i_{\dot\alpha} = \frac{1}{\sqrt{2M}}\,\bar{\mathcal{Q}}^i_{\dot\alpha}\,. \tag{8.28}$$

which satisfy the simple algebra

$$\{A^i_\alpha, \bar{A}^j_{\dot\alpha}\} = \delta^{ij}\delta_{\alpha\dot\alpha}. \tag{8.29}$$

Building the representation is easy. We start with the *Clifford vacuum* $|\Omega\rangle$, which is annihilated by the A^i_α and generate the representation by acting with the creation operators. A typical state would be

$$\bar{A}^{i_1}_{\dot\alpha_1}\cdots\bar{A}^{i_n}_{\dot\alpha_n}\,|\Omega\rangle. \tag{8.30}$$

It is totally antisymmetric under interchange of index-pairs $(\dot\alpha, i) \leftrightarrow (\dot\beta, j)$. Since there are $2\mathcal{N}$ creation operators $\bar{A}^i$ there are $\binom{2\mathcal{N}}{n}$ states at the n-th oscillator level. The total number of states built on *one* vacuum state is

$$\sum_{n=0}^{2\mathcal{N}}\binom{2\mathcal{N}}{n} = (1+1)^{2\mathcal{N}} = 4^{\mathcal{N}}, \tag{8.31}$$

half of them being bosonic and half of them fermionic. If the vacuum sector is degenerate, which happens if the Clifford vacuum $|\Omega\rangle$ is a member of a spin multiplet, then the number of states is

$$\text{NUMBER OF STATES} = 4^{\mathcal{N}}\cdot\text{ DIMENSION OF VACUUM SECTOR.}$$

The *maximal spin* s_{max} is carried by states like

$$\bar{A}^{i_1}_{\dot 1}\cdots\bar{A}^{i_n}_{\dot 1}\,|\Omega\rangle. \tag{8.32}$$

and their spins is equal to the spin s_0 of the ground-state plus $\mathcal{N}/2$,

$$s_{max} = s_0 + \mathcal{N}/2.$$

The *minimal spin* is 0 if $\mathcal{N}/2 \geq s_0$ or $s_0 - \mathcal{N}/2$ otherwise.
Since renormalisability requires *massive matter* to have spin $\leq \frac{1}{2}$. We conclude from the above expression for s_{max} that we must have

• $\mathcal{N} = 1$ *for renormalizable coupling of massive matter.*

We see that in the absence of central charges, the only relevant massive multiplet is that of the massive Wess-Zumino model which has $\mathcal{N} = 1$ and $s_0 = 0$. It contains a scalar, a pseudo-scalar and the two spin states of a massive Majorana spinor

N=1	s^P:	0^+	$\frac{1}{2}$	0^-
	states:	1	2	1

The two spin-zero states correspond to the Clifford vacuum $|\Omega\rangle$ and to $\bar{A}_{\dot 1}\bar{A}_{\dot 2}|\Omega\rangle$.

8.2.2 Massless representations

In this case we can go to the frame $P_\mu = (E, 0, 0, E)$. The anticommutation relations have the form

$$\{\mathcal{Q}^i_\alpha, \bar{\mathcal{Q}}^j_{\dot\alpha}\} = 2E\delta^{ij}(\sigma_0 + \sigma_3)_{\alpha\dot\alpha} = \begin{pmatrix} 4E & 0 \\ 0 & 0 \end{pmatrix} \delta^{ij}\,. \tag{8.33}$$

the rest being zero. Since

$$\{\mathcal{Q}^i_2, \bar{\mathcal{Q}}^i_{\dot 2}\} = \mathcal{Q}^i_2 \mathcal{Q}^{i\dagger}_2 + \mathcal{Q}^{i\dagger}_2 \mathcal{Q}^i_2 = 0 \tag{8.34}$$

they are represented by zero in a unitary theory. Thus we have $\mathcal{N}$ non-trivial creation and annihilation operators

$$A^i = \frac{1}{2\sqrt{E}}\,\mathcal{Q}^i_1 \quad \text{and} \quad \bar{A}^i = \frac{1}{2\sqrt{E}}\,\bar{\mathcal{Q}}^i_{\dot 1}. \tag{8.35}$$

and the representation is $2^{\mathcal{N}}$-dimensional. It is much shorter than the massive one which contains $4^{\mathcal{N}}$ states.
The following Lorentz generators commute with $P_\mu = (E, 0, 0, E)$:

$$J_1 = J_{10} + J_{13}, \quad J_2 = J_{20} + J_{23} \quad \text{and} \quad J_3 = -J_{12}. \tag{8.36}$$

Note that J_3 is just the helicity operator λ defined in (3.35) for a massless particle moving in the 3-direction. The J_i generate the little group E_2 of translations and rotations in the 2-plane,

$$[J_1, J_3] = iJ_2, \quad [J_2, J_3] = -iJ_1 \quad \text{and} \quad [J_1, J_2] = 0. \tag{8.37}$$

In any finite dimensional representation J_1 and J_2 are trivially represented. Note that A^i increases the helicity by $\frac{1}{2}$ and $\bar{A}^i$ decreases it by $\frac{1}{2}$,

$$[\lambda, A^i] = \tfrac{1}{2}A^i \quad \text{and} \quad [\lambda, \bar{A}^i] = -\tfrac{1}{2}\bar{A}^i. \tag{8.38}$$

Now we introduce the Clifford vacuum $|\Omega\rangle$ with maximal helicity λ. It is annihilated by all A^i, $A^i|\Omega = 0\rangle$. The states in an irreducible representation are gotten by acting with the creation operators on this state. For example,

$$\bar{A}^{i_1} \cdots \bar{A}^{i_n}|\Omega\rangle \tag{8.39}$$

has helicity $\lambda - n/2$. This way we get the following states

helicity:	λ	$\lambda - 1/2$	$\dots$	$\lambda - \mathcal{N}/2$
multiplicities:	1	$\binom{\mathcal{N}}{1}$	$\dots$	$\binom{\mathcal{N}}{\mathcal{N}}$

The total number of states in an irreducible massless representation is

$$\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}. \tag{8.40}$$

According to the CPT theorem a physical massless state contains both helicities λ and $-\lambda$. A *single* supermultiplett can contain all massless states for $\lambda = \mathcal{N}/4$. Otherwise the states must be doubled starting with a second Clifford vacuum with helicity $\lambda' = \mathcal{N}/2 - \lambda$. Here we will describe some important examples with helicities up to one.

1. $\mathcal{N} = 1$ **supersymmetry:** The number of CLIFFORD-states in an irreducible multiplet is just $1 + 1$ and we need at least two Clifford vacua to built a CPT invariant model.

   - For the *chiral multiplet* $\lambda = \frac{1}{2}$ and $\lambda' = 0$ and we have the following states

helicity	1/2	1 Majorana spinor
	0	1 complex scalar
	$-1/2$	1 Majorana spinor

     These are the fields of the massless Wess-Zumino model.

   - The *vector multiplet* with $\lambda = 1$ and $\lambda' = -1/2$ consists of

helicity	1	1 gauge field
	1/2	1 Majorana spinor
	$-1/2$	1 Majorana spinor
	-1	1 gauge field

2. $\mathcal{N} = 2$ **supersymmetry:** A irreducible representation of the $\mathcal{N} = 2$-exteded superalgebra contains 3 helicities and $1 + 2 + 1$ 'states'.

   - The simplest multiplet is the *hyper-multiplet* $\lambda = 1/2$ with the following content:

helicity	1/2	1 Majorana spinor
	0	1 complex scalar
	$-1/2$	1 Majorana spinor

   - To built a *vector multiplet* we need two Clifford vacua with $\lambda = 1$ and $\lambda' = 0$ so that it contains

helicity	1	1 gauge field
	1/2	2 Majorana spinors
	0	1 complex scalar
	$-1/2$	2 Majorana spinors
	-1	1 gauge field

     The two Majorana spinors can be combined to a Dirac spinor. This multiplet is considered in the *Seiberg-Witten* solution to $\mathcal{N} = 2$ supersymmetric gauge theories.

3. $\mathcal{N} = 4$ **supersymmetry:**

- A irreducible supermultiplet consists of $1+4+6+4+1=16$ states. The unique multiplet giving rise to a renormalizable field theory in flat spacetime is the *vector multiplet*. It has $\lambda=1$ and contains

helicity	1	1 gauge field
	1/2	4 Majorana spinors
	0	6 real scalars
	$-1/2$	4 Majorana spinors
	-1	1 gauge field

  This multiplet enters in the celebrated AdS/CFT correspondence.

8.2.3 Non-zero central charges

In this case massive supermultipletts can be as short as massless ones. Under a $U(N)$ transformation of the super charges,

$$\mathcal{Q}^i_\alpha \to U^i{}_j \mathcal{Q}^j_\alpha \quad \text{and} \quad \bar{\mathcal{Q}}^i_{\dot\alpha} \to \bar{\mathcal{Q}}^j_{\dot\alpha}(U^\dagger)_j{}^i$$

the antisymmetric central charge matrix $Z=(Z^{ij})$ transforms as

$$Z \longrightarrow UZU^T, \quad \bar{Z} \longrightarrow \bar{U}\bar{Z}U^\dagger.$$

In [40] it was shown that there exists a unitary U such that Z becomes block diagonal[1],

$$\begin{pmatrix} 0 & Z_1 & 0 & 0 & & \dots & \\ -Z_1 & 0 & 0 & 0 & & \dots & \\ 0 & 0 & 0 & Z_2 & & \dots & \\ 0 & 0 & -Z_2 & 0 & & \dots & \\ & \dots & \dots & \dots & \dots & \dots & \\ & \dots & & & & 0 & Z_{N/2} \\ & \dots & & & & -Z_{N/2} & 0 \end{pmatrix}. \tag{8.41}$$

We have labeled the *real positive* eigenvalues by Z_m, $m=1,2,\dots,N/2$. We will split the index $i\to(a,m)$: $a=1,2$ labels positions inside the 2×2 blocks while m labels the blocks. Then

$$\{\mathcal{Q}^{am}_\alpha,\bar{\mathcal{Q}}^{bn}_{\dot\alpha}\}=2M\delta_{\alpha\dot\alpha}\delta^{ab}\delta^{mn} \quad , \quad \{\mathcal{Q}^{am}_\alpha,\mathcal{Q}^{bn}_\beta\}=2Z_n\varepsilon_{\alpha\beta}\varepsilon^{ab}\delta^{mn}\,. \tag{8.42}$$

Now we define the following fermionic oscillators

$$A^m_\alpha=\frac{1}{\sqrt{2}}[\mathcal{Q}^{1m}_\alpha+\varepsilon_{\alpha\beta}\mathcal{Q}^{\dagger 2m}_\beta] \quad , \quad B^m_\alpha=\frac{1}{\sqrt{2}}[\mathcal{Q}^{1m}_\alpha-\epsilon_{\alpha\beta}\mathcal{Q}^{\dagger 2m}_\beta]\,, \tag{8.43}$$

and similarly for the conjugate operators. The anticommutators become

$$\begin{aligned} \{A^m_\alpha,A^n_\beta\} &= \{A^m_\alpha,B^n_\beta\}=\{B^m_\alpha,B^n_\beta\}=0\,, \\ \{A^m_\alpha,A^{\dagger n}_\beta\} &= 2\delta_{\alpha\beta}\delta^{mn}(M+Z_m) \\ \{B^m_\alpha,B^{\dagger n}_\beta\} &= 2\delta_{\alpha\beta}\delta^{mn}(M-Z_m)\,. \end{aligned} \tag{8.44}$$

[1] Consider even N.

Unitarity requires that the right-hand sides in (8.44) to be non-negative. This in turn implies the celebrated BOGOMOL'NYI BOUND

$$M \geq \max\left[Z_n\right] . \tag{8.45}$$

Consider for example the $\mathcal{N} = 4$ theory with $2 \cdot 4$ creation opreators $A_\alpha^{\dagger n}$ and $B_\alpha^{\dagger n}$ and assume that $M = Z_1 > Z_2$. Then

$$\{B_\alpha^1, B_\beta^{\dagger 1}\} = 0 \tag{8.46}$$

implies that the B_α^1 vanish identically and we are left with the following 6 creation and annihilation operators

$$A_\alpha^m,\, A_\alpha^{\dagger m}, \quad m = 1, 2, \quad \text{and} \quad B_\alpha^2,\, B_\alpha^{\dagger 2}.$$

They generate a representation with 2^6 states instead of 2^8 states. More generally, consider $0 \leq r \leq \mathcal{N}/2$ of the Z_m's to be equal to M. Then $2r$ of the B-oscillators vanish identically and we are left with $2\mathcal{N} - 2r$ creation and annihilation operators. The representation has $2^{2\mathcal{N}-2r}$ states.

The maximal case $r = \mathcal{N}/2$ gives rise to the *short BPS multiplet* which has the same number of states as the massless multiplet. The other multiplets with $0 < r < \mathcal{N}/2$ are known as *intermediate BPS multiplets.*

BPS states are important probes of non-perturbative physics in theories with extended ($\mathcal{N} \geq 2$) supersymmetry. The BPS states are special for the following reasons:

- Due to their relation with central charges, and although they are massive, they form multiplets under extended SUSY which are shorter than the generic massive multiplet. Their mass is given in terms of their charges and Higgs (moduli) expectation values.
- They are the only states that can become massless when we vary coupling constants and Higgs expectation values.
- When they are at rest they exert no force on each other.
- Their mass-formula is supposed to be exact if one uses renormalized values for the charges and moduli. The argument is that if quantum corrections would spoil the relation of mass and charges, and if we assume unbroken SUSY at the quantum level then there would be incompatibilities with the dimension of their representations.

We will come back to these remarks later in this lecture.

Kapitel 9

Supersymmetric Gauge Theories

In this chapter we introduce and study supersymmetric gauge gauge theories in flat 4-dimensional space time. For the simplest gauge theory, namely the $\mathcal{N}=1$ Abelian gauge theory containing a photon and a massless Majorana spinor, we explicitly show the invariance of the action, the closing of the supersymmetry algebra on the gauge invariant fields and 'construct' the supercharges. Then we repeat the same for $\mathcal{N}=1$ non-Abelian gauge theories. It follows a discussion of the $\mathcal{N}=2$ and $\mathcal{N}=4$ gauge theories. There has been much progress on $\mathcal{N}=2$-gauge theories due to the seminal work by Witten and Seiberg [41]. The $\mathcal{N}=4$ gauge theories emerge as conformal field theories on the boundary of AdS in the celebrated AdS/CFT-correspondence [44]

9.1 $\mathcal{N}=1$ Abelian gauge theories

We have seen, that the $\mathcal{N}=1$ vector multiplet consists of a gauge field and a Majorana spinor. First we consider the simplest of all gauge theories, namely a $U(1)$ gauge theory. In an off-shell version and the Wess-Zumino gauge (see below) we need in addition to the vector and Majorana fields an uncharged pseudo-scalar field which later may be eliminated. Since a Majorana spinor is uncharged the Lagrangean density takes the simple form

$$\mathcal{L}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\bar{\psi}\partial\!\!\!/\psi+\tfrac{1}{2}\mathcal{G}^2,\qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,\quad \partial\!\!\!/=\gamma^\mu\partial_\mu. \tag{9.1}$$

The dimensions of the fields are

$$[A_\mu]=L^{-1},\quad [\psi]=L^{-3/2},\quad [\mathcal{G}]=L^{-2}, \tag{9.2}$$

and the supersymmetry parameter has dimension $[\alpha]=L^{1/2}$. Also recall that

$$\bar{\alpha}\psi,\ \bar{\alpha}\gamma^{\mu\nu}\psi,\ \bar{\alpha}\gamma_5\gamma^\mu\psi \text{ are hermitean}\quad\text{and}\quad \bar{\alpha}\gamma_5\psi,\ \bar{\alpha}\gamma^\mu\psi \text{ are antihermitean.}$$

We shall need the formulae in (4.76) in which we calculated whether $\bar{\alpha}\gamma^{(n)}\psi$ changes sign when one interchanges the Majorana spinors α and ψ

$$\bar{\alpha}\psi,\ \bar{\alpha}\gamma_5\psi,\ \bar{\alpha}\gamma_5\gamma_\mu\psi \text{ are symmetric}\quad\text{and}\quad \bar{\alpha}\gamma_\mu\psi,\ \bar{\alpha}\gamma_{\mu\nu}\psi \text{ are antisymmetric}$$

Taking the dimensions and hermiticity properties of the various fields into account we could guess the following supersymmetry transformations

$$\begin{aligned}\delta_\alpha A_\mu &= \mathrm{i}\bar\alpha\gamma_\mu\psi\\ \delta_\alpha\psi &= ipF^{\mu\nu}\Sigma_{\mu\nu}\alpha+\mathrm{i}q\mathcal{G}\gamma_5\alpha\\ \delta_\alpha\mathcal{G} &= r\bar\alpha\gamma_5\partial\!\!\!/\psi.\end{aligned}\tag{9.3}$$

As earlier, the supersymmetry parameter α is a constant Majorana spinor which anticommutes with itself and with ψ. We will fix the real parameters p,q,r such that the action is invariant. We shall need

$$\delta\bar\psi=\delta\psi^\dagger\gamma^0=-ipF^{\mu\nu}\alpha^\dagger\Sigma^\dagger_{\mu\nu}\gamma^0-\mathrm{i}q\mathcal{G}\alpha^\dagger\gamma_5\gamma^0=-ipF^{\mu\nu}\bar\alpha\Sigma_{\mu\nu}+\mathrm{i}q\mathcal{G}\bar\alpha\gamma_5,\tag{9.4}$$

where we used $\Sigma^\dagger_{\mu\nu}\gamma^0=\gamma^0\Sigma_{\mu\nu}$.
Next we show that the Lagrangean is, up to surface terms, invariant under the above supersymmetry transformation. The variations of the bosonic terms in the Lagrangean density are

$$-\tfrac14\delta_\alpha\left(F_{\mu\nu}F^{\mu\nu}\right)=-\mathrm{i}F^{\mu\nu}\bar\alpha\gamma_\nu\partial_\mu\psi\qquad\text{and}\quad\tfrac12\delta_\alpha(\mathcal{G}^2)=r\mathcal{G}\bar\alpha\gamma_5\partial\!\!\!/\psi.\tag{9.5}$$

Using that $\bar\alpha\gamma^\rho\gamma_5\psi$ is symmetric under exchange of α and ψ and that

$$\bar\psi\gamma^\rho\Sigma_{\mu\nu}\alpha=\bar\alpha\Sigma_{\mu\nu}\gamma^\rho\psi,\tag{9.6}$$

yields for the variation of the fermionic term in the Lagrangean

$$\tfrac12\delta_\alpha\left(\bar\psi\partial\!\!\!/\psi\right)=\tfrac{q}{2}\,\partial_\rho\left(\bar\alpha\mathcal{G}\gamma_5\gamma^\rho\psi\right)-\tfrac{p}{2}\,\partial_\rho\left(\bar\alpha\Sigma_{\mu\nu}F^{\mu\nu}\gamma^\rho\psi\right)-q\,\mathcal{G}\bar\alpha\gamma_5\partial\!\!\!/\psi+p\,F^{\mu\nu}\bar\alpha\Sigma_{\mu\nu}\partial\!\!\!/\psi.\tag{9.7}$$

Collecting the various terms it follows that one must choose $q=r$ such that the volume terms depending on the auxiliary field in (9.5) and (9.7) cancel, and this choice leads to

$$\delta_\alpha\mathcal{L}=-\mathrm{i}F^{\mu\nu}\bar\alpha\gamma_\nu\partial_\mu\psi+pF^{\mu\nu}\bar\alpha\Sigma_{\mu\nu}\partial\!\!\!/\psi+\text{divergence}.$$

With the help of

$$\Sigma_{\mu\nu}\gamma_\rho=\tfrac12\eta_{\mu\rho}\gamma_\nu-\tfrac12\eta_{\nu\rho}\gamma_\mu-\tfrac12\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5,\qquad\epsilon_{0123}=1,\tag{9.8}$$

the second term on the right hand side can be recast into the form

$$F^{\mu\nu}\bar\alpha\Sigma_{\mu\nu}\partial\!\!\!/\psi=\mathrm{i}\bar\alpha F^{\mu\nu}\gamma_\nu\partial_\mu\psi-\tfrac12\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\bar\alpha\gamma^\sigma\gamma_5\partial^\rho\psi.\tag{9.9}$$

Clearly, we must take $p=1$ in which case

$$\delta_\alpha\mathcal{L}=-\tfrac12\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\bar\alpha\gamma^\sigma\gamma_5\partial^\rho\psi+\text{divergence}.$$

Using the BIANCHI identity $\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu,\rho}=0$, the first term on the right is seen to be a divergence as well,

$$-\tfrac12\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\bar\alpha\gamma^\sigma\gamma_5\partial^\rho\psi=-\tfrac12\partial^\rho(\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}\bar\alpha\gamma^\sigma\gamma_5\psi)\tag{9.10}$$

so that under the transformations (9.3) the complete Lagrangean density (9.1) transforms into a divergence,

$$\delta_\alpha \mathcal{L} = -\tfrac{1}{2}\bar{\alpha}\partial^\rho \left(F^{\mu\nu}\left(\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 + \Sigma_{\mu\nu}\gamma_\rho\right)\psi\right) + \tfrac{q}{2}\bar{\alpha}\partial_\rho\left(\mathcal{G}\gamma_5\gamma^\rho\psi\right).$$

Now we use (9.8) and

$$\gamma_\rho\Sigma_{\mu\nu} = -\tfrac{1}{2}\eta_{\mu\rho}\gamma_\nu + \tfrac{1}{2}\eta_{\nu\rho}\gamma_\mu - \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 \tag{9.11}$$

to further simplify this result. Using these identities finally yields the following rather simple expression for the variation of the Lagrangean density

$$\begin{aligned}\delta_\alpha\mathcal{L} = \bar{\alpha}\partial_\mu V^\mu, \quad V^\mu &= \tfrac{1}{2}\gamma^\mu\Sigma_{\rho\sigma}F^{\rho\sigma}\psi + \tfrac{q}{2}\mathcal{G}\gamma_5\gamma^\mu\psi \\ &= \tfrac{1}{2}\left({}^*F^{\mu\nu}\gamma_5 - \mathrm{i}F^{\mu\nu}\right)\gamma_\nu\psi + \tfrac{q}{2}\mathcal{G}\gamma_5\gamma^\mu\psi\end{aligned} \tag{9.12}$$

under the supersymmetry transformations

$$\begin{aligned}\delta_\alpha A_\mu &= \mathrm{i}\bar{\alpha}\gamma_\mu\psi \\ \delta_\alpha\psi &= \mathrm{i}F^{\mu\nu}\Sigma_{\mu\nu}\alpha + \mathrm{i}q\mathcal{G}\gamma_5\alpha \\ \delta_\alpha\mathcal{G} &= q\bar{\alpha}\gamma_5\partial\!\!\!/\psi.\end{aligned} \tag{9.13}$$

Note that the real parameter q is not fixed. Only later, when we want to recover the known superalgebra are we forced to choose $q \in \{1,-1\}$.

9.1.1 The closing of the algebra

Now we are going to repeat what we have done for the Wess-Zumino model and calculate the commutators of two supersymmetry transformations. The commutator acting on the bosonic fields is easily computed. Using (9.6) we calculate

$$\begin{aligned}[\delta_{\alpha_1},\delta_{\alpha_2}]A_\mu &= -F^{\rho\sigma}\left(\bar{\alpha}_2\gamma_\mu\Sigma_{\rho\sigma}\alpha_1 - \bar{\alpha}_1\gamma_\mu\Sigma_{\rho\sigma}\alpha_2\right) \\ &= F^{\rho\sigma}\bar{\alpha}_2[\Sigma_{\rho\sigma},\gamma_\mu]\alpha_1 = 2\mathrm{i}\bar{\alpha}_2\gamma^\nu\alpha_1 F_{\mu\nu} \\ &= -2\mathrm{i}\left(\bar{\alpha}_2\gamma^\rho\alpha_1\right)\partial_\rho A_\mu + 2\mathrm{i}\partial_\mu\left(A_\rho\bar{\alpha}_2\gamma^\rho\alpha_1\right).\end{aligned} \tag{9.14}$$

The first term is the expected infinitesimal translation of the vector field generated by the momentum operator. The second one we did not encounter in the Wess-Zumino model. It is just a field dependent gauge transformation with gauge parameter

$$\lambda = 2\mathrm{i}A_\rho\bar{\alpha}_2\gamma^\rho\alpha_1. \tag{9.15}$$

Similarly, using (9.11), one finds

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\mathcal{G} = -2\mathrm{i}q^2\bar{\alpha}_2\gamma^\rho\alpha_1\partial_\rho\mathcal{G} + \mathrm{i}q\epsilon_{\mu\nu\rho\sigma}\bar{\alpha}_2\gamma^\sigma\alpha_1\partial^\rho F^{\mu\nu}.$$

The second term vanishes because of the Bianchi identity. If we take $q \in \{-1,1\}$ then we find the by now familiar commutator

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\mathcal{G} = -2\mathrm{i}\left(\bar{\alpha}_2\gamma^\rho\alpha_1\right)\partial_\rho\mathcal{G}. \tag{9.16}$$

To calculate the commutator of two transformations on the fermion field is a bit more tricky. First one obtains ($q^2=1$)

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = \left(-2\,(\bar\alpha_1\gamma^\nu\partial^\mu\psi)\,\Sigma_{\mu\nu}\alpha_2 + \mathrm{i}\,(\bar\alpha_1\gamma_5\gamma^\rho\partial_\rho\psi)\,\gamma_5\alpha_2\right) - (\alpha_1\leftrightarrow\alpha_2).$$

With the help of (6.32) one arrives at the following intermediate result

$$\begin{aligned}[\delta_{\alpha_1},\delta_{\alpha_2}]\psi &= \left(-\tfrac{1}{2}\gamma_\rho\partial\!\!\!/\psi - \Sigma_{\mu\nu}\gamma_\rho\gamma^\nu\partial^\mu\psi\right)(\bar\alpha_2\gamma^\rho\alpha_1)\\ &+ \left(2\Sigma_{\mu\nu}\gamma_{\rho\sigma}\gamma^\nu\partial^\mu\psi - \mathrm{i}\gamma_{\rho\sigma}\partial\!\!\!/\psi\right)(\bar\alpha_2\gamma^{\rho\sigma}\alpha_1).\end{aligned}$$

A lengthy but straightforward calculation where one uses the Clifford algebra, the Lorentz algebra (3.10) and the relation (9.8) one ends up with the expected result

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = -2\mathrm{i}(\bar\alpha_2\gamma^\rho\alpha_1)\,\partial_\rho\psi. \tag{9.17}$$

As we did for the chiral multiplet we introduce the *supercharges*

$$\delta_\alpha(..) = \bar\alpha Q(..).$$

Then the superalgebra reads

$$\{Q_\alpha,\bar Q^\beta\} = 2\mathrm{i}\,(\gamma^\mu)_\alpha^{\ \beta}\,\partial_\mu - G_\alpha^{\ \beta}(A), \tag{9.18}$$

where $G_\alpha^{\ \beta}$ is the gauge transformation $A_\mu\to A_\mu+\partial_\mu\lambda$ with $\lambda = 2\mathrm{i}(A\!\!\!/)_\alpha^{\ \beta}$

9.1.2 Noether charge

To find the Noether charge we must first calculate

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta_\alpha A_\nu + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta_\alpha\psi = -F^{\mu\nu}\delta_\alpha A_\nu - \tfrac{1}{2}\delta_\alpha\bar\psi\gamma^\mu\psi. \tag{9.19}$$

Using the above expressions for the supersymmetry transformations this can be written as (we take $q=1$)

$$\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu}\delta_\alpha A_\nu + \frac{\delta\mathcal{L}}{\delta\partial_\mu\psi}\delta_\alpha\psi = -\tfrac{3i}{2}F^{\mu\nu}\bar\alpha\gamma_\nu\psi + \tfrac{1}{2}\,{}^*F^{\mu\nu}\bar\alpha\gamma_\nu\gamma_5\psi + \tfrac{1}{2}\mathcal{G}\bar\alpha\gamma_5\gamma^\mu\psi. \tag{9.20}$$

Subtracting V^μ in (9.12) we find the conserved Noether current and Noether charge

$$\begin{aligned} J^\mu &= -\left({}^*F^{\mu\nu}\gamma_5 + \mathrm{i}F^{\mu\nu}\right)\gamma_\nu\psi\\ Q &= -\int \mathrm{d}^3x\,\left({}^*F^{0i}\gamma_5 - \mathrm{i}F^{0i}\right)\gamma_i\psi. \end{aligned} \tag{9.21}$$

Inserting the explicit form of the field strength tensor and its dual,

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3\\ E_1 & 0 & -B_3 & B_2\\ E_2 & B_3 & 0 & -B_1\\ E_3 & -B_2 & B_1 & 0\end{pmatrix}\quad,\quad ({}^*F^{\mu\nu}) = \begin{pmatrix} 0 & B_1 & B_2 & B_3\\ -B_1 & 0 & -E_3 & E_2\\ -B_2 & E_3 & 0 & -E_1\\ -B_3 & -E_2 & E_1 & 0\end{pmatrix} \tag{9.22}$$

the gauge invariant Noether charge take the form

$$Q = \int \mathrm{d}^3x\,\left(\mathrm{i}\pi_i - \gamma_5 B_i\right)\gamma_i\psi,\quad\text{where}\quad \pi_i = E_i$$

is the momentum field conjugate to A_i.

9.2 $\mathcal{N}=1$ Non-Abelian gauge theories

We consider $SU(N)$ gauge theories with a massless vector field A_μ, a massless Majorana spinor ψ and an auxiliary field $\mathcal{G}$. All field transform according to the adjoint representation of the gauge group $SU(N)$,

$$A_\mu = A_\mu^a T_a, \quad \psi = \psi^a T_a \quad \text{and} \quad \mathcal{G} = \mathcal{G}^a T_a. \tag{9.23}$$

The real structure constants in

$$[T_a, T_b] = \mathrm{i} f_{ab}^{\ \ c} T_c \tag{9.24}$$

are totally *antisymmetric*. We normalize the generators by $\mathrm{Tr}\, T_a T_b = \delta_{ab}$.
The gauge and matter fields transform under gauge transformations as follows

$$\begin{aligned} A &\rightarrow gAg^{-1} + \mathrm{i} g dg^{-1} \\ \psi &\rightarrow g\psi g^{-1} \\ \mathcal{G} &\rightarrow g\mathcal{G}g^{-1} \end{aligned} \tag{9.25}$$

the infinitesimal versions of which read with $g \sim \mathbb{1} + i\lambda$

$$\delta A_\mu = D_\mu \lambda, \quad \delta\psi = \mathrm{i}[\lambda, \psi] \quad \text{and} \quad \delta\mathcal{G} = \mathrm{i}[\lambda, \mathcal{G}]. \tag{9.26}$$

The covariant derivative of the spinor field is

$$D_\mu \psi = \partial_\mu \psi - \mathrm{i}[A_\mu, \psi] \tag{9.27}$$

and the gauge covariant field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}[A_\mu, A_\nu], \tag{9.28}$$

both transform according to the adjoint represention. Note that the field strength is quadratic in the potential. The Lagrangean density reads

$$\mathcal{L} = -\tfrac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mathrm{Tr}\,(\bar{\psi} \not{D} \psi) + \tfrac{1}{2}\mathrm{Tr}\,\mathcal{G}^2. \tag{9.29}$$

The supersymmetry transformations are gotten from those of the Abelian model if we only replace ordinary derivatives by covariant ones,

$$\begin{aligned} \delta_\alpha A_\mu &= \mathrm{i}\bar{\alpha}\gamma_\mu \psi \\ \delta_\alpha \psi &= \mathrm{i}F^{\mu\nu}\Sigma_{\mu\nu}\alpha + i\mathcal{G}\gamma_5\alpha \\ \delta_\alpha \mathcal{G} &= \bar{\alpha}\gamma_5 \not{D}\psi. \end{aligned} \tag{9.30}$$

Note that for a Majorana spinor α the objects

$$\mathrm{i}\Sigma_{\mu\nu}\alpha, \quad \mathrm{i}\gamma_5\alpha, \quad \gamma_5\gamma^\mu\alpha \quad \text{and} \quad \mathrm{i}\gamma^\mu\alpha \tag{9.31}$$

are Majorana spinors as well. To calculate the variation of the bosonic part of the Lagrangean density we use

$$\delta F_{\mu\nu} = \mathrm{i}\bar{\alpha}\left(\gamma_\nu D_\mu - \gamma_\mu D_\nu\right)\psi$$

and obtain

$$-\tfrac{1}{4}\delta_\alpha \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) = -\mathrm{i}\mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\gamma_\nu D_\mu\psi\right) \quad \text{and} \quad \tfrac{1}{2}\delta_\alpha(\mathcal{G}^2) = \mathrm{Tr}\left(\mathcal{G}\bar{\alpha}\gamma_5 \not{D}\psi\right). \tag{9.32}$$

To calcuate the variation of the Dirac term we need the two formulae

$$\begin{aligned} \delta_\alpha\bar{\psi} &= -\mathrm{i}\bar{\alpha}F^{\mu\nu}\Sigma_{\mu\nu} + \mathrm{i}\bar{\alpha}\mathcal{G}\gamma_5 \\ \delta(D_\mu\psi) &= D_\mu\delta\psi - \mathrm{i}[\delta A_\mu,\psi] = iD_\mu F^{\rho\sigma}\Sigma_{\rho\sigma}\alpha + iD_\mu\mathcal{G}\gamma_5\alpha + [\bar{\alpha}\gamma_\mu\psi,\psi]. \end{aligned} \tag{9.33}$$

The variation of the Dirac term can be calculated similarly as in the Abelian case with the result

$$\begin{aligned} \tfrac{1}{2}\delta_\alpha \mathrm{Tr}\left(\bar{\psi}\not{D}\psi\right) &= \tfrac{1}{2}\partial_\rho \mathrm{Tr}\left(\mathcal{G}\bar{\alpha}\gamma_5\gamma^\rho\psi\right) - \mathrm{Tr}\left(\mathcal{G}\bar{\alpha}\gamma_5\not{D}\psi\right) \\ &- \tfrac{1}{2}\partial_\rho \mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\Sigma_{\mu\nu}\gamma^\rho\psi\right) + \mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\Sigma_{\mu\nu}\not{D}\psi\right) + \tfrac{1}{2}\mathrm{Tr}\left(\bar{\psi}\gamma^\mu[\bar{\alpha}\gamma_\mu\psi,\psi]\right) \end{aligned}$$

Using the Bianchi identity $D_{[\rho}F_{\mu\nu]} = 0$ we find

$$\begin{aligned} -\mathrm{i}\mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\gamma_\nu D_\mu\psi\right) + \mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\Sigma_{\mu\nu}\not{D}\psi\right) &= -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\gamma^\sigma\gamma_5 D^\rho\psi\right) \\ &= -\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\rho \mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\gamma^\sigma\gamma_5\psi\right). \end{aligned}$$

and end up with

$$\begin{aligned} \delta_\alpha\mathcal{L} &= -\tfrac{1}{2}\bar{\alpha}\partial^\rho\,\mathrm{Tr}\left\{F^{\mu\nu}(\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 + \Sigma_{\mu\nu}\gamma_\rho\psi)\right\} + \tfrac{1}{2}\bar{\alpha}\,\partial_\rho \mathrm{Tr}\left(\mathcal{G}\gamma_5\gamma^\rho\psi\right) \\ &+\tfrac{1}{2}\mathrm{Tr}\left(\bar{\psi}\gamma^\mu[\bar{\alpha}\gamma_\mu\psi,\psi]\right). \end{aligned} \tag{9.34}$$

If we can show that the last term vanishes, then we have proved that the variation of $\mathcal{L}$ is a total divergence. It is not straightforward to show this, so let me give the proof. First we expand $\psi = \psi^a T_a$, $\bar{\psi} = \bar{\psi}^a T_a$ and calculate

$$+\tfrac{1}{2}\mathrm{Tr}\left(\bar{\psi}\gamma^\mu[\bar{\alpha}\gamma_\mu\psi,\psi]\right) = \tfrac{1}{2}f_{abc}(\bar{\psi}^a\gamma^\mu\psi^b)(\bar{\alpha}\gamma_\mu\psi^c). \tag{9.35}$$

Now we may insert the Fierz relation (4.85)

$$4\psi^c\bar{\psi}^a = -(\bar{\psi}^a\psi^c) - \gamma_\rho(\bar{\psi}^a\gamma^\rho\psi^c) + \tfrac{1}{2}\gamma_{\rho\sigma}(\bar{\psi}^a\gamma^{\rho\sigma}\psi^c) + \gamma_5\gamma_\rho(\bar{\psi}^a\gamma_5\gamma^\rho\psi^c) - \gamma_5(\bar{\psi}^a\gamma_5\psi^c) \tag{9.36}$$

in the right hand side of the identity

$$(\bar{\psi}^a\gamma^\mu\psi^b)(\bar{\alpha}\gamma_\mu\psi^c) = \bar{\alpha}\gamma_\mu(\psi^c\bar{\psi}^a)\gamma^\mu\psi^b. \tag{9.37}$$

Next we use the relations

$$\begin{aligned} \gamma_\mu\gamma_\rho\gamma^\mu &= (2-d)\gamma_\rho \\ \gamma_\mu\gamma_{\rho\sigma}\gamma^\mu &= (d-4)\gamma_{\rho\sigma} \\ \gamma_\mu\gamma_5\gamma_\rho\gamma^\mu &= (2-d)\gamma_\rho\gamma_5 \end{aligned} \tag{9.38}$$

and arrive at the useful identity

$$\begin{aligned}(\bar{\psi}^a\gamma^\mu\psi^b)(\bar{\alpha}\gamma_\mu\psi^c) &= -(\bar{\alpha}\psi^b)(\bar{\psi}^a\psi^c)+\tfrac{1}{2}(\bar{\alpha}\gamma_\rho\psi^b)(\bar{\psi}^a\gamma^\rho\psi^c)\\ &\quad -\tfrac{1}{2}(\bar{\alpha}\gamma_\rho\gamma_5\psi^b)(\bar{\psi}^a\gamma_5\gamma^\rho\psi^c)+(\bar{\alpha}\gamma_5\psi^b)(\bar{\psi}^a\gamma_5\psi^c).\end{aligned} \tag{9.39}$$

Note that all but the second term on the right hand side are symmetric in a and c. Hence

$$f_{abc}(\bar{\psi}^a\gamma^\mu\psi^b)(\bar{\alpha}\gamma_\mu\psi^c)=\tfrac{1}{2}f_{abc}(\bar{\psi}^a\gamma^\mu\psi^c)(\bar{\alpha}\gamma_\mu\psi^b)=-\tfrac{1}{2}f_{abc}(\bar{\psi}^a\gamma^\mu\psi^b)(\bar{\alpha}\gamma_\mu\psi^c)$$

which implies, that the expression on the left vanishes. Actually, the in a and c antisymmetric part of (9.39) can be rewritten as

$$(\bar{\psi}^a\gamma^\mu\psi^b)(\bar{\alpha}\gamma_\mu\psi^c)+(\bar{\psi}^b\gamma^\mu\psi^c)(\bar{\alpha}\gamma_\mu\psi^a)+(\bar{\psi}^c\gamma^\mu\psi^a)(\bar{\alpha}\gamma_\mu\psi^b)=0\quad\text{if}\quad \psi^a_c=\psi^a. \tag{9.40}$$

Hence the last term in (9.34) is absent and using (9.8) and (9.13) the result (9.34) further simplifies

$$\delta_\alpha\mathcal{L}=\bar{\alpha}\partial_\mu V^\mu,\quad V^\mu=\tfrac{1}{2}\mathrm{Tr}\left\{\left({}^*F^{\mu\nu}\gamma_5-\mathrm{i}F^{\mu\nu}\right)\gamma_\nu\psi\right\}+\tfrac{1}{2}\mathrm{Tr}\left(\mathcal{G}\gamma_5\gamma^\mu\psi\right). \tag{9.41}$$

To calculate the Noether current we need

$$\begin{aligned}&\frac{\delta\mathcal{L}}{\delta\partial_\mu A_{a\nu}}\delta_\alpha A_{a\nu}+\frac{\delta\mathcal{L}}{\delta\partial_\mu\psi_a}\delta_\alpha\psi_a\\ &\quad=-\tfrac{3i}{2}\mathrm{Tr}\left(F^{\mu\nu}\bar{\alpha}\gamma_\nu\psi\right)+\tfrac{1}{2}\,\mathrm{Tr}\left({}^*F^{\mu\nu}\bar{\alpha}\gamma_\nu\gamma_5\psi\right)+\tfrac{1}{2}\,\mathrm{Tr}\left(\mathcal{G}\bar{\alpha}\gamma_5\gamma^\mu\psi\right)\end{aligned} \tag{9.42}$$

so that the conserved current and the supercharge have the simple forms

$$\begin{aligned}J^\mu &= -\mathrm{Tr}\left({}^*F^{\mu\nu}\gamma_5+\mathrm{i}F^{\mu\nu}\right)\gamma_\nu\psi\\ Q &= \int \mathrm{d}^3x\,\mathrm{Tr}\left\{(\mathrm{i}\pi_i-B_\mathrm{i}\gamma_5)\gamma_i\psi\right\}.\end{aligned} \tag{9.43}$$

The commutator of two supersymmetry transformations on the gauge potential is

$$[\delta_{\alpha_1},\delta_{\alpha_2}]A_\mu=2\mathrm{i}\bar{\alpha}_2\left(F_{\mu\nu}\gamma^\nu\right)\alpha_1=-2\mathrm{i}\left(\bar{\alpha}_2\gamma^\nu\alpha_1\right)\partial_\nu A_\mu+D_\mu\lambda,\quad \lambda=2\mathrm{i}(\bar{\alpha}_2\gamma^\nu\alpha_1)A_\nu. \tag{9.44}$$

Similarly as in the Abelian case the commutator of two supersymmetry transformations yields a translation plus an infinitesimal gauge transformation. The gauge parameter λ depends on the gauge field A_μ and the susy parameters α_i.
To calculate the commutator on the dummy field $\mathcal{G}$ one needs the relations

$$\bar{\alpha}\gamma_5\gamma^\rho\Sigma_{\mu\nu}\psi=\bar{\psi}\Sigma_{\mu\nu}\gamma^\rho\gamma_5\alpha\quad\text{and}\quad \mathrm{i}\gamma_5[\gamma_\mu,\gamma_\nu]+\epsilon_{\mu\nu\rho\sigma}\gamma^\rho\gamma^\sigma=0,$$

where we used the definition $\gamma_5=-\mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ and the identity (6.32). After some lengthy but straightforward manipulations one obtains

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\mathcal{G}=-2\mathrm{i}(\bar{\alpha}_2\gamma^\rho\alpha_1)\left(D_\rho\mathcal{G}-D_\sigma\,{}^*F^\sigma{}_\rho\right)=-2\mathrm{i}\left(\bar{\alpha}_2\gamma^\rho\alpha_1\right)\partial_\rho\mathcal{G}+\mathrm{i}[\lambda,\mathcal{G}], \tag{9.45}$$

where once more we used the BIANCHI identity. Here λ it the field dependent gauge parameter introduced above. The commutator on the fermionic field reads

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = -2\Sigma_{\mu\nu}M\gamma^\nu D^\mu\psi + \mathrm{i}\gamma_5 M\gamma_5 \not{D}\psi, \tag{9.46}$$

where M has been introduced in (6.32). Using the relations (4.13,9.8,6.32) and $2\mathrm{i}\Sigma_{\mu\nu}\gamma^\nu = 3\gamma_\mu$ one ends up with the expected result

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = -2\mathrm{i}(\bar{\alpha}_2\gamma^\rho\alpha_1)D_\rho\psi = -2\mathrm{i}(\bar{\alpha}_2\gamma^\rho\alpha_1)\partial_\rho\psi + \mathrm{i}[\lambda,\psi]. \tag{9.47}$$

As in the Abelian case the superalgebra closes only on gauge invariant fields.

9.3 $\mathcal{N}=2$ Supersymmetric Gauge Theory

In this section we will be considering theories with $\mathcal{N}=2$ spacetime supersymmetries. Realistic (i.e. chiral) models of particle interactions have only $\mathcal{N}=1$ supersymmetry. However, there are good reasons for discussing $\mathcal{N}=2$, the main being that the dynamics of this theory is under much better control and this allows one to make statements about the spectrum which are valid non-perturbatively. This model may possess a non-vanishing central charge and, as we have seen, the Bogomol'nyi bound applies. Magnetic monopoles in $\mathcal{N}=2$ supersymmetric theories were first discussed in [42].
The rather simple $\mathcal{N}=2$ model has played in important role in recent developments about confinement in 4 dimensional asymptotically free gauge theories. SEIBERG and Witten have got an analytic expression for the low energy effective action (the leading two terms in a derivative expansion) of this theory [41]. On shell this model contains a vector field A_μ, two Majorana spinors χ_1,χ_2, which can be combined to a Dirac spinor ψ, a scalar field A and a pseudo-scalar field B. All fields transform according to the adjoint representation of a gauge group.
Note that a Dirac spinor can always be written as a linear combination of two Majorana spinors,

$$\psi = \frac{1}{\sqrt{2}}(\chi_1 + \mathrm{i}\chi_2) \quad , \quad \psi_c = \frac{1}{\sqrt{2}}(\chi_1 - \mathrm{i}\chi_2) \tag{9.48}$$

or the sum and difference of a Dirac spinor and its charge conjugate define Majorana spinors,

$$\chi_1 = \frac{1}{\sqrt{2}}(\psi + \psi_c) \quad , \quad \chi_2 = \frac{1}{\mathrm{i}\sqrt{2}}(\psi - \psi_c). \tag{9.49}$$

Of course, a similar decomposition exists for the supersymmetry parameter

$$\alpha = \frac{1}{\sqrt{2}}(\epsilon_1 + \mathrm{i}\epsilon_2) \quad \text{and} \quad \alpha_c = \frac{1}{\sqrt{2}}(\epsilon_1 - \mathrm{i}\epsilon_2). \tag{9.50}$$

It follows, that

$$\mathrm{i}\bar{\psi}\not{D}\psi = \tfrac{1}{2}\sum\left(\bar{\chi}_i\not{D}\chi_i\right) - \tfrac{1}{2}\partial_\rho\left(\bar{\chi}_1\gamma^\rho\chi_2\right) \quad \text{and} \quad \bar{\alpha}M\psi + \bar{\alpha}_c M\psi_c = \sum \bar{\epsilon}_i M\chi_i \tag{9.51}$$

hold true for any matrix M. Now we could return to the $\mathcal{N}=1$ vector multiplet and obtain the following transformation rule for the gauge potential,

$$\delta_\alpha A_\mu = \mathrm{i}\sum \bar{\epsilon}_\mathrm{i}\gamma_\mu\chi_i = \mathrm{i}\bar{\alpha}\gamma_\mu\psi + \mathrm{i}\bar{\alpha}_c\gamma_\mu\psi_c. \tag{9.52}$$

By dimensional arguments there are no other terms which are linear combinations of the χ_i contracted with the supersymmetry parameters.

9.3.1 Action and field equations

We just add the Wess-Zumino Lagrangean (6.39) for massless spin-0 fields, with $\not\partial$ replaced by $\not{D}$ and A, B in the adjoint representation, to the gauge model Lagrangean (9.29). If A, B are in the adjoint representation, then the self interaction term in (6.39), namely

$$-g\bar{\psi}\left(A - \mathrm{i}\gamma_5 B\right)\psi - \tfrac{1}{2}g^2\left(A^2+B^2\right)^2$$

should be replaced by something like

$$-g\mathrm{Tr}\left(\bar{\psi}[A,\psi]\right) + \mathrm{i}g\mathrm{Tr}\left(\bar{\psi}\gamma_5[B,\psi]\right) + \tfrac{1}{2}g^2\,\mathrm{Tr}\left([A,B]^2\right).$$

Hence we would guess that the on-shell Lagrangean of the $\mathcal{N}=2$ vector multiplet has the form

$$\begin{aligned}\mathcal{L} \;=\;& -\tfrac{1}{4}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\mathrm{Tr}(D_\mu A)^2 + \tfrac{1}{2}\mathrm{Tr}(D_\mu B)^2 + \tfrac{1}{2}g^2\,\mathrm{Tr}([A,B]^2)\\ &+\mathrm{i}\mathrm{Tr}\bar{\psi}\not{D}\psi - g\mathrm{Tr}(\bar{\psi}[A,\psi]) + \mathrm{i}g\mathrm{Tr}(\bar{\psi}\gamma_5[B,\psi]).\end{aligned} \tag{9.53}$$

Up to a surface term, the second line can be rewritten as

$$\mathcal{L}_\psi = \mathrm{Tr}\left\{\tfrac{\mathrm{i}}{2}\bar{\chi}_i\not{D}\chi_i - \mathrm{i}g\bar{\chi}_1[A,\chi_2] - g\bar{\chi}_1\gamma_5[B,\chi_2]\right\}. \tag{9.54}$$

Expanding every field as $A = A^a T_a$ such that

$$\begin{aligned}F^a_{\mu\nu} &= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^a{}_{bc}A^b_\mu A^c_\nu\\ (D_\mu A)^a &= \partial_\mu A^a + gf^a{}_{bc}A^b_\mu A^c,\end{aligned}$$

we obtain for the component fields

$$\begin{aligned}\mathcal{L} \;=\;& -\tfrac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} + \tfrac{1}{2}(D_\mu A)^a(D^\mu A)_a + \tfrac{1}{2}(D_\mu B)^a(D^\mu B)_a - \tfrac{1}{2}g^2 f^a{}_{bc}f_{apq}(A^bA^pB^cB^q)\\ &+\mathrm{i}\bar{\psi}^a\gamma^\mu\left(\partial_\mu\psi_a + gf_{abc}A^b_\mu\psi^c\right) + \mathrm{i}gf_{abc}A^a\bar{\psi}^b\psi^c + gf_{abc}B^a\bar{\psi}^b\gamma_5\psi^c.\end{aligned} \tag{9.55}$$

The model contains only *one coupling constant* g: the self-coupling of the scalar fields and the Yukawa couplings are determined by g. There is a potential term in the Lagrangean but it has flat directions whenever $[A,B]=0$. Classically it is scale invariant. This invariance will be spontaneously broken by a non-zero expectation value of the scalar fields.
The EULER-LAGRANGE equations are the following set of hyperbolic field equations:

The variation with respect to the gauge potential yields the YANG-MILLS EQUATIONS

$$D_\mu F^{\mu\nu} = ig[A, D^\nu A] + \mathrm{i}g[B, D^\nu B] - g[\bar\psi, \gamma^\nu\psi], \quad \text{where} \quad [\bar\psi, M\psi] \equiv i f^a{}_{bc}\bar\psi^b M\psi^c T_a \tag{9.56}$$

The variations with respect to the scalar and pseudo-scalar field implies the KLEIN-GORDON-type equations,

$$\begin{aligned} D_\mu D^\mu A &= -g[\bar\psi, \psi] + g^2[B, [B, A]] \\ D_\mu D^\mu B &= -\mathrm{i}g[\bar\psi, \gamma_5\psi] + g^2[A, [A, B]], \end{aligned} \tag{9.57}$$

and the variation of the Dirac field yields the Dirac *equation:*

$$i\not{D}\psi = g[A - \mathrm{i}\gamma_5 B, \psi]. \tag{9.58}$$

A particular class of solutions is obtained if we set

$$\psi = 0 \quad \text{and} \quad B = 0,$$

in which case the field equations for the remaining fields reduce to

$$D_\mu F^{\mu\nu} = ig[A, D^\nu A] \quad \text{and} \quad D_\mu D^\mu A = 0. \tag{9.59}$$

These reduced equations admit *magnetic monopole solutions.* The mass of the monopoles are proportional to the asymptotic value of $|A|$ and inversely proportional to the gauge coupling constant g. These monopoles are relevant for a field theoretic understanding of the *central charge* of the $\mathcal{N}=2$ gauge theory.
If we now further set $A = 0$ then we get the pure Yang-Mills field equations

$$D_\mu F^{\mu\nu} = 0 \tag{9.60}$$

which in the EUCLIDean sector possess (anti)selfdual *Instanton solutions.* Later we shall see, that instantons break half of the supersymmetry: the instanton background is left invariant by a subset of the $\mathcal{N}=2$ supersymmetry transformations which generate a $\mathcal{N}=1$ supersymmetry.
To calculate the Hamiltonian density we note that the momenta conjugate to the vector, scalar and fermion fields are

$$\pi_i = -F^{0i} = E_i, \quad \pi_A = D_0 A, \quad \pi_B = D_0 B \quad \text{and} \quad \pi_\psi = i\psi^\dagger.$$

When discussing the Hamiltonian structure of gauge theories it is advantageous to choose the Weyl gauge $A_0 = 0$ for which

$$\pi_i = \dot A_i, \quad \pi_A = \dot A, \quad \pi_B = \dot B \quad \text{and} \quad \pi_\psi = i\psi^\dagger.$$

The Hamiltonian density is gotten as LEGENDRE transform from the Lagrangean density and it reads

$$\begin{aligned} \mathcal{H} &= \mathrm{Tr}\Big\{\tfrac12\left(\boldsymbol{\pi}^2 + \pi_A^2 + \pi_B^2\right) + \tfrac12\boldsymbol{B}^2 + \tfrac12(\boldsymbol{D}A)^2 + \tfrac12(\boldsymbol{D}B)^2 - i\psi^\dagger\gamma^0\gamma^i D_i\psi \\ &\qquad + g\bar\psi[A, \psi] - \mathrm{i}g\bar\psi\gamma_5[B, \psi] - \frac{g^2}{2}[A, B]^2\Big\}. \end{aligned} \tag{9.61}$$

This density is (formally) hermitean and the bosonic part is bounded below.

9.3.2 Supersymmetry transformations and invariance of S

Now we shall fix the variations of the remaining fields A, B and ψ such that $\mathcal{L}$ is invariant up to surface terms. The calculations are very similar as in the Abelian case and I need not give many details here. We start from the general formulae

$$\begin{aligned} \delta D_\mu A = D_\mu \delta A - \mathrm{i}g[\delta A_\mu, A] \quad &, \quad \delta D_\mu B = D_\mu \delta B - \mathrm{i}g[\delta A_\mu, B] \\ \delta F_{\mu\nu} = D_\mu \delta A_\nu - D_\mu \delta A_\mu \quad &, \quad \delta \not{D}\psi = \not{D}\delta\psi - \mathrm{i}g[\delta A_\mu, \gamma^\mu \psi] \end{aligned}$$

and deduce the following transformation rules for the various terms in the Lagrangean density:

$$\begin{aligned} \tfrac{1}{4}\delta \mathrm{Tr}\,(F_{\mu\nu}F^{\mu\nu}) &= \mathrm{Tr}\,(F^{\mu\nu}D_\mu \delta A_\nu) \\ \tfrac{1}{2}\delta \mathrm{Tr}\,(D_\mu A D^\mu A) &= \mathrm{Tr}\,\{D_\mu A(D_\mu \delta A - \mathrm{i}g[\delta A_\mu, A])\} \\ \tfrac{1}{2}\delta \mathrm{Tr}\,(D_\mu B D^\mu B) &= \mathrm{Tr}\,\{D_\mu B(D_\mu \delta B - \mathrm{i}g[\delta A_\mu, B])\} \\ \delta \mathrm{Tr}\,([A,B]^2) &= 2\mathrm{Tr}\,\{[A,B]([\delta A, B] + [A, \delta B])\} \\ \tfrac{\mathrm{i}}{2}\delta \mathrm{Tr}\,(\bar\chi \not{D}\chi) &= \tfrac{1}{2}\mathrm{Tr}\,\{\mathrm{i}\delta\bar\chi\not{D}\chi + i\bar\chi\not{D}\delta\chi + g\bar\chi[\delta A_\mu, \gamma^\mu\chi]\} \\ \delta \mathrm{Tr}\,(\bar\chi_1[A,\chi_2]) &= \mathrm{Tr}\,\{\delta\bar\chi_1[A,\chi_2] + \bar\chi_1[A,\delta\chi_2] + \bar\chi_1[\delta A, \chi_2]\} \\ \delta \mathrm{Tr}\,(\bar\chi_1\gamma_5[B,\chi_2]) &= \mathrm{Tr}\,\{\delta\bar\chi_1\gamma_5[B,\chi_2] + \bar\chi_1\gamma_5[B,\delta\chi_2] + \bar\chi_1\gamma_5[\delta B,\chi_2]\} \end{aligned}$$

To fix the transformation properties of the scalar fields we first study the terms in $\delta\mathcal{L}$ which are *trilinear* in the Dirac field:

$$g\mathrm{Tr}\,\{\tfrac{1}{2}\bar\chi_{\mathrm{i}}[\delta A_\mu, \gamma^\mu\chi_i] - \mathrm{i}\bar\chi_1[\delta A, \chi_2] - \bar\chi_1\gamma_5[\delta B, \chi_2]\}$$

Using the explicit form of δA_μ and that $\mathrm{Tr}\,(\bar\chi[\bar\epsilon\gamma_\mu\chi, \gamma^\mu\chi]) = 0$ we obtain

$$= g f_{abc}\{\tfrac{1}{2}(\bar\chi_1^a\gamma^\mu\chi_1^b)(\bar\epsilon_2\gamma_\mu\chi_2^c) + \tfrac{1}{2}(\bar\chi_2^a\gamma^\mu\chi_2^b)(\bar\epsilon_1\gamma_\mu\chi_1^c) - (\chi_1^a\chi_2^b)\delta A^c + \mathrm{i}(\bar\chi_1^a\gamma_5\chi_2^b)\delta B^c\}$$

We rearrange the first two terms with the help of the FIERZ identities

$$\begin{aligned} \tfrac{1}{2}(\bar\epsilon_1\gamma_\mu\chi_1^a)(\bar\chi_2^b\gamma^\mu\chi_2^c) - \tfrac{1}{2}(\bar\epsilon_1\gamma_5\gamma_\mu\chi_1^a)(\bar\chi_2^b\gamma_5\gamma^\mu\chi_2^c) &= -(\bar\chi_1^a\chi_2^b)(\bar\epsilon_1\chi_2^c) + (\bar\chi_1^a\gamma_5\chi_2^b)(\bar\epsilon_1\gamma_5\chi_2^c) \\ \tfrac{1}{2}(\bar\epsilon_2\gamma_\mu\chi_2^b)(\bar\chi_1^a\gamma^\mu\chi_1^c) - \tfrac{1}{2}(\bar\epsilon_2\gamma_5\gamma_\mu\chi_2^b)(\bar\chi_1^a\gamma_5\gamma^\mu\chi_1^c) &= -(\bar\chi_1^a\chi_2^b)(\bar\epsilon_2\chi_1^c) + (\bar\chi_1^a\gamma_5\chi_2^b)(\bar\epsilon_2\gamma_5\chi_1^c) \end{aligned}$$

The second terms on the left hand sides do not contribute, since they are symmetric in b, c or a, c, respectively. Now we can read off how the scalar fields must transform such that the terms trilinear in $\delta\mathcal{L}$ chancel:

$$\begin{aligned} \delta_\alpha A &= -(\bar\epsilon_1\chi_2 - \bar\epsilon_2\chi_1) = \mathrm{i}(\bar\alpha\psi - \bar\alpha_c\psi_c) \\ \delta_\alpha B &= \mathrm{i}(\bar\epsilon_1\gamma_5\chi_2 - \bar\epsilon_2\gamma_5\chi_1) = \bar\alpha\gamma_5\psi - \bar\alpha_c\gamma_5\psi_c. \end{aligned} \tag{9.62}$$

Again, the variations of A and B can only be linear combinations of the χ_i contracted with the supersymmetry parameters. Let us see, whether we can arrange for a transformation of the Majorana spinors such the terms trilinear in the bosonic fields A, B chancel in $\delta\mathcal{L}$. Such term can only arise from

$$g\mathrm{Tr}\Big\{g[A,B]([\delta A, B] + [A, \delta B]) - i\delta\bar\chi_1[A,\chi_2] - \mathrm{i}\bar\chi_1[A,\delta\chi_2] - \delta\bar\chi_1\gamma_5[B,\chi_2] - \bar\chi_1\gamma_5[B,\delta\chi_2]\Big\}.$$

So let us assume, that

$$\delta\chi_i = g[A,B]\gamma_5\epsilon_i + \dots, \qquad \delta\bar\chi_i = g\bar\epsilon_{\rm i}\gamma_5[A,B] + \dots$$

Then the last four terms in the second to last equation read

$$g\mathrm{Tr}\{[A,B]\big(-\mathrm{i}[A,\bar\epsilon_1\gamma_5\chi_2-\bar\epsilon_2\gamma_5\chi_1]+[\bar\epsilon_1\chi_2-\bar\epsilon_2\chi_1,B]\big)\} \tag{9.63}$$

and exactly cancel the first two terms. From our experience with the Wess-Zumino model we also expect terms

$$\delta\chi_i = -\mathrm{i}\not{D}(A-\mathrm{i}\gamma_5 B)\kappa_i + \dots, \qquad \delta\bar\chi_i = \mathrm{i}\bar\kappa_i\not{D}(A+\mathrm{i}\gamma_5 B) + \dots \tag{9.64}$$

which are fixed by considering the contributions which are quadratic in the scalar fields. These contributions can only come from

$$\begin{aligned}\delta\mathcal{L} \;=\; & -\mathrm{i}g\mathrm{Tr}\{D_\mu A[\delta A_\mu,A]+D_\mu B[\delta A_\mu,B]\}+\tfrac12\mathrm{Tr}\{\delta\bar\chi_i\not{D}\chi_i+\bar\chi_i\not{D}\delta\chi_i\}\\ & -\mathrm{i}g\mathrm{Tr}\{\delta\bar\chi_1[A,\chi_2]+\bar\chi_1[A,\delta\chi_2]\}-g\mathrm{Tr}\{\delta\bar\chi_1\gamma_5[B,\chi_2]+\bar\chi_1\gamma_5[B,\delta\chi_2]\}+\dots\end{aligned}$$

In the first two terms we insert the variation of the gauge potential, in the next two terms the result (9.63) and in the last two terms the expected results (9.64). This then leads to

$$\begin{aligned}=\;&\tfrac12 g\{[A,B]D_\mu(\bar\epsilon_{\rm i}\gamma_5\gamma^\mu\chi_i)-D_\mu[A,B](\bar\epsilon_{\rm i}\gamma_5\gamma^\mu\chi_i)\}\\ &+g\{[A,D_\mu A]+[B,D_\mu B]\}\{\bar\epsilon_{\rm i}\gamma^\mu\chi_i-(\bar\kappa_1\gamma^\mu\chi_2-\bar\kappa_2\gamma^\mu\chi_1)\}\\ &+\mathrm{i}g\{[A,D_\mu B]+[D_\mu A,B]\}\{\bar\kappa_1\gamma_5\gamma^\mu\chi_2-\bar\kappa_2\gamma_5\gamma^\mu\chi_1)\}+\dots,\end{aligned}$$

where $\dots$ indicates the terms which are not quadratic in A,B. Clearly, we must choose $\kappa_1=\epsilon_2$ and $\kappa_2=-\epsilon_1$ such that the second line vanishes and this becomes a total divergence

$$\tfrac12\partial_\mu\mathrm{Tr}\{[A,B]\bar\epsilon_{\rm i}\gamma_5\gamma^\mu\chi_i\}.$$

Collecting the various contribution we conclude, that

$$\begin{aligned}\delta_\alpha\psi &= \mathrm{i}F^{\mu\nu}\Sigma_{\mu\nu}\alpha-\not{D}(A-\mathrm{i}\gamma_5B)\alpha+g\,[A,B]\gamma_5\alpha\\ \delta_\alpha\bar\psi &= -\mathrm{i}\bar\alpha F^{\mu\nu}\Sigma_{\mu\nu}-\bar\alpha\not{D}(A+\mathrm{i}\gamma_5B)+g\,\bar\alpha\gamma_5[A,B].\end{aligned} \tag{9.65}$$

Let us summarize our findings (what has not been proven here, we leave as an exercise). The action (9.53) is invariant with respect to the following supersymmetry transformations:

$$\begin{aligned}\delta_\alpha A_\mu &= \mathrm{i}\,(\bar\alpha\gamma_\mu\psi+\bar\alpha_c\gamma_\mu\psi_c)\\ \delta_\alpha A &= \mathrm{i}\,(\bar\alpha\psi-\bar\alpha_c\psi_c)\\ \delta_\alpha B &= \bar\alpha\gamma_5\psi-\bar\alpha_c\gamma_5\psi_c\\ \delta_\alpha\psi &= \mathrm{i}F^{\mu\nu}\Sigma_{\mu\nu}\alpha-\not{D}\,(A-\mathrm{i}\gamma_5B)\,\alpha+g\,[A,B]\gamma_5\alpha.\end{aligned} \tag{9.66}$$

Note that a constant A-field background is left invariant by an arbitrary supersymmetry transformation. It does not 'break supersymmetry'.
We have already done most of the calculation in order to show that S is invariant. The explicit result reads

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu\left(\bar\alpha V^\mu + \bar\alpha_c V_c^\mu\right) \\ V^\mu &= \mathrm{Tr}\left\{(ig[A,B]\gamma_5\gamma^\mu + 2D_\nu(A-iB\gamma_5)\Sigma^{\nu\mu} + {}^*F^{\mu\nu}\gamma_5\gamma_\nu)\psi\right\} \\ V_c^\mu &= -\mathrm{i}\mathrm{Tr}\left\{(D^\mu(A-iB\gamma_5) + F^{\mu\nu}\gamma_\nu)\psi_c\right\} \end{aligned}$$

To construct the Noether current we need to calculate

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\delta A_\nu + \ldots = \mathrm{Tr}\{-F^{\mu\nu}\delta A_\nu + \mathrm{i}\bar\psi\gamma^\mu\delta\psi + D^\mu A\delta A + D^\mu B\delta B\}$$

and subtract $\bar\alpha V^\mu + \bar\alpha_c V_c^\mu$. The result is the following hermitean Noether current:

$$\begin{aligned} J^\mu &= \bar\alpha\left\{{}^*F^{\mu\nu}\gamma_\nu\gamma_5 - \mathrm{i}F^{\mu\nu}\gamma_\nu + i\not{D}(A+iB\gamma_5)\gamma^\mu + \mathrm{i}g[A,B]\gamma^\mu\gamma_5\right\}\psi \\ &+ \bar\psi\left\{{}^*F^{\mu\nu}\gamma_\nu\gamma_5 + \mathrm{i}F^{\mu\nu}\gamma_\nu - \mathrm{i}\gamma^\mu\not{D}(A-iB\gamma_5) + \mathrm{i}g[A,B]\gamma^\mu\gamma_5\right\}\alpha \end{aligned}$$

The corresponding Noether charge reads

$$\begin{aligned} Q &= \bar\alpha\int \mathrm{d}^3x\,(R+S)\,\psi + \int \mathrm{d}^3x\,\bar\psi\,(R-S)\,\alpha \qquad \text{with} \\ R &= \gamma_i B_\mathrm{i}\gamma_5 + \mathrm{i}\gamma^0\gamma_i D_i(A-iB\gamma_5) + \mathrm{i}\gamma^0\gamma_5[A,B] \\ S &= \mathrm{i}\gamma_\mathrm{i}\pi_i + \mathrm{i}\pi_A + \pi_B\gamma_5 \end{aligned}$$

Here $\pi_i = E_i$ is the momentum conjugate to A_i, $\pi_A = D_0 A$ the momentum conjugate to A and $\pi_B = D_0 B$ the momentum conjugate to B. One should compare the charge of the $\mathcal{N}=2$ vector multiplet with the one in (6.48) of the Wess-Zumino model and the charge after (9.22) of the $\mathcal{N}=1$ vector multiplet.
Exercise: Check, that the commutator of two supersymmetry transformations is

$$\begin{aligned} [\delta_{\alpha_1},\delta_{\alpha_2}]A_\mu &= a^\rho\partial_\rho A_\mu + D_\mu\lambda \qquad & [\delta_{\alpha_1},\delta_{\alpha_2}]\psi &= a^\rho\partial_\rho\psi + \mathrm{i}[\lambda,\psi] \\ [\delta_{\alpha_1},\delta_{\alpha_2}]A &= a^\rho\partial_\rho A + \mathrm{i}[\lambda,A] \qquad & [\delta_{\alpha_1},\delta_{\alpha_2}]B &= a^\rho\partial_\rho B + \mathrm{i}[\lambda,B] \end{aligned}$$

where

$$a^\rho = -2\mathrm{i}\left(\bar\alpha_2\gamma^\rho\alpha_1 - \bar\alpha_1\gamma^\rho\alpha_2\right)$$

is the parameter for the infinitesimal translation generated by the commutator of two susy transformations and

$$\lambda = 2\mathrm{i}\left\{(\bar\alpha_2\gamma^\rho\alpha_1 - \bar\alpha_1\gamma^\rho\alpha_2)A_\rho - \bar\alpha_2(A-\mathrm{i}\gamma_5 B)\alpha_1 + \bar\alpha_1(A-\mathrm{i}\gamma_5 B)\alpha_2\right\}$$

the field dependent gauge parameter.

9.3.3 Chiral basis

We rewrite the previous results in terms of Weyl fermions

$$\psi=\begin{pmatrix}\lambda_\alpha\\ \bar\chi^{\dot\alpha}\end{pmatrix}\qquad \bar\psi=(\,\chi^\alpha,\bar\lambda_{\dot\alpha}\,)\quad\text{and}\quad \phi=(A+iB)/\sqrt{2}.$$

Using the conventions in the appendix we see that

$$\begin{aligned}\bar\psi\not{D}\psi &= \chi^\alpha\sigma_{\mu\alpha\dot\alpha}D^\mu\bar\chi^{\dot\alpha}+\bar\lambda_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\alpha}D_\mu\lambda_\alpha\equiv\chi(\sigma D)\bar\lambda+\bar\lambda(\tilde\sigma D)\lambda\\ \bar\psi[A,\psi] &= \chi^\alpha[A,\lambda_\alpha]+\bar\lambda_{\dot\alpha}[A,\bar\chi^{\dot\alpha}]=\chi[A,\lambda]+\bar\lambda[A,\bar\chi]\\ \bar\psi\gamma_5[B,\psi] &= -\chi^\alpha[B,\lambda_\alpha]+\bar\lambda_{\dot\alpha}[B,\bar\chi^{\dot\alpha}]=-\chi[B,\lambda]+\bar\lambda[B,\bar\chi]\\ [\phi,\phi^\dagger] &= -\mathrm{i}[A,B]\,.\end{aligned}$$

In terms of these Weyl spinors the action takes the form

$$\begin{aligned}\mathcal{L} &= -\tfrac14\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}+\mathrm{Tr}\,(D_\mu\phi D^\mu\phi^\dagger)-\tfrac12 g^2\,\mathrm{Tr}\,([\phi,\phi^\dagger]^2)\\ &\quad+\mathrm{Tr}\,\Big\{\mathrm{i}\chi(\sigma D)\bar\chi+i\bar\lambda(\tilde\sigma D)\lambda-\sqrt{2}g\chi[\phi,\lambda]-\sqrt{2}g\bar\lambda[\phi^\dagger,\bar\chi]\Big\}\,.\end{aligned}\tag{9.67}$$

To rewrite the susy-transformations we also change from Dirac- to Weyl spinors for the supersymmetry parameters,

$$\alpha=\begin{pmatrix}\theta_\alpha\\ \bar\eta^{\dot\alpha}\end{pmatrix}\qquad\bar\alpha=(\,\eta^\alpha,\bar\theta_{\dot\alpha}\,)\,.$$

In the chiral representation we may choose as charge conjugation matrix

$$\mathcal{C}=\mathrm{i}\gamma^0\gamma^2=\begin{pmatrix}-i\sigma_2&0\\0&i\sigma_2\end{pmatrix}=\begin{pmatrix}(\varepsilon_{\alpha\beta})&0\\0&(\varepsilon^{\dot\alpha\dot\beta})\end{pmatrix}\tag{9.68}$$

such that

$$\psi_c=\mathcal{C}\bar\psi^T=\begin{pmatrix}\chi_\alpha\\ \bar\lambda^{\dot\alpha}\end{pmatrix},\quad\bar\psi_c=(\lambda^\alpha,\bar\theta_{\dot\alpha}),\quad\alpha_c=\begin{pmatrix}\eta_\alpha\\ \bar\theta^{\dot\alpha}\end{pmatrix},\quad\bar\alpha_c=(\theta^\alpha,\bar\eta_{\dot\alpha})\,.\tag{9.69}$$

Now we can rewrite the fermionic bilinears in the chiral basis

$$\begin{aligned}\bar\alpha\psi-\bar\alpha_c\psi_c &= \eta\lambda+\bar\theta\bar\chi-\theta\chi-\bar\eta\bar\lambda\\ \bar\alpha\gamma_5\psi-\bar\alpha_c\gamma_5\psi_c &= -\eta\lambda+\bar\theta\bar\chi+\theta\chi-\bar\eta\bar\lambda\\ \bar\alpha\gamma_\mu\psi+\bar\alpha_c\gamma_\mu\psi_c &= \eta\sigma_\mu\bar\chi+\bar\theta\tilde\sigma_\mu\lambda+\theta\sigma_\mu\bar\lambda+\bar\eta\tilde\sigma_\mu\chi\end{aligned}$$

Inserting these relations into (9.66) yields

$$\begin{aligned}\delta A_\mu &= \mathrm{i}(\eta\sigma_\mu\bar\chi+\bar\theta\tilde\sigma_\mu\lambda+\theta\sigma_\mu\bar\lambda+\bar\eta\tilde\sigma_\mu\chi)\\ \delta\phi &= i\sqrt{2}(\bar\theta\bar\chi-\bar\eta\bar\lambda),\quad\delta\phi^\dagger=i\sqrt{2}(\eta\lambda-\theta\chi)\\ \delta\lambda &= \tfrac12F^{\mu\nu}\sigma_{\mu\nu}\theta-\sqrt{2}\,(\sigma_\mu D^\mu\phi^\dagger)\,\bar\eta\,-\mathrm{i}g[\phi,\phi^\dagger]\theta\\ \delta\bar\chi &= \tfrac12F^{\mu\nu}\tilde\sigma_{\mu\nu}\bar\eta-\sqrt{2}\,(\tilde\sigma_\mu D^\mu\phi)\theta+\mathrm{i}g[\phi,\phi^\dagger]\bar\eta\,.\end{aligned}$$

9.3.4 Bogomol'nyi bound, Monopoles and Jackiw-Rebbi modes

Let us come back to the monopole and instanton solutions in this model. Assume that we have constructed a solution with $B=0$ and $\psi=0$. Such a solution is left invariant by the supersymmetry transformations if

$$\delta\psi = 0 \Longleftrightarrow \mathrm{i}F^{\mu\nu}\Sigma_{\mu\nu}\alpha = \not{D}A\alpha \quad \text{or} \quad \tfrac{1}{2}F^{\mu\nu}\gamma_{\mu\nu}\alpha = \not{D}A\alpha \tag{9.70}$$

which in a chiral basis reads

$$\tfrac{1}{2}F^{\mu\nu}(\sigma_{\mu\nu})_\alpha{}^\beta\,\theta_\beta = D_\mu A(\sigma^\mu)_{\alpha\dot\beta}\,\bar\eta^{\dot\beta} \quad \text{and} \quad \tfrac{1}{2}F^{\mu\nu}(\tilde\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\,\bar\eta^{\dot\beta} = D_\mu A(\tilde\sigma^\mu)^{\dot\alpha\beta}\,\theta_\beta\,.$$

Let us see what second order equation is implied by the condition (9.70). For that we act with $\not{D}$ on this equation which leads to

$$\tfrac{1}{2}\gamma_\rho\gamma_{\mu\nu}D^\rho F^{\mu\nu}\alpha = D^2A\alpha - \frac{\mathrm{i}g}{2}\gamma^{\rho\mu}[F_{\rho\mu},A]\alpha.$$

Now we use (9.11) and the Bianchi identity to simplify the left hand side,

$$\gamma_\nu D_\mu F^{\mu\nu}\alpha = D^2A\alpha - \frac{\mathrm{i}g}{2}\gamma^{\rho\mu}[F_{\rho\mu},A]\alpha = D^2A\alpha - \mathrm{i}g\gamma^\mu[D_\mu A,A]\alpha,$$

where in the last step we used the invariance condition (9.70). This finally yields the *second order equation*

$$\left(D_\mu F^{\mu\nu} - \mathrm{i}g[A,D^\nu A]\right)\gamma^\nu\alpha = (D^2A)\alpha. \tag{9.71}$$

Next we argue that both sides must vanish separately if α is arbitrary. This way one arrives at the field equations for the scalar field and the gauge potential in cases where the pseudoscalar and Dirac field vanish. To see this we note that $\bar\alpha\gamma^\mu\alpha$ is imaginary and $\bar\alpha\alpha$ is real for a Majorana parameter α. Hence if we multiply (9.71) with $\bar\alpha$ from the left and assume that α is Majorana, then the left hand side becomes antihermitean, whereas the right hand side becomes hermitean. It follows that both sides must vanish. This then proves that the first order equation (9.70) implies the second order field equations (9.59). Let us now assume, that (A_μ,A) is a (static) magnetic monopole solution. Then the first order equation (9.70) reduces to

$$0 = \delta\psi = i\begin{pmatrix} B_i\tau_i & -\tau_i D_i A \\ \tau_i D_i A & B_i\tau_i \end{pmatrix}\alpha = \begin{pmatrix} \mathrm{i}\boldsymbol{\tau}\cdot\boldsymbol{B}\theta - \boldsymbol{\tau}\cdot\boldsymbol{D}A\,\bar\eta \\ i\boldsymbol{B}\cdot\boldsymbol{\tau}\bar\eta + \boldsymbol{\tau}\cdot\boldsymbol{D}A\,\theta \end{pmatrix}.$$

This system of equations is satisfied if

$$i\theta_\alpha = (\sigma_0)_{\alpha\dot\beta}\,\bar\eta^{\dot\beta} \quad \text{and} \quad \boldsymbol{B} = \boldsymbol{D}A \tag{9.72}$$

hold true. The first condition means that only a $\mathcal{N}=1$ supersymmetry is left intact and the second condition is the Bogomol'nyi bound. To make this more clear, consider the energy or mass (9.61) for a static background (A_i,A)

$$M = \tfrac{1}{2}\int \mathrm{d}^3x\,\mathrm{Tr}\left(\boldsymbol{B}^2 + (\boldsymbol{D}A)^2\right) = \tfrac{1}{2}\int \mathrm{d}^3x\,\mathrm{Tr}\,(\boldsymbol{B}\pm\boldsymbol{D}A)^2 \mp \int \mathrm{Tr}\,(\boldsymbol{B}\cdot\boldsymbol{D}A)\,.$$

Clearly we have the bound

$$M \geq \Big| \int \mathrm{Tr}\,(\boldsymbol{B}\cdot\boldsymbol{D}A) \Big|$$

which is just the celebrated Bogomol'nyi *bound*. The Bianchi identity for the 'chromo-magnetic' field reads

$$\boldsymbol{D}\cdot\boldsymbol{B}=0 \Longrightarrow \nabla\,\mathrm{Tr}\,(\boldsymbol{B}\,A)=\mathrm{Tr}\,(\boldsymbol{B}\cdot\boldsymbol{D}A).$$

Inserting this into the Bogomol'nyi bound and using the Gauss theorem yields

$$M \geq \Big| \oint \mathrm{Tr}\,(\boldsymbol{B}\,A)d\boldsymbol{S} \Big|. \tag{9.73}$$

To continue we assume that the gauge group is $SU(2)$ with generators

$$T_a=\frac{1}{\sqrt{2}}\tau_a, \quad \text{such that} \quad f_{abc}=\epsilon_{abc}.$$

If v denotes the length of A at infinity, then the last surface integral is just the *magnetic flux* of the $U(1)$-magnetic field in the direction of A times v such that

$$M \geq v\Phi.$$

Since $|\boldsymbol{D}A|$ tends to zero at spatial infinity it follows that

$$A_i^a=-\frac{1}{gv^2}\epsilon_{abc}A^b\partial_iA^c+A^aC_i$$

holds true with C_i arbitrary. If we now compute the leading order behaviour of the non-Abelian gauge fields we find

$$F_{ij}^a \sim \frac{1}{v}A^af_{ij},$$

where we introduced

$$f_{ij}=-\frac{1}{gv^3}\epsilon_{abc}A^a\partial_iA^b\partial_jA^c+\partial_iC_j-\partial_jC_i$$

and the equation of motion imply $\partial_i f^{ij}=\partial_i\,{}^*f^{ij}\sim 0$. Outside the core of the monopole the non-Abelian field points in the direction of the Higgs field, that is, in the direction of the unbroken $U(1)$. The *magnetic charge* of this field configuration is given by the flux of the magnetic field $B_i=\frac{1}{2}\epsilon_{ijk}f_{jk}$ through a sphere at infininty,

$$\Phi=\oint \boldsymbol{B}d\boldsymbol{S}=\lim_{R\to\infty}\frac{R^2}{2gv^3}\oint \epsilon_{ijk}\epsilon_{abc}A^a\partial^jA^b\partial^kA^cn^id\Omega=\frac{4\pi N}{g} \tag{9.74}$$

with N the winding number of the Higgs field, i.e. the winding of the map

$$S^2 \ni \hat{x} \longrightarrow A(\hat{x})=\lim_{r\to\infty}A(r\hat{x})\in S^2.$$

Thus we obtain the celebrated Dirac quantitation condition

$$g\Phi = 4\pi N \tag{9.75}$$

relating the gauge coupling to the magnetic charge of the monopoles. The corresponding BOGOMOL'NYI bound

$$M \geq \frac{4\pi N}{g} v \tag{9.76}$$

saturates for $B_i = \pm D_i A$, that is for BPS-monopoles. The solutions to (9.72) are well-known. In the BPS limit monopoles neither attract nor repel each other. This must be the case, since the lower bound is attained independent of the distance between the monopoles. When one changes the collective coordinates the energy does not change and hence the monopole cannot increase or decrease the energy if they move relative to each other. This behaviour is typical for BPS states which saturate a BOGOMOL'NYI type bound.
Now we shall see that supersymmetry automatically generates the so-called Jackiw-Rebbi modes. Note that the Dirac equation in the gauge $A_0 = 0$ reads

$$\mathrm{i}\dot{\psi} = -\mathrm{i}\alpha^i D_i \psi + g\gamma^0 [A, \psi], \qquad \alpha^i = \gamma^0 \gamma^i.$$

For the static monopole background (A_i, A) we may separate off the energy dependence and arrive at the time-independent Dirac equation

$$E\psi = -\mathrm{i}\alpha^i D_i \psi + g\gamma^0 [A, \psi] = H\psi.$$

Now we shall prove that $\delta\psi$ is a zero-mode of H if (A_i, A) satisfy the BOGOMOL'NYI equation. For $\boldsymbol{B} = \boldsymbol{D}A$ the supersymmetry variation of the Dirac spinor simplifies to

$$\delta\psi = D_i A \begin{pmatrix} \mathrm{i}\tau_i\,\kappa \\ \tau_i\,\kappa \end{pmatrix}, \quad \text{where} \quad \kappa = \theta + \mathrm{i}\bar{\eta}.$$

It follows that

$$\begin{aligned} \gamma^j D_j \delta\psi &= D_j D_i A \begin{pmatrix} \tau_j \tau_i \kappa \\ -\mathrm{i}\tau_j \tau_i \kappa \end{pmatrix} = \tfrac{1}{2}[D_j, D_i] A \begin{pmatrix} \mathrm{i}\epsilon_{jik}\,\kappa \\ \epsilon_{jik}\,\kappa \end{pmatrix} \\ &= \mathrm{i}[B_k, A] \begin{pmatrix} \mathrm{i}\tau_k\,\kappa \\ \tau_k\,\kappa \end{pmatrix} = \mathrm{i}[D_k A, A] \begin{pmatrix} \mathrm{i}\tau_k\,\kappa \\ \tau_k\,\kappa \end{pmatrix} = -\mathrm{i}[A, \delta\psi]\,, \end{aligned}$$

where we used $D^2 A = 0$ in the second step. This is just the equation

$$H\delta\psi = 0$$

for a zero mode. Thus we have found an explicit expression for the JACKIW-REBBI zero-modes about a magnetic monopoles. Supersymmetry opens up an elegant way to characterize and construct BPS-monopoles, it also gives the associated JACKIW-REBBI modes. For the physical consequences of these modes I refer to [43].

9.3.5 $\mathcal{N}=2$-SYM in Euclidean spacetime

In Euclidean spacetime the gamma matrices must be hermitean so that we choose

$$(\gamma_0,\gamma_i)_M=(\gamma_0,-\mathrm{i}\gamma_i)_E. \tag{9.77}$$

From

$$\mathcal{L}_M\sim\tfrac{1}{2}\mathrm{Tr}\,(F_{0i})^2-\tfrac{1}{4}\mathrm{Tr}\,(F_{ij})^2+\dots$$

we see, that the Lagrangean becomes negative definite, irrespective whether we multiply the time coordinate or the space coordinates with i. Because

$$\mathcal{L}_M=\tfrac{1}{2}\mathrm{Tr}\,(D_0A)^2-\tfrac{1}{2}\mathrm{Tr}\,(D_iA)^2$$

we conclude that we should continue the time coordinate for these terms to become negative definite. Hence we should choose

$$(\partial_0,\partial_i)_M=(i\partial_0,\partial_i)_E,\quad\text{such that}\quad(A_0,A_i,F_{0i},F_{ij})_M=(iA_0,A_i,iF_{0i},F_{ij})_E. \tag{9.78}$$

With this choices we have

$$\not{D}_M=i\not{D}_E.$$

In Euclidean spacetime the Dirac term must have the form

$$\pm\mathrm{i}\psi^\dagger\not{D}\psi$$

in order to be hermitean and $SO(4)$ invariant. This tells us, that

$$(\psi,\bar\psi)_M=(\psi,i\psi^\dagger)_E\quad\text{such that}\quad(\mathrm{i}\bar\psi\not{D}\psi)_M=-\mathrm{i}(\psi^\dagger\not{D}\psi)_E. \tag{9.79}$$

For

$$A_M=A_E\quad\text{and}\quad B_M=B_E \tag{9.80}$$

the Yukawa interaction term

$$-g\mathrm{Tr}\,(\bar\psi[A,\psi])=-\mathrm{i}g\mathrm{Tr}\,(\psi^\dagger[A,\psi])$$

becomes antihermitean in Euclidean spacetime,

$$\{-\mathrm{i}g\mathrm{Tr}\,(\psi^\dagger[A,\psi])\}^\dagger=(gf_{abc}\psi^{a\dagger}A^b\psi^c)^\dagger=(gf_{abc}\psi^{c\dagger}A^b\psi^a)=ig\mathrm{Tr}\,(\psi^\dagger[A,\psi]).$$

More generally, one can show that there are no consistent transformation rules of the coordinates, γ-matrices and fields such the resulting Euclidean action is hermitean and bounded from below.

This is not as bad as it seems, since in Euclidean spacetime the mass term in $\psi^\dagger(i\not{D}+im)\psi$ is not hermitean either. But although the eigenvalues of $i\not{D}+im$ are not real, they come in pairs $\pm\lambda+im$ and the partition function stays real, since $(\lambda+im)(-\lambda+im)$ is real. Since after spontaneous symmetry breaking the Yukawa terms may lead to mass terms for the fermions we even expect an antihermitean Yukawa interaction. Hence we may accept the transformation rules (9.77-9.80) in which case

$$\begin{aligned}\mathcal{L}_E &= \tfrac{1}{4}\mathrm{Tr}\,F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}\mathrm{Tr}\,(D_\mu A)^2+\tfrac{1}{2}\mathrm{Tr}\,(D_\mu B)^2-\tfrac{1}{2}g^2\,\mathrm{Tr}\,([A,B]^2)\\ &\quad +\mathrm{i}\mathrm{Tr}\,\psi^\dagger \not{D}\psi+\mathrm{i}g\mathrm{Tr}\,(\psi^\dagger[A,\psi])+g\mathrm{Tr}\,(\psi^\dagger\gamma_5[B,\psi]),\end{aligned}\tag{9.81}$$

where the indices are raised and lowered with the EUCLIDean metric $\delta_{\mu\nu}$. This seems to be a completely consistent EUCLIDean model. In the chiral representation the hermitean matrices γ-matrices and the hermitean γ_5 take the form

$$\gamma_0=\begin{pmatrix}0&\tau_0\\ \tau_0&0\end{pmatrix},\quad \gamma_i=\begin{pmatrix}0&\mathrm{i}\tau_i\\ -\mathrm{i}\tau_i&0\end{pmatrix}\quad\text{and}\quad \gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3=\begin{pmatrix}\tau_0&0\\ 0&-\tau_0\end{pmatrix}.$$

Also note that

$$\gamma_{0i}=\begin{pmatrix}-\mathrm{i}\tau_i&0\\ 0&\mathrm{i}\tau_i\end{pmatrix}\quad\text{and}\quad \gamma_{ij}=\mathrm{i}\epsilon_{ijk}\begin{pmatrix}\tau_k&0\\ 0&\tau_k\end{pmatrix},$$

so that $\sigma_{\mu\nu}$ in

$$\gamma_{\mu\nu}=\begin{pmatrix}\sigma_{\mu\nu}&0\\ 0&\tilde\sigma_{\mu\nu}\end{pmatrix}$$

is anti-selfdual and $\tilde\sigma_{\mu\nu}$ selfdual.
The gauge- and general covariant derivative of the fields are

$$D_\mu\Phi=\partial_\mu\Phi-\mathrm{i}[A_\mu,\Phi].$$

If there is a supersymmetry, then the symmetry transformations should have the form

$$\delta_\alpha A_\mu=\mathrm{i}(\alpha^\dagger\gamma_\mu\psi-\psi^\dagger\gamma_\mu\alpha),\quad \delta_\alpha A=-\alpha^\dagger\psi+\psi^\dagger\alpha,\quad \delta_\alpha B=\mathrm{i}(\alpha^\dagger\gamma_5\psi-\psi^\dagger\gamma_5\alpha).\tag{9.82}$$

Now we have got a serious problem. The second formula shows that A should be antihermitean. But with an antihermitean A the terms $\mathrm{Tr}\,(DA)^2$ and the potential term quartic in the scalar fields become unbounded from below. The corresponding model in unstable and probably not consistent. Note however, that for an antihermitean A also $D_\mu A$ is antihermitean.
The remaining transformation of the Dirac spinor field reads

$$\begin{aligned}\delta_\alpha\psi &= -\mathrm{i}F^{\mu\nu}\Sigma_{\mu\nu}\alpha-\mathrm{i}\not{D}(A-\mathrm{i}\gamma_5 B)\alpha+g\,[A,B]\gamma_5\alpha\\ \delta_\alpha\psi^\dagger &= \mathrm{i}\alpha^\dagger F^{\mu\nu}\Sigma_{\mu\nu}-\mathrm{i}\alpha^\dagger\not{D}(A+i\gamma_5 B)+g\alpha^\dagger\,[A,B]\gamma_5.\end{aligned}\tag{9.83}$$

The invariance of S is proven as follows. The terms in $\delta\mathcal{L}$ containing the scalar fields is easy to calculate and leads to $\partial_\mu\mathrm{Tr}\,(K^\mu)$ with

$$K^\mu=-\mathrm{i}g[A,B](\alpha^\dagger\gamma^\mu\gamma_5\psi)+\alpha^\dagger D^\mu(A-\mathrm{i}\gamma_5 B)\psi+\psi^\dagger D^\mu(A-\mathrm{i}\gamma_5 B)\alpha-\alpha^\dagger\gamma^\mu\not{D}(A-\mathrm{i}\gamma_5 B)\psi.$$

As always, the term containing only the fermions is the hard one.
To continue our calculation we note, that (4.85) holds with our conventions in EUCLIDean spacetime, i.e.

$$4+\mathrm{i}\chi^\dagger = -(\chi^\dagger\psi) - \gamma_\mu(\chi^\dagger\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\chi^\dagger\gamma^{\mu\nu}\psi) + \gamma_5\gamma_\mu(\chi^\dagger\gamma_5\gamma^\mu\psi) - \gamma_5(\chi^\dagger\gamma_5\psi). \quad (9.84)$$

But the formulae (9.8) and (9.11) now read

$$\begin{aligned} 2\Sigma_{\mu\nu}\gamma_\rho &= i\delta_{\mu\rho}\gamma_\nu - i\delta_{\nu\rho}\gamma_\mu + \mathrm{i}\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 \\ 2\gamma_\rho\Sigma_{\mu\nu} &= -i\delta_{\mu\rho}\gamma_\nu + i\delta_{\nu\rho}\gamma_\mu + i\epsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_5 \end{aligned} \quad (9.85)$$

where $\epsilon_{0123}=1$. Using the last set of relations together with the BIANCHI identity one arrives at the intermediate result

$$\begin{aligned} &-i\partial_\mu \mathrm{Tr}\,(F^{\mu\nu}\psi^\dagger\gamma_\nu\alpha) - i\partial_\mu\mathrm{Tr}\,({}^*F^{\mu\nu}\alpha^\dagger\gamma_\nu\gamma_5\psi) \\ &+gf_{abc}\{(\psi_a^\dagger\gamma^\mu\psi_b)(\alpha^\dagger\gamma_\mu\psi^c) - (\psi_a^\dagger\psi_b)(\alpha^\dagger\psi_c) + (\psi_a^\dagger\gamma_5\psi_b)(\alpha^\dagger\gamma_5\psi_c\} \\ &-gf_{abc}\{(\psi_a^\dagger\gamma^\mu\psi_b)(\psi_c^\dagger\gamma_\mu\alpha) - (\psi_a^\dagger\psi_b)(\psi_c^\dagger\alpha) + (\psi_a^\dagger\gamma_5\psi_b)(\psi_c^\dagger\gamma_5\alpha)\} \end{aligned} \quad (9.86)$$

Now we apply the FIERZ-identity (9.84) and find

$$\begin{aligned} \gamma^\mu\chi\psi^\dagger\gamma_\mu &= -(\psi^\dagger\chi) + \tfrac{1}{2}\gamma_\mu(\psi^\dagger\gamma^\mu\chi) + \tfrac{1}{2}\gamma_5\gamma_\mu(\psi^\dagger\gamma_5\gamma^\mu\chi) + \gamma_5(\psi^\dagger\gamma_5\chi) \\ \chi\psi^\dagger - \gamma_5\chi\psi^\dagger\gamma_5 &= -\tfrac{1}{2}\gamma_\mu(\psi^\dagger\gamma^\mu\chi) + \tfrac{1}{2}\gamma_5\gamma_\mu(\psi^\dagger\gamma_5\gamma^\mu\chi) \end{aligned}$$

which yield

$$\gamma^\mu\chi\psi^\dagger\gamma_\mu - \chi\psi^\dagger + \gamma_5\chi\psi^\dagger\gamma_5 = \gamma_\mu(\psi^\dagger\gamma^\mu\chi) - (\psi^\dagger\chi) + \gamma_5(\psi^\dagger\gamma_5\chi). \quad (9.87)$$

With this result the terms in (9.86) containing the ψ^3-terms can be rewritten as

$$\begin{aligned} &+gf_{abc}\{(\psi_a^\dagger\gamma_\mu\psi_c)(\alpha^\dagger\gamma^\mu\psi_b) - (\psi_a^\dagger\psi_c)(\alpha^\dagger\psi_b) + (\psi_a^\dagger\gamma_5\psi_c)(\alpha^\dagger\gamma_5\psi_b)\} \\ &-gf_{abc}\{(\psi_c^\dagger\gamma^\mu\psi_b)(\psi_a^\dagger\gamma_\mu\alpha) - (\psi_c^\dagger\psi_b)(\psi_a^\dagger\alpha) + (\psi_c^\dagger\gamma_5\psi_b)(\psi_a^\dagger\gamma_5\alpha)\} \end{aligned}$$

It follows that the terms in curly brackets are symmetric in (b,c) and (a,c), respectively. Because of the antisymmetry of the structure constants the contribution of the 3-fermion terms to the variation of the Lagrangean vanishes. Thus we remain with the following result for the variation of $\mathcal{L}$ under supersymmetry transformations:

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu \mathrm{Tr}\,V^\mu \quad \text{where} \\ V^\mu &= -\mathrm{i}g[A,B](\alpha^\dagger\gamma^\mu\gamma_5\psi) + \alpha^\dagger D^\mu(A-\mathrm{i}\gamma_5 B)\psi + \psi^\dagger D^\mu(A-\mathrm{i}\gamma_5 B)\alpha \\ &\quad -\alpha^\dagger\gamma^\mu \not{D}(A-\mathrm{i}\gamma_5 B)\psi - \mathrm{i}F^{\mu\nu}\psi^\dagger\gamma_\nu\alpha - i\,{}^*F^{\mu\nu}\alpha^\dagger\gamma_\nu\gamma_5\psi \end{aligned} \quad (9.88)$$

Now it is easy to construct the Noether current. One finds

$$\begin{aligned} J^\mu = &\quad \alpha^\dagger\{\mathrm{i}\gamma_\nu({}^*F^{\mu\nu}\gamma_5 + F^{\mu\nu}) + \mathrm{i}g[A,B]\gamma^\mu\gamma_5 - \not{D}(A+iB\gamma_5)\gamma^\mu\}\psi \\ &+ \psi^\dagger\{\mathrm{i}\gamma_\nu({}^*F^{\mu\nu}\gamma_5 - F^{\mu\nu}) + \mathrm{i}g[A,B]\gamma^\mu\gamma_5 + \gamma^\mu\not{D}(A-iB\gamma_5)\}\alpha \end{aligned} \quad (9.89)$$

9.3.6 Selfdual instantons and small fluctuations

As previously we are interested in background configurations which preserve part of the supersymmetry. We assume that all fields with the exception of the gauge potential vanish. This condition is preserved by a supersymmetry transformations if

$$\delta\psi = -\mathrm{i}F^{\mu\nu}\Sigma_{\mu\nu}\alpha = 0, \quad \alpha = \begin{pmatrix} \theta \\ \bar{\theta} \end{pmatrix} \quad \Longleftrightarrow \quad F^{\mu\nu}\sigma_{\mu\nu}\theta = F^{\mu\nu}\tilde{\sigma}_{\mu\nu}\bar{\theta} = 0 \tag{9.90}$$

There are two ways to fulfill these conditions:

$$\theta = 0,\ F^{\mu\nu}\tilde{\sigma}_{\mu\nu} = 0 \quad \text{or} \quad \bar{\theta} = 0,\ F^{\mu\nu}\sigma_{\mu\nu} = 0. \tag{9.91}$$

In the first case only the righthanded supersymmetry survives and $F^{\mu\nu}$ is anti-selfdual, in the second case the lefthanded supersymmetry survives and $F^{\mu\nu}$ is selfdual. As expected these selfdual and anti-selfdual gauge fields are solutions of the classical field equations which are the Yang-Mills, Klein-Gordon and Dirac equations

$$\begin{aligned} D_\nu F^{\mu\nu} &= ig[A, D^\mu A] + \mathrm{i}g[B, D^\mu B] + g[\psi^\dagger, \gamma^\mu\psi] \\ D^2 A &= ig[\psi^\dagger, \psi] - g^2[[A, B], B]] \\ D^2 B &= g[\bar{\psi}, \gamma_5\psi] - g^2[A, [A, B]] \\ \mathrm{i}\not{D}\psi &= -\mathrm{i}g[A - \mathrm{i}\gamma_5 B, \psi]. \end{aligned} \tag{9.92}$$

Similarly to the previously considered monopoles we can interprete the (anti)selfdual instantons as BPS-states which preserve only half of the supersymmetry.
Now we pick a selfdual instanton configuration $\bar{A}_\mu$ and consider its supersymmetry variation

$$\delta_\alpha A_\mu = 0, \quad \delta_\alpha A = 0, \quad \delta_\alpha B = 0, \quad \delta_\alpha\psi = -\mathrm{i}F^{\mu\nu}\Sigma_{\mu\nu}\alpha.$$

Since $\bar{A}_\mu$ is a classical solution we have

$$S[\bar{A}_\mu] = S[\bar{A}_\mu, \delta_\alpha\psi] \sim S[\bar{A}_\mu] + (\delta_\alpha\psi^\dagger, i\not{D}\,\delta_\alpha\psi)$$

which should imply that $\delta_\alpha\psi$ is a zero-mode of the Dirac operator. This is easily proven:

$$\begin{aligned} \gamma^\mu\delta_\alpha\psi &= \gamma_\nu({}^*F^{\mu\nu}\gamma_5 - F_{\mu\nu})\alpha \Longrightarrow \\ \mathrm{i}\not{D}\delta_\alpha\psi &= \mathrm{i}\gamma_\nu D_\mu({}^*F^{\mu\nu}\gamma_5 - F^{\mu\nu})\alpha = -\mathrm{i}\gamma_\nu D_\mu F^{\mu\nu} = 0, \end{aligned}$$

where we have used the Bianchi identity and the Yang-Mills equation. This proves that $\delta_\alpha\psi$ is a zeromode of the Dirac operator. Since

$$\delta_\alpha\psi = -\tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \tilde{\sigma}F \end{pmatrix}\alpha,$$

this zero mode is right-handed. Actually, from the index theorem we infer that in an $SU(2)$-instanton background with charge q the operator $i\not{D}$ possesses

$$\tfrac{2}{3}(2j+1)(j+1)j\cdot q \tag{9.93}$$

righthanded zero-modes in the spin-j representation. In particular for adjoint fermions there are $4q$ zero modes.
Now we wish to relate the various fluctuation fields about selfdual instantons [46]. We start with perturbing an arbitrary background field by small fluctuations

$$A_\mu = \bar{A}_\mu + a_\mu, \quad A = \bar{A} + a, \quad B = \bar{B} + b. \tag{9.94}$$

We perform the corresponding Taylor expansion of the (unstable) EUCLIDean action

$$S = S_0 + S_1 + S_2 + \dots,$$

where S_i is of order i in the fluctuation fields. In what follows we only work with the background fields and the perturbation and hence may safely omit the bar for the background fields. The first term S_1 leads to the field equations (9.92). The expression for S_2 for general background fields is rather complicated. But for backgrounds with vanishing A, B and ψ it becomes rather simple

$$S_2 = \int a_\nu \{ -D^2\delta_{\nu\mu} + D_\nu D_\mu - 2ig[F^{\mu\nu}, .]\} a_\mu - aD^2 a - bD^2 b + i\psi^\dagger \not{D}\psi. \tag{9.95}$$

We read off the following fluctuation operators

$$M_{\nu\mu} = -D^2\delta_{\nu\mu} + D_\nu D_\mu + 2ig\,\mathrm{ad}(F_{\nu\mu}), \quad M_A = -D^2, \quad M_B = -D^2, \quad M_\psi = i\not{D}, \tag{9.96}$$

where all covariant derivatives are to be taken in the adjoint representation.
In a one-loop approximation to the partition function one needs the eigenvalues of these fluctuation operators,

$$M_{\nu\mu} a_\mu = \lambda^2 a_\mu, \quad M_A a = \lambda^2 a, \quad M_B b = \lambda^2 b \quad \text{and} \quad M_\psi \psi = \lambda\psi. \tag{9.97}$$

Let us assume that ψ is an eigenmode of the Dirac operator,

$$i\not{D}\psi = \lambda\psi \Longrightarrow iD_\mu \psi^\dagger \gamma^\mu = -\lambda\psi^\dagger. \tag{9.98}$$

The squared Dirac operator,

$$\not{D}^2 = \gamma_\mu\gamma_\nu D_\mu D_\nu = (\delta_{\mu\nu} + \gamma_{\mu\nu}) D_\mu D_\nu = D^2 + \tfrac{1}{2}\gamma_{\mu\nu}[D_\mu, D_\nu] = D^2 + \Sigma_{\mu\nu}\mathrm{ad}(F_{\mu\nu}), \tag{9.99}$$

simplifies for selfdual gauge fields as follows:

$$(i\not{D})^2 = -D^2 + \tfrac{1}{2}\begin{pmatrix} \sigma\mathrm{ad}(F) & 0 \\ 0 & \tilde{\sigma}\mathrm{ad}(F) \end{pmatrix} \longrightarrow -D^2 + \tfrac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & \tilde{\sigma}\mathrm{ad}(F) \end{pmatrix}. \tag{9.100}$$

We see that the 'left-handed' fermions

$$\psi_L = P_L\psi, \qquad P_L = \tfrac{1}{2}(1+\gamma_5) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{9.101}$$

fulfill the equation

$$-D^2\psi_L = \lambda^2\psi_L$$

which is identical to the eigenvalue equation for the scalar and pseudoscalar field. Therefore any eigenmode ψ_L of $-\not{D}^2$ with eigenvalue λ^2 transforms under susy transformations into eigenmodes

$$a=-\alpha_L^\dagger\psi_L+\psi_L^\dagger\alpha_L \quad \text{and} \quad b=i(\alpha_L^\dagger\psi_L-\psi_L^\dagger\alpha_L) \tag{9.102}$$

of $-D^2$ with the same eigenvalue.
The same procedure applies to the eigenvalue equation of the vector fluctuations. We start with a righthanded eigenmode of $-\not{D}^2$ which fulfills

$$(-D^2+\tfrac{1}{2}\tilde{\sigma}\mathrm{ad}\,(F))\psi_R=\lambda^2\psi_R.$$

Now we multiply this equation with σ_ν from the left and use the first of the following two identities

$$\sigma_\rho\tilde{\sigma}_{\mu\nu}=\delta_{\rho\mu}\sigma_\nu-\delta_{\rho\nu}\sigma_\mu+\epsilon_{\mu\nu\rho\sigma}\sigma_\sigma \quad , \quad \tilde{\sigma}_\rho\sigma_{\mu\nu}=\delta_{\rho\mu}\tilde{\sigma}_\nu-\delta_{\rho\nu}\tilde{\sigma}_\mu-\epsilon_{\mu\nu\rho\sigma}\tilde{\sigma}_\sigma,$$

which follow from (9.85). This leads to

$$\{-D^2\delta_{\nu\mu}+i\mathrm{ad}\,(F_{\nu\mu}+{}^*F_{\nu\mu})\}(\sigma_\mu\psi_R)=\lambda^2(\sigma_\nu\psi_R). \tag{9.103}$$

For a selfdual gauge field this equation becomes

$$\begin{aligned}\{-D^2\delta_{\nu\mu}+2\mathrm{i}\,\mathrm{ad}(F_{\nu\mu})\}\,\sigma_\mu\psi_R &= \lambda^2\gamma_\nu\psi_R\\ \{-D^2\delta_{\nu\mu}+2\mathrm{i}\,\mathrm{ad}(F_{\nu\mu})\}\,\psi_R^\dagger\tilde{\sigma}_\mu &= \lambda^2\psi_R^\dagger\tilde{\sigma}_\nu\end{aligned} \tag{9.104}$$

and it implies

$$-D^2(\sigma D)\psi_R=\lambda^2(\sigma D)\psi_R. \tag{9.105}$$

Now we take the vector field which is gotten as supersymmetry transformation of a right-handed eigenmode,

$$a_\mu=\mathrm{i}(\alpha^\dagger\sigma_\mu\psi_R-\psi_R^\dagger\tilde{\sigma}_\mu\alpha). \tag{9.106}$$

For $\lambda\neq 0$ this vector field has a non-vanishing divergence,

$$D_\mu a_\mu=\lambda(\alpha_L^\dagger\psi_L+\psi_L^\dagger\alpha_L). \tag{9.107}$$

However, out of a_μ we can construct the following source free vector field

$$b_\mu=a_\mu+\frac{1}{\lambda^2}D_\mu D_\nu a_\nu$$

which satisfies the so-called *background gauge condition*

$$D_\mu b_\mu=0. \tag{9.108}$$

The difference between b_μ and a_μ is an infinitesimal gauge transformation. The proof is simple and makes use of (9.105):

$$\begin{aligned} D_\mu b_\mu &= D_\mu a_\mu + \frac{1}{\lambda^2} D^2 \{\alpha_L^\dagger \mathrm{i}(\sigma D)\psi_R - iD_\nu \psi_R^\dagger \tilde{\sigma}_\nu \alpha_L\} \\ &= D_\mu a_\mu - \{\alpha_L^\dagger \mathrm{i}(\sigma D)\psi_R - iD_\nu \psi_R^\dagger \tilde{\sigma}_\nu \alpha_L\} = D_\mu a_\mu - D_\nu a_\nu = 0. \end{aligned}$$

After this preparations we come back to the fluctuation operator $M_{\nu\mu}$ for the gauge bosons in (9.97). It is just the sum of the the operator on the left in (9.104) plus $D_\nu D_\mu$ which annihilates b_μ. Hence b_μ is eigenmode of the fluctuation operator with eigenvalues λ^2,

$$M_{\nu\mu} b_\mu = \lambda^2 b_\nu, \qquad D_\mu b_\mu = 0.$$

Hence we have already diagonalized the fluctuation operator on fluctuations fulfilling the background gauge condition.
We summarize the main results of this section: the bosonic fluctuation operators and $-\not{D}^2$ have all the same spectrum and the eigenmodes are related by supersymmetry. Now we turn to calculating the partition function in the one-loop approximation.

9.3.7 One-loop partition function

After our previous investigations it seems proper to use background gauge fixing. Hence we add a gauge fixing term $\frac{1}{2}(D_\mu a_\mu)^2$ to the LAGRANGEan, and the first eigenvalue equation in (9.97) is then modifies to

$$(M_{\nu\mu} - D_\nu D_\mu) a_\mu = \lambda^2 a_\nu.$$

We use the zeta-function regularization to 'calculate' the determinant of an selfadjoint and non-negative operator A:

$$\log\det A = -\frac{d}{ds}\zeta_A(s)|_{s=0}, \quad \zeta_A(s) = \sum_{\lambda_n > 0} \lambda_n^{-s}.$$

The so defined determinant has a simple scaling property,

$$\log\det \frac{1}{\lambda} A = \log\det A - \log\lambda \cdot \zeta_A(0).$$

Using the fact that the fluctuation operators have the same non-zero eigenvalues

$$\log\det{}'(-\not{D}^2) = 4\log\det{}'(-D^2) = \log\det{}'(M_{\nu\mu}),$$

where in the last equations we assumed the background gauge condition $D_\mu b_\mu = 0$. Now we insert our result in the general formula for the one-loop partition function in the q-instanton sector[47]

$$Z_q(V,g) = e^{W_q(V,g)} = \frac{1}{\mathcal{N}} \int \mathcal{D}(a_\mu, \psi^\dagger, \psi, A, B) e^{-S + \int(\psi^\dagger \eta + \eta^\dagger \psi)}$$

$$= \quad e^{-S(\bar A_\mu)}\left[g\sqrt{\frac{2\pi}{V}}\right]^{d_H}\frac{1}{V_H}\prod_1^p d\gamma_r(\det J)^{1/2}\frac{\det'^{1/2}(-\not{D}^2)\det' M_{gh}}{\{\det'(-D^2)\det'(-D^2)\det'(-M_{\nu\mu}\}^{1/2}}$$
$$\times\prod_n(\eta^\dagger,\psi_n)(\psi_n^\dagger,\eta)\exp\big(-\int\eta^\dagger G'\eta\big).$$

For the background gauge fixing $M_{gh}=-D^2$. Here d_H and V_H are the dimensions and volume of the stability group H which commutes with the $su(2)$-algebra defined by the instanton [48]. For $G=SU(2)$ we have $d_H=0$. The fluctuation operator $M_{\nu\mu}$ may possess p additional zero-modes arising from the variation of the collective parameters $\{\gamma_r\}$. J denotes the Jacobian when one converts p expansion parameters (in the expansion of the gauge field) into the collective parameters.
Now we consider the rescaled theory with

$$\bar A(\lambda x)=\bar A(x)$$

to the original one. Again we refer to [47] and take the general result

$$\begin{aligned}W_q[\lambda V,g] &= W_q(V,g)+\frac{\log\lambda}{16\pi^2}\int\mathrm{Tr}\,(a_2^A(x)+a_2^B(x)-2a_2^{gh}(x)-a_2^\psi(x)+a_2^{A_\mu}(x))\\ &= W_q(V,g)+\frac{\log\lambda}{16\pi^2}\int\mathrm{Tr}\,(-a_2^\psi(x)+a_2^{A_\mu}(x)),\end{aligned}$$

where the second Seeley-deWitt coefficients of the various fluctuation operators appear. These coefficients are

$$\begin{aligned}a_2^{A_\mu}(x) &= -\tfrac{5}{3}g^2\mathrm{Tr}_A F^{\mu\nu}F_{\mu\nu}\\ a_2^A(x) &= \tfrac{1}{12}g^2\mathrm{Tr}_A F^{\mu\nu}F_{\mu\nu}\\ a_2^\psi(x) &= -\tfrac{2}{3}\mathrm{Tr}_A g^2F^{\mu\nu}F_{\mu\nu},\end{aligned}\tag{9.109}$$

where Tr_A is the trace in the adjoint representation, we obtain

$$W_q[\lambda V,g] = W_q(V,\eta,g)-\frac{\log\lambda g^2}{16\pi^2}\int\mathrm{Tr}_A(F^{\mu\nu}F_{\mu\nu})=W_q(V,g)-T_A\frac{\log\lambda g^2}{16\pi^2}\int(F^a_{\mu\nu}F^{\mu\nu}_a),$$

where T_A is the second-order Casimir of the adjoint representation. This then implies the following scaling law for the effective action,

$$\Gamma_q(\lambda V,g)=\frac{1}{4g^2}\Big(1-T_A\frac{\log\lambda g^2}{4\pi^2}\Big)\int F_a^{\mu\nu}F^a_{\mu\nu}.$$

The effective action is invariant, if we rescale the field and coupling constant according to

$$A_\mu\to\sqrt{Z_3}A_\mu,\quad g^2(\lambda)=\frac{g^2}{Z_3}\quad\text{with}\quad Z_3=1-\frac{\log\lambda}{4\pi^2}T_Ag^2.$$

For example, for $G=SU(2)$ we have $T_A=2$ and the β-function and anomalous dimension of the gauge fields are found to be

$$\beta(g)=\mu\frac{\partial}{\partial\mu}g(\mu)=-\frac{1}{4\pi^2}g^3(\mu)\quad\text{and}\quad\gamma_A(g)=\mu\frac{\partial}{\partial\mu}\log Z_3=\frac{1}{2\pi^2}g^2(\mu),\qquad\mu=\frac{1}{\lambda}.$$

We see, that the theory is asymptotically free, similarly as QCD with a not too big number of fermions. This ends the first part of our discussion of the Seiberg-Witten model. To make further progress we need the superfield formulation which we shall introduce soon. The non-perturbative structure of the $\mathcal{N}=2$-SYM can much easier seen in this formalism. But before we turn to superfields we shall discuss a new technique to construct supersymmetric theories, namely the dimensional reduction.

9.4 Closing of the algebra on fermions.

We show explicitly that the superalgebra (9.82,9.83) in Euclidean spacetime close on the fermions. This calculation is rather tricky and we skipped it in the main body of the chapter. When calculating the commutator of two susy transformations on ψ one obtains three types of terms. Those which are linear in A, those which are linear in B and those containing neither the scalar nor the pseudo-scalar field,

$$[\delta_{\alpha_1},\delta_{\alpha_2}]\psi = \mathrm{i}g[A,X_A]+g[B,X_B]+R.$$

One finds

$$\begin{aligned} X_A &= \{(\alpha_1^\dagger\gamma_5\psi)-(\psi^\dagger\gamma_5\alpha_1)\}\gamma_5\alpha_2+\{(\alpha_1^\dagger\gamma_\mu\psi)-(\psi^\dagger\gamma_\mu\alpha_1)\}\gamma^\mu\alpha_2-\Big\{\alpha_1\leftrightarrow\alpha_2\Big\}\\ X_B &= \{(\alpha_1^\dagger\psi)-(\psi^\dagger\alpha_1)\}\gamma_5\alpha_2+\{(\alpha_1^\dagger\gamma_\mu\psi)-(\psi^\dagger\gamma_\mu\alpha_1)\}\gamma^\mu\gamma_5\alpha_2-\Big\{\alpha_1\leftrightarrow\alpha_2\Big\} \end{aligned}$$

and

$$\begin{aligned} R = \quad & 2\{(\alpha_1^\dagger\gamma_\nu D_\mu\psi)-(D_\mu\psi^\dagger\gamma_\nu\alpha_1)\}\Sigma_{\mu\nu}\alpha_2\\ & +\mathrm{i}\{(\alpha_1^\dagger D_\mu\psi)-(D_\mu\psi^\dagger\alpha_1)\}\gamma^\mu\alpha_2\\ & -\mathrm{i}\{(\alpha_1^\dagger\gamma_5 D_\mu\psi)-(D_\mu\psi^\dagger\gamma_5\alpha_1)\}\gamma^\mu\gamma_5\alpha_2\\ & -\Big\{\alpha_1\leftrightarrow\alpha_2\Big\}. \end{aligned}$$

Using the Fierz identity the he terms in R containing ψ (and not $\psi^\dagger$) can be rewritten as

$$\begin{aligned} &\tfrac{3\mathrm{i}}{4}(\alpha_1^\dagger\alpha_2)\not{D}\psi+\mathrm{i}(\alpha_1^\dagger\gamma^\mu\alpha_2)D_\mu\psi-\tfrac{3\mathrm{i}}{4}(\alpha_1^\dagger\gamma^\rho\alpha_2)\gamma_\rho\not{D}\psi\\ &-\tfrac{\mathrm{i}}{4}(\alpha_1^\dagger\gamma^{\rho\sigma}\alpha_2)\gamma_{\rho\sigma}\not{D}\psi-3\mathrm{i}(\alpha_1^\dagger\gamma_5\gamma^\mu\alpha_2)\gamma_5 D_\mu\psi\\ &+\tfrac{7\mathrm{i}}{4}(\alpha_1^\dagger\gamma_5\gamma^\rho\alpha_2)\gamma_5\gamma_\rho\not{D}\psi+\tfrac{3\mathrm{i}}{4}(\alpha_1^\dagger\gamma_5\alpha_2)\gamma_5\not{D}\psi. \end{aligned}$$

To be done!

Kapitel 10

$N = 4$ Super-Yang-Mills Theory

The largest supersymmetry that can be represented on a particle multiplet with spins ≤ 1 is $N = 4$. Correspondingly the $N = 4$ model [51] is called *maximally extended*. All $N = 4$ models must be constructed from gauge multiplets. This makes the particles necessarily massless and there will be no central charge.
Our interest in the $N = 4$ SYM is twofold. On the one hand, the theory is believed to be S-dual, one expects that the complete effective action, including all instanton and anti-instanton effects organize into an $SL(2, Z)$ invariant expression. On the other hand, not unrelated to the previous, we are motivated by the AdS/CFT correspondence.
On-shell the theory contains a gauge boson A_μ, 4 Majorana (or equivalently 4 Weyl) fermions which maybe grouped into two Dirac spinors, and 6 scalar fields. All fields transform according to the adjoint representation of the gauge group.

10.1 Scale invariance in 1 loop

Without knowledge of the precise action we can already calculate the $1 - loop$ β-function of this theory. Under scale transformations the Schwinger functional changes as

$$\begin{aligned} W_q[\lambda V, g] &= W_q(V, g) + \frac{\log \lambda}{16\pi^2} \int \mathrm{Tr}\,(6a_2^A(x) - 2a_2^{gh}(x) - 2a_2^\psi(x) + a_2^{A_\mu}(x)) \\ &= W_q(V, g) + \frac{\log \lambda}{16\pi^2} \int \mathrm{Tr}\,\{4a_2^A(x) - 2a_2^\psi(x) + a_2^{A_\mu}(x)\}, \end{aligned}$$

where the Seeley-deWitt coefficients have been given in [50]. We have used the field content of the theory that all fields transform under the adjoint representation. One finds, that the expression between the curly brackets vanishes. Thus the 1-loop action is scale invariant and the β-function and wave function renormalization vanishes,

$$\beta(g) = 0 \quad \text{and} \quad Z_3(g) = 0. \tag{10.1}$$

In [52] it was found that for the $N = 4$ theory the β-function remained zero up to three loops and that therefore there were no divergent graphs at all to that order. This led everyone to suspect that the theory may be a finite field theoretical model in four space-time dimensions and arguments were put forward to proof the finiteness to all orders in perturbation theory [53]. Some arguments are based on the relation between the trace anomaly in the energy-momentum tensor, the β-function and conformal invariance, other arguments used the explicit matching of bosonic and fermionic counting in the light-cone gauge and yet other arguments were based on the non-renormalisation theorem and the background gauge.

10.2 Kaluza-Klein-Reduction

The relevant models with extended supersymmetry are intimately related to Lagrangian field theories in more than four space-time dimensions. The idea of employing higher dimensions to understand the complexities of extended supersymmetry has been pioneered by J. Scherk [54]. As an exercise in getting familiar with higher dimensions, I shall show how $N = 4$ supersymmetric Yang-Mills theories fit into a world in higher dimensions. We derive this theory by a sort of Kaluza-Klein reduction of a gauge theory in higher dimensions. After compactification all but four components of the vector potential become scalar fields. Since the $N = 4$ SYM theory has 6 scalar fields and the gauge field has 4 components, this implies that we must reduce a gauge theory with a $4+6 = 10$-component gauge potential, that is a 10-dimensional gauge theory.

10.2.1 Spinors in 10 dimensions

In 10 dimensions a Dirac spinor has 32 complex components which are four times the 4×4 real components of the 4 Majorana spinors in the $N = 4$ theory in 4 dimensions. Hence we must assume that the spinors in 10 dimensions are both Weyl and Majorana. Such spinors do exists as we have discussed in chapter 3.
We start with the Dirac matrices γ^μ in 4-dimensional Minkowski space and give an explicit realization for the matrices $\Gamma^m, m = 0, \dots, 9$ in 10 dimensions: We make the ansatz

$$\Gamma_\mu = \Delta \otimes \gamma_\mu, \quad \Gamma_{3+a} = \Delta_a \otimes \gamma_5, \quad \mu = 0, 1, 2, 3, \quad a = 1, \dots, 6.$$

To see what are the condition on the Δ-factors such that

$$\{\Gamma_m, \Gamma_n\} = 2\eta_{mn}, \qquad \eta = \mathrm{diag}(1, -1, \dots, -1) \tag{10.2}$$

holds, one uses that for $[A, B] = 0$ on has

$$\{A \otimes C, B \otimes D\} = AB \otimes \{C, D\} \quad \text{and} \quad \{C \otimes A, D \otimes B\} = \{C, D\} \otimes AB. \tag{10.3}$$

and for $\{A, B\} = 0$ one has

$$\{A \otimes C, B \otimes D\} = AB \otimes [C, D] \quad \text{and} \quad \{C \otimes A, D \otimes B\} = [C, D] \otimes AB. \tag{10.4}$$

Now it is easy to see that we must demand

$$\Delta^2 = \mathbb{1}_8, \quad [\Delta, \Delta_a] = 0, \qquad \{\Delta_a, \Delta_b\} = -2\delta_{ab}\mathbb{1}_8.$$

for (10.2) to hold true. Since Γ^0 must be hermitean and the $\Gamma^{m>0}$ antihermitean and since $\gamma_5=-i\gamma_0\gamma_1\gamma_2\gamma_3$ is hermitean it also follows from

$$(A\otimes B)^\dagger = A^\dagger\otimes B^\dagger.$$

that

$$\Delta^\dagger=\Delta,\quad \Delta_a^\dagger=-\Delta_a.$$

Other useful identities which we shall sometimes need are

$$(A\otimes B)^T=A^T\otimes B^T\quad\text{and for vectors}\quad (x\otimes y,u\otimes v)=(x,u)(y,v). \tag{10.5}$$

Since Δ commutes with all matrices and squares to $\mathbb{1}_8$ we may choose it to be the identity,

$$\Delta=\mathbb{1}_8.$$

Note that the hermitean $i\Delta_a$ generate the Euclidean Clifford algebra in 6 dimensions and that the $[\Delta_a,\Delta_b]$ generate the group *spin*(6). Earlier we have shown that in 6 Euclidean dimensions there is a Majorana representation. Hence we may choose Δ_a to be real and hence antisymmetric. To construct this Majorana representation we make for the Δ_a the following ansatz

$$\Delta_i=i\tau_1\otimes\alpha_i\quad\text{and}\quad \Delta_{3+i}=i\tau_3\otimes\tilde\alpha_i,\qquad i=1,2,3,$$

so that

$$\{\alpha_i,\alpha_j\}=\{\tilde\alpha_i,\tilde\alpha_j\}=2\delta_{ij}\mathbb{1}_4,\quad [\alpha_i,\tilde\alpha_j]=0,\quad \alpha_i^\dagger=\alpha_i,\quad \tilde\alpha_i^\dagger=\tilde\alpha_i.$$

A possible solution is

$$\begin{aligned}\alpha_1&=\tau_2\otimes\tau_1,\quad \alpha_2=\tau_0\otimes\tau_2,\quad \alpha_3=\tau_2\otimes\tau_3\\ \tilde\alpha_1&=\tau_1\otimes\tau_2,\quad \tilde\alpha_2=-\tau_3\otimes\tau_2,\quad \tilde\alpha_3=\tau_2\otimes\tau_0\end{aligned}$$

These hermitean, imaginary and hence antisymmetric matrices obey

$$\alpha_i\alpha_j=\delta_{ij}+i\epsilon_{ijk}\alpha_k\quad\text{and}\quad \tilde\alpha_i\tilde\alpha_j=\delta_{ij}+i\epsilon_{ijk}\tilde\alpha_k,\quad [\alpha_i,\tilde\alpha_j]=0.$$

They lead to *real and antisymmetric* Δ_a. With our earlier convention the hermitean $\Gamma_{11}=-\Gamma_0\cdots\Gamma_9$ takes the form

$$\Gamma_{11}=\Gamma_*\otimes\gamma_5,\qquad \Gamma_*=-i\Delta_1\cdots\Delta_6,\qquad \Gamma_*^\dagger=\Gamma_*=-\Gamma_*^T,\quad \gamma_5^\dagger=\gamma_5. \tag{10.6}$$

If we would choose a Majorana representation for the 4-dimensional γ_μ then they would be purely imaginary (see section 3.5) and so would be $\gamma_5=-i\gamma_0\gamma_1\gamma_2\gamma_3$. Since the Δ_a are real it follows that the Γ_m are purely imaginary as well and that the charge conjugation matrix is $\mathcal{C}_{10}=-\Gamma_0=-\mathbb{1}_8\otimes\gamma_0$. Hence we are lead to take the following antisymmetric charge conjugation matrix

$$\mathcal{C}_{10}=\mathbb{1}_8\otimes\mathcal{C}_4 \tag{10.7}$$

where $\mathcal{C}_4 = -\gamma^0$ is the antisymmetric charge conjugation matrix in 4 dimensions. Since the Δ_a are antisymmetric it is easily seen that

$$\mathcal{C}_{10}\Gamma_m\mathcal{C}_{10}^{-1} = -\Gamma_m^T. \tag{10.8}$$

We see that the constants ϵ and η are both 1, similarly as in 4 spacetime dimensions. Let us first see how Majorana spinors look like. Since

$$\mathcal{B}_{10} = \mathcal{C}_{10}\Gamma_0^T = \mathbb{1}_8 \otimes \mathcal{C}_4\gamma_0^T = \mathbb{1}_8 \otimes \mathcal{B}_4$$

the Majorana condition reads

$$\Psi = \xi \otimes \chi = \Psi_c \Longleftrightarrow \xi \in \mathbb{R}^8,\ \chi_c = \chi, \tag{10.9}$$

and an arbitrary Majorana spinor in 10 space-time dimensions has the expansion

$$\Psi = \sum_{r=1}^{8} E_r \otimes \chi_r, \tag{10.10}$$

where the χ_r are Majorana spinors in 4 dimensions and the E_r form a (real) base in $\mathbb{R}^8$. A spinor has positive chirality if

$$\Psi = \Gamma_{11}\Psi = (\Gamma_* \otimes \gamma_5)\Psi. \tag{10.11}$$

To parameterize Weyl spinors we expand the factor ξ in $\Psi = \xi \otimes \chi$ in eigenvectors of Γ_*. First we take the 4 (necessarily complex) eigenvectors F_p of Γ_* with eigenvalue 1. For convenience we assume that the F_p are orthonormal. Since Γ_* is purely imaginary the vectors F_p^* have then eigenvalue -1 and together with the F_p form a basis. With this choice a Weyl-spinor with $\Gamma_{11} = 1$ has the expansion

$$\Psi = \sum_{p=1}^{4} (F_p \otimes P_+\chi_p + F_p^* \otimes P_-\chi_{4+p}), \tag{10.12}$$

where $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$ are the projections onto the spinors with fixed chirality. A Majorana-Weyl (MW) spinor in 10 dimensions must at the same time have the expansion (16.9). By using

$$\gamma_5\mathcal{B}_4 = -\mathcal{B}_4\gamma_5^* \quad \text{or} \quad \mathcal{B}_4 P_+^* = P_-\mathcal{B}_4, \quad \mathcal{B}_4 P_-^* = P_+\mathcal{B}_4$$

one finds that the Weyl-spinor (10.12) fulfills the Majorana conditions in 10 dimensions if $P_+\chi_p + P_-\chi_{4+p}$ is a Majorana spinor in 4 dimensions. We denote this spinor again by χ_p. This way we arrive at the following expansion for a MW-spinor

$$\Psi = \sum_{p=1}^{4} (F_p \otimes P_+\chi_p + F_p^* \otimes P_-\chi_p) \quad , \quad \bar{\Psi} = \sum_p (F_p^\dagger \otimes \bar{\chi}_p\, P_- + F_p^T \otimes \bar{\chi}_p\, P_+), \tag{10.13}$$

where the χ_p are Majorana-spinors in 4 space-time dimensions and $\bar{\chi} = \chi^\dagger \gamma_0$ denotes the Dirac conjugate of χ. Indeed, using the real basis

$$E_p = \frac{1}{\sqrt{2}}(F_p + F_p^*) \quad \text{and} \quad E_{4+p} = \frac{i}{\sqrt{2}}(F_p - F_p^*)$$

the above expansions can be rewritten as

$$\begin{aligned} \Psi &= \frac{1}{\sqrt{2}} \sum_{p=1}^{4} \{E_p \otimes \chi_p + \Gamma_* E_p \otimes \gamma_5 \chi_p\} \\ \bar{\Psi} &= \frac{1}{\sqrt{2}} \sum_{p=1}^{4} \{E_p^T \otimes \bar{\chi}_p - E_p^T \Gamma_* \otimes \bar{\chi}_p \gamma_5\}. \end{aligned} \tag{10.14}$$

Since $i\Gamma_*$ is real and $i\gamma_5\chi$ is a Majorana spinor if χ has this property, this expansion has the form (16.9), as required.

10.2.2 Reduction of Yang-Mills term

In 10 spacetime dimensions a gauge field and gauge coupling constant have the dimensions

$$[A_m] = L^{-4}, \quad [g_{10}] = L^3 \Longrightarrow [g_{10} A_m] = L^{-1}.$$

We may absorb the coupling constant in the gauge potential, $A_m \to g_{10} A_m$ such that the 10-dimensional coupling constant appears only in front of the the Yang-Mills action,

$$S_{YM} = -\frac{1}{4g_{10}^2} \int d^{10}x \operatorname{Tr} F_{mn} F^{mn}. \tag{10.15}$$

Now we perform the Kaluza-Klein reduction of the Yang-Mills action on $\mathbb{R}^4 \times T^6$. As internal space we choose the 6-dimensional torus T^6 with volume V_6. We write

$$A_m = (A_\mu, \Phi_a), \qquad m = 0, \dots, 9; \quad a = 1, \dots, 6, \tag{10.16}$$

and assume all fields are *independent of the internal coordinates* $x^4, \dots, x^9$. We find the following decomposition of the field strength,

$$F_{\mu,3+a} = \partial_\mu \Phi_a - i[A_\mu, \Phi_a] \quad , \quad F_{3+a,3+b} = -i[\Phi_a, \Phi_b]$$

Inserting this into the 10-dimensional Yang-Mills action we find

$$S_{YM} \to \frac{1}{4g^2} \int d^4x \operatorname{Tr} \left(-F_{\mu\nu} F^{\mu\nu} + 2 \sum_a D^\mu \Phi_a D_\mu \Phi_a + \sum_{ab} [\Phi_a, \Phi_b]^2 \right), \tag{10.17}$$

where we took into account, that $\Phi_a = -\Phi^a$ and where the dimensionful coupling constant g_{10} and the *dimensionless coupling constant* g are related as

$$g^2 = g_{10}^2 / V_6. \tag{10.18}$$

Now we may rescale the fields with g to obtain after the Kaluza-Klein reduction the following Lagrangian in 4-dimensional Minkowski spacetime:

$$\mathcal{L}_{YM} \to \mathcal{L}_{YMH} = \mathrm{Tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\sum_a D^\mu\Phi_a D_\mu\Phi_a + \tfrac{1}{4}g^2\sum_{ab}[\Phi_a,\Phi_b]^2\right). \quad (10.19)$$

The covariant derivative is $D_\mu = \partial_\mu - ig\ \mathrm{ad}A_\mu$. Not unexpectedly we have gotten the action for a four-dimensional Yang-Mills-Higgs theory with 6 Higgs fields in the adjoint representation.

10.2.3 Reduction of the Dirac term

In 10 and 4 dimensions a spinor field has the dimension

$$[\Psi] = L^{-9/2} \quad \text{and} \quad [\chi] = L^{-3/2},$$

respectively. We start with the general expansions (10.13,10.14) for a 10-dimensional Majorana-Weyl spinor and its adjoint. We rescale the spinors such that the χ_p in

$$\Psi = \frac{1}{\sqrt{V_6}}\sum_{p=1}^4 \left(F_p\otimes P_+\chi_p + F_p^*\otimes P_-\chi_p\right) \quad , \quad \bar\Psi = \frac{1}{\sqrt{V_6}}\sum_{p=1}^4\left(F_p^\dagger\otimes\bar\chi_p P_- + F_p^T\otimes\bar\chi_p P_+\right)$$

have the dimension of a spinorfield in 4-dimensional Minkowski spacetime. The spinor should be independent of the internal coordinates. Again we absorb the 10-dimensional gauge coupling constant in the gauge potential. We find

$$\begin{aligned}
D_\mu\Psi &= \frac{1}{\sqrt{V_6}}\sum_p\left(F_p\otimes D_\mu P_+\chi_p + F_p^*\otimes D_\mu P_-\chi_p\right), \qquad \mu=0,1,2,3\\
D_{3+a}\Psi &= -\frac{i}{\sqrt{V_6}}\sum_p\left(F_p\otimes[\Phi_a,P_+\chi_p] + F_p^*\otimes[\Phi_a,P_-\chi_p]\right), \quad a=1,\dots,6.
\end{aligned}$$

Now we may rewrite the Dirac term in 10 dimensions as follows:

$$\begin{aligned}
\int d^{10}x\,\mathrm{Tr}\,\bar\Psi\Gamma^m D_m\Psi &= \int d^4x\,\mathrm{Tr}\,\bar\chi_p \not{D}\chi_p\\
&- i\int d^4x\,\mathrm{Tr}\Big\{(\Delta_+^a)_{pq}\bar\chi_p P_+[\Phi_a,\chi_q] - (\Delta_-^a\Gamma_*)_{pq}\bar\chi_p P_-[\Phi_a,\chi_q]\Big\},
\end{aligned}$$

where one should sum over the indices on the right. We have introduced

$$(\Delta_+^a)_{pq} = (F_p^*,\Delta^a F_q) \quad \text{and} \quad (\Delta_-^a)_{pq} = (F_p,\Delta^a F_q^*) \quad \text{with} \quad \Delta_+^a = (\Delta_-^a)^*. \quad (10.20)$$

Putting the various terms together we end up with the following $N=4$ supersymmetric gauge theory in 4-dimensional Minkowski spacetime:

$$\begin{aligned}
\mathcal{L} &= \mathrm{Tr}\Big(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}D^\mu\Phi_a D_\mu\Phi_a + \tfrac{1}{4}g^2[\Phi_a,\Phi_b]^2\Big)\\
&+\tfrac{1}{2}\mathrm{Tr}\Big(i\bar\chi_p\not{D}\chi_p + g(\Delta_+^a)_{pq}\bar\chi_p P_+[\Phi_a,\chi_q] - g(\Delta_-^a)_{pq}\bar\chi_p P_-[\Phi_a,\chi_q]\Big), \quad (10.21)
\end{aligned}$$

where $a=1,\dots,6$ and $p=1,\dots,4$.

10.2.4 R-symmetry

In 10 dimensions a vector- and spinor field transform under Lorentz transformations as

$$A(x) \longrightarrow \Lambda_{10} A(\Lambda_{10}^{-1} x) \quad \text{and} \quad \Psi(x) \longrightarrow S_{10} \Psi(\Lambda_{10}^{-1} x), \quad \bar{\Psi}(x) \longrightarrow \bar{\Psi}(\Lambda_{10}^{-1} x) S_{10}^{-1},$$

where

$$S_{10} = \exp\left(\tfrac{1}{2}\omega_{mn}\Gamma^{mn}\right) \quad \text{and} \quad (\Lambda_{10})^m_{\ n} = (e^{\omega})^m_{\ n}, \qquad \Gamma_{mn} = \tfrac{1}{2}[\Gamma_m, \Gamma_n]$$

are the spinrotation and Lorentz transformation with parameter ω_{mn} in 10-dimensional spacetime. They are related via

$$S_{10}^{-1}\Gamma^m S_{10} = (\Lambda_{10})^m_{\ n}\Gamma^n.$$

The Lagrangian density is as scalar field such that the corresponding action is lorentz invariant. For example,

$$(\bar{\Psi}\Gamma^m A_m \Psi)(x) \longrightarrow (\bar{\Psi} S_{10}^{-1}\Gamma^m S_{10} (\Lambda_{10})^m_{\ n} A_n \psi)(\Lambda_{10}^{-1} x) = (\bar{\Psi}\Gamma^m A_m \Psi)(\Lambda_{10}^{-1} x).$$

When we reduce the theory to $\mathbb{R}^4$ we must require that the fields do not depend on the internal coordinates. Clearly this condition is not compatible with the 10 dimensional Lorentz invariance. Only those Lorentz transformations survive which do not mix the coordinates on $\mathbb{R}^4$ with those in the internal space, hence

$$SO(1,9) \longrightarrow SO(1,3) \times SO(6) \quad \text{or} \quad \Lambda_{10} \longrightarrow \begin{pmatrix} \Lambda & 0 \\ 0 & R \end{pmatrix},$$

where Λ is a 4-dimensional Lorentz transformation and $R \in O(6)$[1]. With our choice for the Γ_m the generators of the corresponding spin transformations in 10 dimensions read

$$\Gamma_{\mu\nu} = \mathbb{1}_8 \otimes \gamma_{\mu\nu} \quad , \quad \Gamma_{3+a,3+b} = \Delta_{ab} \otimes \mathbb{1}_4 \tag{10.22}$$

The Δ_{pq} generate the $spin(6) \sim su(4)$ subalgebra of $spin(1,9)$. Since the $\Gamma_{\mu\nu}$ act trivially on the first factor in the decomposition $\Psi = \xi \otimes \chi$ and the other generators in (10.22) act trivially on the second factor, the remaining spin rotations after the dimensional reduction decompose as follows:

$$\begin{aligned} &S_{10} = S_6 \otimes S, \qquad S_6 : \xi \longrightarrow S_6 \xi, \quad S : \chi \longrightarrow S\chi \\ &S_6^{-1}\Delta^a S_6 = R^a_{\ b}\Delta^b, \qquad S^{-1}\gamma^\mu S = \Lambda^\mu_{\ \nu}\gamma^\nu, \end{aligned}$$

where S is a spin rotation acting on spinors in 4-dimensional Minkowski spacetime. To summarize, the remaining Lorentz transformation act on the various fields as follows

$$\begin{array}{llll} S_6 : & \Phi_a(x) \longrightarrow R_a^{\ b}\Phi_b(x), & \Psi(x) \longrightarrow S_6\xi \otimes \chi, & A_\mu(x) \to A_\mu(x) \\ S : & \Phi_a(x) \longrightarrow \Phi_a(\Lambda^{-1}x), & \Psi(x) \longrightarrow \xi \otimes S\chi(\Lambda^{-1}x), & A_\mu(x) \longrightarrow \Lambda_\mu^{\ \nu} A_\nu(\Lambda^{-1}x). \end{array}$$

[1] actually, only $\det\Lambda \cdot \det R = 1$ is required.

We see that the spin rotations S_6 and the associated $O(6)$-rotations R have become *internal symmetries* which rotate the fields without changing the coordinates (on $\mathbb{R}^4$).
In the reduced 4-dimensional theory the first factor in $\Psi=\xi\otimes\chi$ is absent and we should reinterpret the transformation $\Psi\to S_6\xi\otimes\chi$ as transformation of the 4-dimensional spinor χ. To find this transformation we note that the real spin rotations S_6 commute with Γ_* such that

$$S_6F_p=U_{qp}F_q\quad\text{and}\quad S_6F_p^*=U_{qp}^*F_q^*,\qquad U,U^*\in SU(4).$$

These relations between S_6 and U or U^* are just the isomorphisms $spin(6)\sim SU(4)$. Therefore, under the R-symmetry the spinors transform as

$$P_+\chi_p\longrightarrow\sum_q U_{pq}P_+\chi_q\quad,\quad P_-\chi_p\longrightarrow\sum_q U_{pq}^*P_-\chi_q$$

and the corresponding transformations of the Dirac conjugates read

$$\bar\chi_pP_+\longrightarrow\sum_q\bar\chi_qP_+(U^T)_{qp}\quad,\quad \bar\chi_pP_-\longrightarrow\sum_q\bar\chi_qP_-U^\dagger_{qp}.$$

Note that for Majorana spinors χ_p the linear combinations

$$\sum_q\left(U_{pq}P_+\chi_q+U_{pq}^*P_-\chi_p\right)$$

are Majorana spinors as well.
Now it is not difficult to prove that the action (16.32) of the $N=4$ extended SYM-theory is invariant under R-symmetry transformations. For the terms in (16.32) containing no fermions the invariance is manifest. The Diracterm is also invariant: for example

$$\sum_p\bar\chi_pP_-\not{D}\chi_p\longrightarrow\sum_{pqr}\bar\chi_qP_-U^\dagger_{qp}\not{D}U_{pr}\chi_r=\sum_q\bar\chi_qP_-\not{D}\chi_q$$

is left invariant by $SU(4)$-rotations of the fermions with positive chirality. To show the invariance of the Yukawa terms is a bit more difficult. First we calculate

$$\begin{aligned}(\Delta^a_+)_{pq}\bar\chi_pP_+[\Phi_a,\chi_q]&\longrightarrow(U^T\Delta^a_+U)_{pq}\bar\chi_pP_+[R_a{}^b\Phi_b,\chi_q]\\(\Delta^a_-)_{pq}\bar\chi_pP_-[\Phi_a,\chi_q]&\longrightarrow(U^\dagger\Delta^a_-U^*)_{pq}\bar\chi_pP_-[R_a{}^b\Phi_b,\chi_q].\end{aligned}$$

Now we use $S_6^{-1}\Delta^aS_6=R^a{}_b\Delta^b$ and the definitions in (16.31) and these give rise to the following isomorphisms between $SO(6)$ and $SU(4)$:

$$U^T\Delta^a_+U=R^a{}_b\Delta^b_+\quad\text{and}\quad U^\dagger\Delta^a_-U^*=R^a{}_b\Delta^b_-.$$

These relations imply that the Yukawa terms are both invariant under R-transformations.

10.3 Susy transformations in $d=10$

The susy transformations in 10 dimensions are

$$\delta_\alpha A_m = i\bar\alpha\Gamma_m\Psi,\quad \delta_\alpha\Psi = iF^{mn}\Sigma_{mn}\alpha \quad\text{and}\quad \delta_\alpha\bar\Psi = -i\bar\alpha F^{mn}\Sigma_{mn}. \tag{10.23}$$

The calculation parallel those for the $N=1$ theory in 4 dimensions. In particular

$$\begin{aligned} -\tfrac14\delta_\alpha \mathrm{Tr}\,(F_{mn}F^{mn}) &= -i\mathrm{Tr}\,(F^{mn}\bar\alpha\Gamma_n D_m\Psi) \qquad (10.24)\\ \delta_\alpha \not{D}\psi &= i(D^pF^{mn})\Gamma_p\Sigma_{mn}\alpha + \Gamma^m[(\bar\alpha\Gamma_m\Psi,\Psi] \\ \delta_a \mathrm{Tr}\,(\tfrac{i}{2}\bar\Psi\not{D}\Psi) &= \tfrac12 \mathrm{Tr}\Big(F^{mn}\bar\alpha\Sigma_{mn}\Gamma_pD^p\Psi - (D^pF^{mn})\bar\Psi\Gamma_p\Sigma_{mn}\alpha + i\bar\Psi\Gamma^m[(\bar\alpha\Gamma_m\Psi),\Psi]\Big). \end{aligned}$$

Since we have $\epsilon=\eta=1$ (as we had in 4 dimensions) it follows that

$$\bar\Psi\Gamma_p\Sigma_{mn}\alpha = \bar\alpha\Sigma_{mn}\Gamma_p\Psi$$

which implies

$$\begin{aligned} \delta_a \mathrm{Tr}\,(\tfrac{i}{2}\bar\Psi\not{D}\Psi) &= -\tfrac12\partial_p \mathrm{Tr}\,(F^{mn}\bar\alpha\Sigma_{mn}\Gamma_p\Psi) + \mathrm{Tr}\,(F^{mn}\bar\alpha\Sigma_{mn}\Gamma_pD^p\Psi)\\ &\quad + \tfrac{i}{2}\mathrm{Tr}\,(\bar\Psi\Gamma^m[(\bar\alpha\Gamma_m\Psi),\Psi]). \end{aligned} \tag{10.25}$$

To continue we use

$$\begin{aligned} \Sigma_{mn}\Gamma_p &= \tfrac{i}{2}\eta_{mp}\Gamma_n - \tfrac{i}{2}\eta_{np}\Gamma_m - \tfrac{i}{2}\Gamma_{mnp}\\ \Gamma_p\Sigma_{mn} &= -\tfrac{i}{2}\eta_{mp}\Gamma_n + \tfrac{i}{2}\eta_{np}\Gamma_m - \tfrac{i}{2}\Gamma_{mnp} \end{aligned} \tag{10.26}$$

The term (10.24) and the second term in (10.25) add up to

$$\begin{aligned} \mathrm{Tr}\,F^{mn}\bar\alpha(\Sigma_{mn}\Gamma_p - \tfrac{i}{2}\eta_{mp}\Gamma_n + \tfrac{i}{2}\eta_{np}\Gamma_m)D^p\Psi &= -\tfrac{i}{2}\mathrm{Tr}\,(F^{mn}\bar\alpha\Gamma_{mnp}D^p\Psi)\\ &= -\tfrac{i}{2}\partial^p \mathrm{Tr}\,(F^{mn}\bar\alpha\Gamma_{mnp}\Psi), \end{aligned} \tag{10.27}$$

where we have used the Bianchi-identity in the last step. To prove that the 10-dimensional action is invariant we need to show that the last term in (10.25) vanishes. For that we expand the Majorana field Ψ in terms of an orthonormal base T_a of the Lie algebra and find

$$\tfrac{i}{2}\mathrm{Tr}\,(\bar\Psi\Gamma^m[(\bar\alpha\Gamma_m\Psi),\Psi]) = \tfrac12 f_{abc}(\bar\Psi^a\Gamma^m\Psi^b)(\bar\alpha\Gamma_m\Psi^c) \tag{10.28}$$

To proceed we need some Fierz relations in 10 dimensions: The relation (4.83) becomes

$$\begin{aligned} 32\Psi\bar\chi &= -\bar\chi\psi - \Gamma_m(\bar\chi\Gamma^m\Psi) + \tfrac{1}{2!}\Gamma_{mn}(\bar\chi\Gamma^{mn}\Psi) + \tfrac{1}{3!}\Gamma_{mnp}(\bar\chi\Gamma^{mnp}\Psi) - \tfrac{1}{4!}\Gamma_{mnpq}(\bar\chi\Gamma^{mnpq}\Psi)\\ &\quad - \tfrac{1}{5!}\Gamma_{mnpqr}(\bar\chi\Gamma^{mnpqr}\Psi) - \tfrac{1}{4!}\Gamma_{11}\Gamma_{mnpq}(\bar\chi\Gamma_{11}\Gamma^{mnpq}\Psi) - \tfrac{1}{3!}\Gamma_{11}\Gamma_{mnp}(\bar\chi\Gamma_{11}\Gamma^{mnp}\Psi)\\ &\quad + \tfrac{1}{2!}\Gamma_{11}\Gamma_{mn}(\bar\chi\Gamma_{11}\Gamma^{mn}\Psi) + \Gamma_{11}\Gamma_m(\bar\chi\Gamma_{11}\Gamma^m\Psi) - \Gamma_{11}(\bar\chi\Gamma_{11}\Psi) \end{aligned}$$

We used (4.11) and

$$\epsilon_{m_1\dots m_a n_1\dots n_b}\epsilon^{m_1\dots m_a p_1\dots p_b} = -a!\,b!\,\delta^{[p_1}_{[n_1}\cdots\delta^{p_b]}_{n_b]}.$$

Next we use the identities

$$\begin{array}{l}\Gamma_s\Gamma_m\Gamma^s=(2-d)\Gamma_m \quad , \quad \Gamma_s\Gamma_{mn}\Gamma^s=(d-4)\Gamma_{mn} \quad , \quad \Gamma_s\Gamma_{mnp}\Gamma^s=(6-d)\Gamma_{mnp}\\ \Gamma_s\Gamma_{mnpq}\Gamma^s=(d-8)\Gamma_{mnpq} \quad , \quad \Gamma_s\Gamma_{mnpqr}\Gamma^s=(10-d)\Gamma_{mnpqr}\end{array}$$

and

$$\begin{array}{l}\Gamma_s\Gamma_{11}\Gamma_m\Gamma^s=-(2-d)\Gamma_{11}\Gamma_m \quad , \quad \Gamma_s\Gamma_{11}\Gamma_{mn}\Gamma^s=-(d-4)\Gamma_{11}\Gamma_{mn}\\ \Gamma_s\Gamma_{11}\Gamma_{mnp}\Gamma^s=-(6-d)\Gamma_{11}\Gamma_{mnp} \quad , \quad \Gamma_s\Gamma_{11}\Gamma_{mnpq}\Gamma^s=-(d-8)\Gamma_{11}\Gamma_{mnpq}\end{array}$$

Taking into account that for Majorana spinors

$$(\bar\Psi^a\Gamma^{(n)}\Psi^c)=(\bar\Psi^c\Gamma^{(n)}\Psi^a) \quad \text{for} \quad n=0,3,4,7,8$$

and for Weyl spinors

$$(\bar\Psi^a\Gamma^{(n)}\Psi^c)=0 \quad \text{for} \quad n=0,2,4,6,8,10$$

we may rewrite (twice) the term on the right hand side of (10.28) as

$$\begin{aligned} f_{abc}(\bar\Psi^a\Gamma^m\Psi^b)(\bar\alpha\Gamma_m\Psi^c) &= \tfrac{1}{32}f_{abc}\Big(8(\bar\alpha\Gamma_m\Psi^b)(\bar\Psi^a\Gamma^m\Psi^c)+8(\bar\alpha\Gamma_{11}\Gamma_m\Psi^b)(\bar\Psi^a\Gamma_{11}\Gamma^m\Psi^c)\Big)\\ &= \tfrac12 f_{abc}(\bar\alpha\Gamma_m\Psi^b)(\bar\Psi^a\Gamma^m\Psi^c)=-\tfrac12 f_{abc}(\bar\Psi^a\Gamma^m\Psi^b)(\bar\alpha\Gamma_m\Psi^c)\end{aligned}$$

which proves that the left hand side vanishes. Hence we end up with

$$\delta_\alpha\mathcal{L}=\bar\alpha\partial_m V^m, \qquad V^m=-\tfrac{i}{2}\mathrm{Tr}\,(F^{mn}\Gamma_n\Psi)-\tfrac{i}{4}\mathrm{Tr}\,(F_{pq}\Gamma^{pqm}\Psi) \tag{10.29}$$

Since

$$\frac{\partial\mathcal{L}}{\partial(\partial_m A_{an})}\delta_\alpha A_{an}+\frac{\partial\mathcal{L}}{\partial(\partial_m\Psi_a)}\delta_\alpha\Psi_a=-\tfrac{3i}{2}\mathrm{Tr}\,(F^{mn}\bar\alpha\Gamma_n\Psi)+\tfrac{i}{4}\mathrm{Tr}\,(F_{pq}\bar\alpha\Gamma^{pqm}\Psi)$$

the Noether current takes the simple form

$$J^m=-i\mathrm{Tr}\,(F^{mn}\Gamma_n\Psi)+\tfrac{i}{2}\mathrm{Tr}\,(F_{pq}\Gamma^{pqm}\Psi) \tag{10.30}$$

10.4 Susy transformation of reduced theory

After we have gotten the supersymmetry transformations for the $N=1$-model in 10 dimensions we are now in the position to derive the symmetry transformations of the $N=4$ extended SYM-theory in 4 dimensions. To that goal we insert the expansion (10.13) for a Majorana-Weyl spinor into the supersymmetry transformations (10.23)

$$\begin{aligned}\delta_\alpha A_\mu &= i\bar\alpha(\mathbb{1}_8\otimes\gamma_\mu)\Psi \quad , \quad \delta_\alpha\Phi_a=i\bar\alpha(\Delta_a\otimes\gamma_5)\Psi\\ \delta_\alpha\Psi &= \Big(iF_{\mu\nu}\mathbb{1}_8\otimes\Sigma^{\mu\nu}+D_\mu\Phi_a(\Delta^a\otimes\gamma^\mu\gamma_5)+\frac{g_{10}}{2i}[\Phi_a,\Phi_b]\Delta^{ab}\otimes\mathbb{1}_4\Big)\alpha\end{aligned}$$

as well as the corresponding expansion for the supersymmetry parameter α:

$$\alpha = \sum_{p=1}^{4} (F_p \otimes P_+\theta_p + F_p^* \otimes P_-\theta_p\} \quad , \quad \bar\alpha = \sum_{p=1}^{4} \{F_p^\dagger \otimes \bar\theta_p P_- + F_p^T \otimes \bar\theta_p P_+\}.$$

Using (10.13) we find for the variations of the vector potential and scalar fields in 4 dimensions

$$\delta_\alpha A_\mu = i\sum_{p=1}^{4} \bar\theta_p \gamma_\mu \chi_p \quad \text{and} \quad \delta_\alpha \Phi_a = i\sum_p (\Delta_a)_{pq} \bar\theta_p \gamma_5 \chi_q + i\sum_{pq} (\Delta_a \Gamma_*)_{pq} \bar\theta_p \chi_q, \tag{10.31}$$

where we did not make the rescaling by $\sqrt{V_6}$ explicit. To get the susy variation of the 4 Majorana spinors is a bit trickier. We insert into the above formula for

$$\delta_\alpha \Psi = \sum_p (F_p \otimes P_+ \delta_\alpha \chi_p + F_p^* \otimes P_- \delta_\alpha \chi_p)$$

the expansion for α and compare coefficients. After one introduces

$$(\Delta_+^{ab})_{pq} \equiv (F_p, \Delta^{ab} F_q) \quad \text{and} \quad (\Delta_-^{ab})_{pq} \equiv (F_p^*, \Delta^{ab} F_q^*) \quad \text{with} \quad \Delta_+^{ab} = (\Delta_-^{ab})^*,$$

the variations of the 4-dimensional Majorana spinors can be written as

$$\begin{aligned} \delta\chi_p &= iF_{\mu\nu}\Sigma^{\mu\nu}\theta_p + \not{D}\Phi_a \sum_q \Big((\Delta_+^a)_{pq} P_+\theta_q - (\Delta_-^a)_{pq} P_-\theta_q\Big) \\ &\quad + \frac{g}{2i}\,[\Phi_a, \Phi_b] \sum_q \Big((\Delta_+^{ab})_{pq} P_+\theta_q + (\Delta_-^{ab})_{pq} P_-\theta_q\Big). \end{aligned} \tag{10.32}$$

The Noether current for the extended YM theory can be gotten from the current (10.30) by dimensional reduction. The 4 Noether currents are linear in the χ_p, the field strength and its dual but also linear in the covariant derivatives of the scalar fields Φ_a. Their explicit form reads

$$\begin{aligned} J_p^\mu &= -\mathrm{Tr}\,({}^*F^{\mu\nu}\gamma_5 + iF^{\mu\nu})\gamma_\nu\chi_p \\ &\quad -i\mathrm{Tr}\Big\{D_\alpha\Phi_a\Big((\Delta_+^a)_{pq} P_+\gamma^\alpha\gamma^\mu\chi_q - (\Delta_-^b)_{pq} P_-\gamma^\alpha\gamma^\mu\chi_q\Big)\Big\} \\ &\quad -\tfrac{1}{2}\mathrm{Tr}\Big\{\Phi_a, \Phi_b]\Big((\Delta_+^{ab})_{pq} P_-\gamma^\mu\chi_q + (\Delta_-^{ab})_{pq} P_+\gamma^\mu\chi_q\Big)\Big\}. \end{aligned} \tag{10.33}$$

From these 4 currents one can get the 4 supercharges of the $N = 4$ supersymmetric Yang-Mills theory. They fulfill the anticommutation relations (7.101) with $N = 4$.

At the beginning of the section we have argued that $N = 4$ super-YM is scale invariant since the β-function vanishes to all orders in perturbation theory. Thus the theory should be conformally invariant and the supersymmetry algebra maybe extended to the $N = 4$ superconformal algebra. This enlarged symmetry leads to stringent conditions for the spectrum of the theory. We would need to discuss the representations of the superconformal algebras to understand the spectrum of $N = 4$ super-YM. When one tries to argue in favor of the AdS-CFT correspondence one needs this spectrum. Unfortunately, at this point I must refer to the literature, since time is running out and there are other important topics we must discuss. In the next chapter we turn to the superspace formalism.

Kapitel 11

Superspaces

There are great advantages to constructing supersymmetric field theories in the superspace/superfield formalism, just as there are great advantages to constructing relativistic quantum field theories in a manifestly Lorentz covariant formalism. The following detour into superspace and superfield constructions pay off nicely when one constructs supersymmetric actions.

11.1 Minkowski spacetime as coset space

Let g be an arbitrary element of a group G which contains a subgroup H. We define equivalence classes in G: two elements g and g' are considered equivalent if they can be connected by a right multiplication with an element $h \in H$:

$$g' = g \circ h \quad \text{or} \quad g^{-1} \circ g' \in H. \tag{11.1}$$

This equivalence class is called the *coset* of g with respect to H. The set of all cosets is a manifold denoted by G/H. This way one obtains a fiber bundle with total space G, base manifold G/H and typical fiber H. A section $L(a)$, labeled by $\dim(G/H)$ parameters, parameterize the manifold if each coset contains exactly one of the L's. Once we have chosen a (local) section $L(a)$, each group element g can be uniquely decomposed into a product

$$g = L(a) \circ h.$$

A product of g with another group element, and in particular with $L(a)$, will define another L and h according to

$$g \circ L(a) = L(a') \circ h \quad \text{or} \quad g = L(a') \circ h \circ L^{-1}(a), \tag{11.2}$$

where a' and h are in general functions of both g and a. In particular, every Poincaré transformation can be uniquely decomposed as product of a translation and a Lorentz transformation,

$$(\Lambda, a) = (\mathbb{1}, a) \circ (\Lambda, 0)$$

and the Minkowski spacetime can be considered as coset manifold

$$\text{Minkowski spacetime} = \text{Poincaré group / Lorentz group.}$$

Choosing $g = (\Lambda, b)$ in (11.2) we find

$$(\Lambda, a) = (\mathbb{1}, \Lambda b + a) \circ (\Lambda, 0) \circ (\mathbb{1}, -b) \quad \text{such that} \quad a' = a + \Lambda b \quad \text{and} \quad h = \Lambda \tag{11.3}$$

in that equation. In section (5.3) we introduced the *representation*[1]

$$(\Lambda, a) \longrightarrow U(\Lambda, a) = \exp\left(-ia\hat{P} - \tfrac{i}{2}\omega\hat{M}\right)$$

of the Poincaré group. We recall the Poincaré algebra

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}) \\ [P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{11.4}$$

Since $U(a)$ translates the argument of the quantum field,

$$U(a)\Phi(x)U^{-1}(a) = \Phi(x+a), \qquad U(a) = e^{-iaP},$$

we define

$$\Phi(x) = U(x)\Phi(0)U^{-1}(x). \tag{11.5}$$

From the relation (11.3) and representation property it follows that

$$U(a)U(x) = U(a+x) \quad \text{and} \quad U(\omega)U(x)U^{-1}(\omega) = U(e^\omega x), \quad U(\Lambda) \equiv U(\omega) = e^{-i\omega M/2} \tag{11.6}$$

These composition rules for the unitary operators $U(a)$ and $U(\omega)$ also follow directly from the Poincaré algebra (11.4) which in particular implies

$$U(\omega)P_\mu U^{-1}(\omega) = (e^{-\omega})_\mu{}^\nu P_\nu.$$

From these multiplication rules we may now easily extract the transformation properties of the quantum field (11.5) as follows:

$$\begin{aligned} U(a)\Phi(x)U^{-1}(a) &= U(a+x)\Phi(0)U^{-1}(a+x) = \Phi(a+x) \\ U(\omega)\Phi(x)U^{-1}(\omega) &= U(e^\omega x)U(\omega)\Phi(0)U^{-1}(\omega)U^{-1}(e^\omega x) = e^{i\omega\Sigma/2}\Phi(e^\omega x) \end{aligned}$$

where we assumed that

$$U(\omega)\Phi(0)U^{-1}(\omega) = e^{i\omega\Sigma/2}\Phi(0). \tag{11.7}$$

[1] compared to the previous sections we change the signs of P_μ and $M_{\mu\nu}$. The (anti)commutators containing these generators must be changed accordingly

The $\Sigma_{\mu\nu}$ form some matrix representation of the algebra of the $M_{\mu\nu}$. For example, for Dirac spinors

$$\Sigma_{\mu\nu} = \tfrac{i}{2}\gamma_{\mu\nu}, \qquad S(e^{-i\omega\Sigma/2}) = e^{\omega}: \quad e^{i\omega\Sigma/2}\gamma^\rho e^{-i\omega\Sigma/2} = (e^\omega)^\rho{}_\sigma \gamma^\sigma.$$

The action of the Poincaré group on the quantum field is uniquely fixed once we knew how the Lorentz group acted on $\Phi(0)$, The Poincaré/Lorentz coset is not the most general example of a coset space. The translations form an invariant subgroup of the Poincaré group and it follows that the element h in (11.2) is independent of a. The infinitesimal version of this transformation rule reads

$$\delta_a\Phi = -i[aP,\Phi] \equiv ia^\mu r(P_\mu)\Phi, \quad \delta_\omega\Phi = -\tfrac{i}{2}[\omega M,\Phi] \equiv \tfrac{i}{2}\omega^{\mu\nu} r(M_{\mu\nu})\Phi$$

where we have defined the representations $r(P)$ and $r(M)$ of the infinitesimal translations and Lorentz boosts. One finds

$$r(P_\mu) = -i\partial_\mu \quad \text{and} \quad r(M_{\mu\nu}) = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}.$$

11.2 Superspace

Infinitesimal supersymmetry transformations are generated by the supercharges. Indeed in section 5 we have introduced the supercharges by the requirement that

$$\delta_\alpha\Phi = i[\bar\alpha Q,\Phi]$$

where Φ is any field of the theory. The $\bar\alpha Q$ fulfill commutation relations, contrary to the supercharges Q_α which satisfy anti-commutation relations. The spinor parameters α^α anti-commute with everything fermionic (including themselves) and commute with everything bosonic (including, of course, ordinary c-numbers. Hence with these parameters the supersymmetry algebra can be integrated to a group G, the *super-Poincaré group*, with typical group elements

$$U(a,\alpha,\omega) = \exp\big(-i(a,P) + i\bar\alpha Q - \tfrac{i}{2}(\omega,M)\big) \tag{11.8}$$

where we used the conventions

$$(\Lambda,a) = \exp\Big\{\tfrac{i}{2}\omega^{\mu\nu}M_{\mu\nu} + ia^\mu P_\mu\Big\} \equiv \exp\Big\{\tfrac{i}{2}(\omega,M) + i(a,P)\Big\}.$$

Superspace is the coset space

$$\text{Superspace} = \text{super-Poincaré group / Lorentz group.}$$

The most commonly way to parameterize this 'manifold' is

$$U(a,\alpha) \equiv U(a,\alpha,\Lambda=\mathbb{1}) = \exp\left(-iaP + i\bar\alpha Q\right).$$

We rewrite this in terms of Weyl spinors. To calculate the product of group elements we need the Baker-Campbell-Hausdorff formula

$$e^A e^B = \exp\Big(\sum_{n=1}^{\infty}\frac{1}{n!}C_n(A,B)\Big) \tag{11.9}$$

where the C_n are multi-commutators of A and B:

$$C_1 = A+B,\quad C_2 = [A,B],\quad C_3 = \tfrac{1}{2}[[A,B],B] + \tfrac{1}{2}[A,[A,B]]\dots \tag{11.10}$$

We also need the various commutators of the super Poincaré algebra. These are the commutators of the Poincaré algebra plus the supersymmetry algebra

$$\{\bar\beta Q, \bar\alpha\bar Q\} = 2(\bar\beta\gamma^\mu\alpha)P_\mu.$$

With

$$[-bP+\bar\beta Q, -aP+\bar\alpha Q] = 2(\bar\beta\gamma^\mu\alpha)P_\mu$$

we obtain the following composition rule for two sections in the supergroup

$$U(b^\mu,\beta)U(a^\mu,\alpha) = U(b^\mu + a^\mu - i\bar\beta\gamma^\mu\alpha, \beta+\alpha) \tag{11.11}$$

Let us calculate the conjugation of $U(a,\theta,\bar\theta)$ with a Lorentz transformation,

$$U(\omega)U(a,\alpha)U^{-1}(\omega) = \exp\Big(iU(\omega)(-aP+\alpha\bar Q)U^{-1}(\omega)\Big),\qquad U(\omega) = e^{-i\omega\Sigma/2}. \tag{11.12}$$

The conjugation of P_μ with the Lorentz boosts we have calculated above. To calculate the conjugation of the supercharges we use

$$[Q, M_{\mu\nu}] = \Sigma_{\mu\nu},$$

From the commutators we read find that

$$U(\omega)QU^{-1}(\omega) = e^{i\omega\Sigma/2}\,Q \quad\text{and}\quad U(\omega)\bar QU^{-1}(\omega) = \bar Q\,e^{-i\omega\Sigma/2}.$$

The transformations of the operators P, Q can be absorbed by the inverse transformations of the parameters (a,α) in (11.12). This way we find for the conjugation of $U(a,\alpha)$ with an arbitrary element of the Lorentz group

$$\exp\big(\tfrac{i}{2}\omega M\big)U(a,\alpha) = U(a',\alpha')\exp\big(\tfrac{i}{2}\omega M\big) \tag{11.13}$$

with transformed coordinates

$$a'^\mu = (e^\omega)^\mu{}_\nu a^\nu,\quad \alpha' = e^{-i\omega\Sigma/2}\alpha. \tag{11.14}$$

11.3 Representations on superfields

Similarly to $\Phi(x) = U(x)\Phi(0)U^{-1}(x)$ the action of the supersymmetry transformations on a superfield is defined via

$$\Phi(x,\alpha) = U(x,\alpha)\Phi(0,0)U^{-1}(x,\alpha). \tag{11.15}$$

For any group element the action on Φ is given by the coordinate transformation (11.13,11.14) in conjunction with

$$U(\omega)\Phi(0,0)U^{-1}(\omega) = \exp\left(\tfrac{i}{2}\omega\Sigma\right)\Phi(0,0).$$

More explicitly, from (11.12) and (11.13,11.7) we obtain

$$\begin{aligned} U(b,\beta)\Phi(x,\alpha)U^{-1}(b,\beta) &= \Phi(b+x-i\bar\beta\gamma^\mu\alpha, \alpha+\beta) \\ U(\omega)\Phi(x,\alpha)U^{-1}(\omega) &= e^{i\omega\Sigma}\Phi(e^\omega x, e^{-i\omega\Sigma/2}\alpha), \end{aligned} \tag{11.16}$$

the infinitesimal versions of which read

$$\delta\Phi = ib^\mu r(P_\mu)\Phi + i\bar\beta^\alpha r(Q_\alpha)\Phi \quad , \quad \delta\Phi = \tfrac{i}{2}\omega^{\mu\nu} r(M_{\mu\nu})\Phi \tag{11.17}$$

with

$$\begin{aligned} r(P_\mu) &= -i\partial_\mu \quad , \quad r(Q_\alpha) = -i\partial_\alpha - \gamma^\mu\alpha\partial_\mu \\ r(M_{\mu\nu}) &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu} - (\Sigma_{\mu\nu})^\alpha{}_\beta\alpha^\beta\partial_\alpha. \end{aligned} \tag{11.18}$$

Using the anticommuting properties of the α^α and

$$\partial_\alpha\alpha^\beta \equiv \frac{\partial}{\partial\alpha^\alpha}\alpha^\beta = \delta^\beta_\alpha$$

one can prove that the last term in (11.18) fulfills the Lorentz algebra. We rewrite the most relevant relations in terms of Weyl spinors. For that we recall our conventions for spinors and Dirac conjugate spinors:

$$\bar\alpha = (\theta^\alpha, \bar\theta_{\dot\alpha}), \quad Q = \begin{pmatrix} Q_\alpha \\ \bar Q^{\dot\alpha} \end{pmatrix}, \quad \text{such that} \quad \bar\alpha Q = \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha} = \theta Q + \bar\theta\bar Q.$$

The representation of the supercharge reads as follows in the Weyl basis:

$$r(Q_\alpha) = -i\partial_\alpha - (\sigma^\mu\bar\theta)_\alpha\partial_\mu \quad , \quad r(\bar Q^{\dot\alpha}) = -i\bar\partial^{\dot\alpha} - (\tilde\sigma^\mu\theta)^{\dot\alpha}\partial_\mu \tag{11.19}$$

for the representation of the generators as differential operators. We have introduced

$$\partial_\alpha = \frac{\partial}{\partial\theta^\alpha}, \quad \bar\partial^{\dot\alpha} = \frac{\partial}{\partial\bar\theta_{\dot\alpha}}, \quad \text{such that} \quad \partial_\alpha\theta^\beta = \delta^\beta_\alpha, \quad \bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}_{\dot\beta} \tag{11.20}$$

hold true. Because of the anticommuting property of the θ-parameters we must always anticommute θ or $\bar\theta$ to immediately behind the differentiation operator. We also insist on

$$\partial^\alpha = \frac{\partial}{\partial\theta_\alpha}, \quad \bar\partial_{\dot\alpha} = \frac{\partial}{\partial\bar\theta^{\dot\alpha}} \quad \text{to satisfy} \quad \partial^\alpha\theta_\beta = \delta^\alpha_\beta, \quad \bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta^{\dot\beta}_{\dot\alpha}, \tag{11.21}$$

which, for example, implies

$$\partial^\alpha\theta_\beta=\delta^\alpha_\beta=\varepsilon^{\alpha\sigma}\varepsilon_{\sigma\beta}=\epsilon^{\alpha\sigma}\epsilon_{\delta\beta}\delta^\delta_\sigma=-\epsilon^{\alpha\sigma}\epsilon_{\beta\delta}\partial_\sigma\theta^\delta=-\epsilon^{\alpha\sigma}\partial_\sigma\theta_\beta,$$

from which follows, that

$$\partial^\alpha=-\epsilon^{\alpha\sigma}\partial_\sigma. \tag{11.22}$$

We collect the basic notation and properties of N=1 superspace derivatives.

$$\begin{array}{rclllll}\partial^\alpha &=& -\varepsilon^{\alpha\beta}\partial_\beta, & \bar\partial^{\dot\alpha}=-\varepsilon^{\dot\alpha\dot\beta}\bar\partial_{\dot\beta}, & \partial_\alpha=-\varepsilon_{\alpha\beta}\partial^\beta, & \bar\partial_{\dot\alpha}=-\varepsilon_{\dot\alpha\dot\beta}\bar\partial^{\dot\beta} \\ \partial^\alpha\theta^\beta &=& -\varepsilon^{\alpha\beta}, & \partial_\alpha\theta_\beta=-\varepsilon_{\alpha\beta}, & \bar\partial^{\dot\alpha}\bar\theta^{\dot\beta}=-\varepsilon^{\dot\alpha\dot\beta}, & \bar\partial_{\dot\alpha}\bar\theta_{\dot\beta}=-\varepsilon_{\dot\alpha\dot\beta} \\ \partial_\alpha(\theta\theta) &=& 2\theta_\alpha, & \bar\partial_{\dot\alpha}(\bar\theta\bar\theta)=-2\bar\theta_{\dot\alpha}, & \partial^2(\theta\theta)=4 & \bar\partial^2(\bar\theta\bar\theta)=4\end{array} \tag{11.23}$$

We can rewrite (11.19) as follows: From

$$\sigma_2\sigma_\mu\sigma_2=\tilde\sigma_\mu^T \quad\text{or}\quad \sigma^\mu_{\alpha\dot\alpha}=\varepsilon_{\dot\alpha\dot\beta}\varepsilon_{\alpha\beta}\tilde\sigma^{\mu\dot\beta\beta}$$

it follows that

$$r(\bar Q_{\dot\alpha})\equiv\varepsilon_{\dot\alpha\dot\beta}\big(-i\bar\partial^{\dot\beta}-\tilde\sigma^{\mu\dot\beta\sigma}\varepsilon_{\sigma\rho}\theta^\rho\partial_\mu\big)=i\bar\partial_{\dot\alpha}+(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu.$$

Hence we arrive at the following equivalent form for the supercharges,

$$r(Q_\alpha)=-i\partial_\alpha-(\sigma^\mu\bar\theta)_\alpha\partial_\mu \quad,\quad r(\bar Q_{\dot\alpha})=i\bar\partial_{\dot\alpha}+(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu, \tag{11.24}$$

and these representations are mostly used in the literature. It is not difficult to see that

$$\{r(Q_\alpha),r(\bar Q_{\dot\alpha})\}=-2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu=2\sigma^\mu_{\alpha\dot\alpha}r(P_\mu)$$

holds true, as required.

11.3.1 Component fields

The general scalar superfield $\Phi(x,\alpha)\equiv\Phi(x,\theta,\bar\theta)$ is just a scalar function in $N=1$ rigid superspace. Its Taylor expansion in powers of θ and $\bar\theta$ is finite owing to the anticommuting property of these expansion parameters. For example, $\theta^\alpha(\theta\theta)$ vanishes. The coefficients in the expansion are local fields over Minkowski space. The following is already the most general superfield

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= C(x)+\theta\phi(x)+\bar\theta\bar\chi(x)+\theta^2M(x)+\bar\theta^{\,2}N(x)+(\theta\sigma^\mu\bar\theta)\,V_\mu(x)\\ &\quad+\theta^2\,\bar\theta\bar\lambda(x)+\bar\theta^{\,2}\,\theta\psi(x)+\theta^2\bar\theta^{\,2}\,D(x).\end{aligned} \tag{11.25}$$

For example, we have used that $\bar\theta\tilde\sigma^\mu\theta V_\mu=-\theta\sigma^\mu\bar\theta V_\mu$ is not an independent term. This superfield contains as coefficient fields:

4 complex (pseudo) scalar fields	C,M,N,D
4 spinor fields	$\phi,\psi\in(\frac12,0),\quad(\chi,\lambda)\in(0,\frac12)$
1 Lorentz 4-vector field	V_μ

Altogether there are 16 (bosonic) + 16 (fermionic) field components.
The transformation laws for the components of a general $N = 1$ superfield under supersymmetry transformation are calculated by comparing coefficients in the expansion

$$\begin{aligned}\delta\Phi(x,\theta,\bar\theta) &= \delta C(x) + \theta\delta\phi(x) + \bar\theta\delta\bar\chi(x) + \theta^2\delta M(x) + \bar\theta^2\delta N(x) + (\theta\sigma^\mu\bar\theta)\,\delta V_\mu(x)\\ &\quad +\theta^2\,\bar\theta\delta\bar\lambda(x) + \bar\theta^2\,\theta\delta\psi(x) + \theta^2\bar\theta^2\,\delta D(x).\end{aligned} \tag{11.26}$$

with

$$\begin{aligned}&\delta\Phi = i\zeta^\alpha r(Q_\alpha)\Phi + i\bar\zeta_{\dot\alpha} r(\bar Q^{\dot\alpha})\Phi \equiv \delta_\zeta\Phi + \delta_{\bar\zeta}\Phi\\ &\delta_\zeta = \zeta^\alpha\partial_\alpha - i(\zeta\sigma^\mu\bar\theta)\partial_\mu \quad , \quad \delta_{\bar\zeta} = \bar\zeta_{\dot\alpha}\bar\partial^{\dot\alpha} - i(\bar\zeta\tilde\sigma^\mu\theta)\partial_\mu.\end{aligned} \tag{11.27}$$

Using the Fierz relations (4.88) one arrives after some lengthy but straightforward calculations at the following formulae for the supersymmetry transformations of the superfield:

$$\begin{aligned}\delta_\zeta\Phi &= \zeta\phi + 2\theta\zeta M + \bar\theta(i\tilde\sigma^\mu\partial_\mu C - \tilde\sigma^\mu V_\mu)\zeta + \bar\theta^2(\zeta\psi + \tfrac{i}{2}\zeta\sigma^\mu\partial_\mu\bar\chi)\\ &\quad +(\theta\sigma^\mu\bar\theta)(\zeta\sigma_\mu\bar\lambda + \tfrac{i}{2}\partial_\nu\phi\sigma_\mu\tilde\sigma^\nu\zeta)\\ &\quad +i\theta^2\bar\theta\tilde\sigma^\mu\zeta\partial_\mu M + \bar\theta^2\theta(2\zeta D - \tfrac{i}{2}\sigma^\nu\tilde\sigma^\mu\zeta\partial_\mu V_\nu) + \tfrac{i}{2}\theta^2\bar\theta^2\zeta\sigma^\mu\partial_\mu\bar\lambda \qquad (11.28)\\ \delta_{\bar\zeta}\Phi &= \bar\zeta\bar\chi + 2\bar\theta\bar\zeta N + \theta(i\sigma^\mu\partial_\mu C + \sigma^\mu V_\mu)\bar\zeta + \theta^2(\bar\zeta\bar\lambda + \tfrac{i}{2}\bar\zeta\tilde\sigma^\mu\partial_\mu\phi)\\ &\quad +(\theta\sigma^\mu\bar\theta)(\psi\tilde\sigma_\mu\bar\zeta - \tfrac{i}{2}\partial_\nu\bar\chi\tilde\sigma_\mu\sigma^\nu\bar\zeta)\\ &\quad +i\bar\theta^2\theta\sigma^\mu\bar\zeta\partial_\mu N + \theta^2\bar\theta(2\bar\zeta D + \tfrac{i}{2}\tilde\sigma^\nu\sigma^\mu\bar\zeta\partial_\mu V_\nu + \tfrac{i}{2}\theta^2\bar\theta^2\bar\zeta\tilde\sigma^\mu\partial_\mu\psi\end{aligned}$$

By comparing the two expressions (11.27) and (11.28) we obtain the following transformation rules for the component fields

$$\begin{aligned}\delta C &= \zeta\phi + \bar\chi\bar\zeta && (11.29)\\ \delta\phi &= 2\zeta M + (i\sigma^\mu\partial_\mu C + \sigma^\mu V_\mu)\bar\zeta && (11.30)\\ \delta\bar\chi &= 2N\bar\zeta + (i\tilde\sigma^\mu\partial_\mu C - \tilde\sigma^\mu V_\mu)\zeta = 2N\bar\zeta + (\zeta\sigma^\mu\varepsilon)(i\partial_\mu C - V_\mu) && (11.31)\\ \delta M &= \bar\zeta(\bar\lambda + \tfrac{i}{2}\tilde\sigma^\mu\partial_\mu\phi) = \bar\lambda\bar\zeta - \tfrac{i}{2}\partial_\mu\phi\sigma^\mu\bar\zeta && (11.32)\\ \delta N &= \zeta\psi + \tfrac{i}{2}\zeta\sigma^\mu\partial_\mu\bar\chi && (11.33)\\ \delta V_\mu &= \zeta\sigma_\mu\bar\lambda + \psi\sigma_\mu\bar\zeta + \tfrac{i}{2}\partial_\nu\phi\sigma_\mu\tilde\sigma^\nu\zeta - \tfrac{i}{2}\partial_\nu\bar\chi\tilde\sigma_\mu\sigma^\nu\bar\zeta && (11.34)\\ \delta\bar\lambda &= 2D\bar\zeta + \tfrac{i}{2}\tilde\sigma^\nu\sigma^\mu\bar\zeta\partial_\mu V_\nu + i\tilde\sigma^\mu\zeta\partial_\mu M && (11.35)\\ \delta\psi &= 2D\zeta - \tfrac{i}{2}\sigma^\nu\tilde\sigma^\mu\zeta\partial_\mu V_\nu + i\sigma^\mu\bar\zeta\partial_\mu N && (11.36)\\ \delta D &= \tfrac{i}{2}\zeta\sigma^\mu\partial_\mu\bar\lambda + \tfrac{i}{2}\bar\zeta\tilde\sigma^\mu\partial_\mu\psi. && (11.37)\end{aligned}$$

Note, that D transforms into a total derivative.

11.3.2 Real superfields:

So far we have not imposed any conditions on the superfield Φ. As a result the components of Φ form a highly reducible representation of $N = 1$ supersymmetry. We may easily half the degrees of freedom by imposing a reality condition on Φ which is consistent with supersymmetry. So let α, the $(1/2,0)$ component of which is θ, and Q be Majorana spinors, such that

$$(\bar\alpha Q)^\dagger = (\theta Q + \bar Q\bar\theta)^\dagger = \bar Q\bar\theta + \theta Q$$

is hermitian and

$$(U(x,\theta,\bar\theta))^\dagger = U^{-1}(x,\theta,\bar\theta)$$

is unitary. In this case

$$(\theta^\alpha)^* = \bar\theta^{\dot\alpha} \quad \text{and} \quad (\theta_\alpha)^* = \theta_{\dot\alpha}. \tag{11.38}$$

Now we may impose the following *reality condition* on the superfield,

$$(\Phi(x,\theta,\bar\theta))^\dagger = \Phi(x,\theta,\bar\theta), \tag{11.39}$$

and this condition is consistent with supersymmetry: If Φ is a real superfield, then the transformed superfield

$$\Phi(x',\theta',\bar\theta') = U(a,\zeta,\bar\zeta)\Phi(x,\theta,\bar\theta)U^{-1}(a,\zeta,\bar\zeta)$$

is real as well, provided the ζ satisfy the same reality conditions (11.38) as the coordinates θ of superspace. For a real superfield the component fields in (11.25) fulfill the conditions

$$C = C^\dagger, \quad D^\dagger = D, \quad M^\dagger = N, \quad V_\mu = V_\mu^\dagger, \quad \bar\chi_{\dot\alpha} = \phi_\alpha^\dagger \quad \text{and} \quad \bar\lambda_{\dot\alpha} = \psi_\alpha^\dagger. \tag{11.40}$$

The total number of component fields is now only $8+8$. The superfield (11.25) the components of which fulfill (11.40) is called the *real (general) superfield.*
We may rewrite a real superfield in the 4-component notation. For Majorana spinors we have

$$\begin{aligned}
&\bar\alpha\gamma_\mu\alpha = \bar\alpha\gamma_{\mu\nu}\alpha = 0, \quad \bar\alpha\alpha = \theta^2 + \bar\theta^2, \quad \bar\alpha\gamma_5\alpha = \bar\theta^2 - \theta^2\\
&\bar\alpha\gamma_\mu\gamma_5\alpha = \theta\sigma_\mu\bar\theta - \bar\theta\tilde\sigma_\mu\theta = 2\theta\sigma_\mu\bar\theta = -2\bar\theta\tilde\sigma_\mu\theta\\
&(\bar\alpha\alpha)(\bar\alpha\gamma_5\alpha) = (\bar\alpha\alpha)(\bar\alpha\gamma^\mu\alpha) = (\bar\alpha\alpha)(\bar\alpha\gamma_5\gamma^\mu\alpha) = 0.
\end{aligned} \tag{11.41}$$

In the 4-component notation a real superfield takes the form

$$\begin{aligned}
\Phi(x,\alpha) \;=\; & C + \bar\alpha\phi + \tfrac12\bar\alpha\alpha M_1 + \tfrac{i}{2}\bar\alpha\gamma_5\alpha M_2 + \tfrac12(\bar\alpha\gamma^\mu\gamma_5\alpha)V_\mu + (\bar\alpha\alpha)(\bar\alpha\lambda) + \tfrac12(\bar\alpha\alpha)^2 D\\
& \text{where} \quad M_1 \equiv M + M^\dagger,\; M_2 \equiv i(M - M^\dagger),
\end{aligned} \tag{11.42}$$

and

$$\phi = \begin{pmatrix}\phi_\alpha\\ \chi^{\dot\alpha}\end{pmatrix}, \quad \lambda = \begin{pmatrix}\psi_\alpha\\ \lambda^{\dot\alpha}\end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix}\theta_\alpha\\ \bar\theta^{\dot\alpha}\end{pmatrix}$$

are Majorana spinors. If β is the Majorana spinor whose positive chirality part is ζ, then

$$\delta_\beta\Phi(x,\alpha) = \Big(\bar\beta\frac{\partial}{\partial\bar\alpha} - i(\bar\beta\gamma^\mu\alpha)\partial_\mu\Big)\Phi. \tag{11.43}$$

We may read off the transformation laws for the components of a real superfield from (11.29-11.37) by imposing the reality constraints (11.40). Alternatively we may directly compute them by comparing coefficients in the expansion

$$\begin{aligned}\delta\Phi(x,\alpha) &= \delta C + \bar\alpha\delta\phi + \tfrac{1}{2}\bar\alpha\alpha\delta M_1 + \tfrac{i}{2}\bar\alpha\gamma_5\alpha\delta M_2\\ &\quad + \tfrac{1}{2}(\bar\alpha\gamma^\mu\gamma_5\alpha)\delta V_\mu + (\bar\alpha\alpha)(\bar\alpha\delta\lambda) + \tfrac{1}{2}(\bar\alpha\alpha)^2\delta D,\end{aligned}$$

with the $\delta_\beta\Phi$ in (11.43) where one inserts the explicit parameterization (11.42) for the superfield. A straightforward calculation yields

$$\begin{aligned}(\bar\beta\partial_{\bar\alpha} - i(\bar\beta\gamma^\mu\alpha)\partial_\mu)\Phi &= \bar\beta\phi + \bar\beta\alpha M_1 + \bar\beta\gamma_5\alpha M_2 + (\bar\beta\gamma^\mu\gamma_5\alpha)V_\mu - i(\bar\beta\gamma^\mu\alpha)\partial_\mu C\\ &+ 2(\bar\beta\alpha)\bar\alpha\lambda + (\bar\alpha\alpha)\bar\beta\lambda - i(\bar\beta\gamma^\mu\alpha)\bar\alpha\partial_\mu\phi\\ &- \tfrac{i}{2}(\bar\beta\gamma^\mu\alpha)(\bar\alpha\alpha)\partial_\mu M_1 + \tfrac{1}{2}(\bar\beta\gamma^\mu\alpha)(\bar\alpha\gamma_5\alpha)\partial_\mu M_2\\ &- \tfrac{i}{2}(\bar\beta\gamma^\mu\alpha)(\bar\alpha\gamma^\nu\gamma_5\alpha)\partial_\mu V_\nu - i(\bar\beta\gamma^\mu\alpha)(\bar\alpha\alpha)\bar\alpha\partial_\mu\lambda + 2(\bar\beta\alpha)(\bar\alpha\alpha)D.\end{aligned} \tag{11.44}$$

To compare this result with (11.42) we use Fierz identities. For that we recall the Fierz identity (4.85), which for a Majorana spinor α reduces to

$$4\alpha\bar\alpha = -\bar\alpha\alpha + \gamma_5\gamma_\mu(\bar\alpha\gamma_5\gamma^\mu\alpha) - \gamma_5(\bar\alpha\gamma_5\alpha). \tag{11.45}$$

The second line in (11.44), i.e. the terms quadratic in α, can be rewritten as

$$\tfrac{1}{4}(\bar\alpha\alpha)(2\bar\beta\lambda + i\bar\beta\partial\!\!\!/\phi) - \tfrac{1}{4}(\bar\alpha\gamma_5\alpha)(2\bar\beta\gamma_5\lambda + i\bar\beta\gamma_5\partial\!\!\!/\phi) - \tfrac{1}{4}(\bar\alpha\gamma^\mu\gamma_5\alpha)(2\bar\beta\gamma_5\gamma_\mu\lambda - i\bar\beta\gamma^\nu\gamma_5\gamma_\mu\partial_\nu\phi).$$

The last term in the equation maybe recast as follows:

$$\bar\beta\gamma^\nu\gamma_5\gamma_\mu\partial_\nu\phi = \partial_\nu\phi\gamma_\mu\gamma_5\gamma^\nu\beta.$$

Again using the Fierz identity for $\alpha\bar\alpha$ and the relations in the last line in (11.41) we obtain the identities

$$\begin{aligned}(\bar\beta\gamma^\mu\alpha)(\bar\alpha\gamma_5\alpha) &= (\bar\beta\gamma_5\gamma^\mu\alpha)(\bar\alpha\alpha)\\ -i(\bar\beta\gamma^\mu\alpha)(\bar\alpha\alpha)(\bar\alpha\partial_\mu\lambda) &= \tfrac{i}{4}(\bar\alpha\alpha)^2\bar\beta\partial\!\!\!/\lambda\\ (\bar\beta\gamma^\mu\alpha)(\bar\alpha\gamma^\nu\gamma_5\alpha) &= -(\bar\alpha\alpha)(\bar\alpha\gamma_5\gamma^\nu\gamma_\mu\beta),\end{aligned}$$

which allow us to recast the cubic and quartic in α terms in (11.44). Now we are ready to compare the coefficients in (11.42) and (11.44) and extract the following transformation rules for the coefficient functions:

$$\begin{aligned}\delta C &= \bar\beta\phi\\ \delta D &= \tfrac{i}{2}\bar\beta\partial\!\!\!/\lambda\\ \delta V_\mu &= \bar\beta\gamma_\mu\gamma_5\lambda - \bar\beta\gamma_5\partial_\mu\phi + \tfrac{i}{2}\bar\beta\gamma_5\gamma_\mu\partial\!\!\!/\phi\\ \delta M_1 &= \bar\beta\lambda + \tfrac{i}{2}\bar\beta\partial\!\!\!/\phi\\ \delta M_2 &= i\bar\beta\gamma_5\lambda - \tfrac{1}{2}\gamma_5\partial\!\!\!/\phi\\ \delta\phi &= (M_1 + iM_2\gamma_5 - \gamma_5\gamma^\mu V_\mu + i\partial\!\!\!/C)\beta\\ \delta\lambda &= (2D + \tfrac{i}{2}\partial\!\!\!/M_1 + \tfrac{1}{2}\gamma_5\partial\!\!\!/M_2 + \tfrac{i}{2}\gamma_5\gamma^\nu\partial\!\!\!/V_\nu)\beta\end{aligned} \tag{11.46}$$

This transformation laws agree with those in (11.29-11.37) after imposing the reality constraints (11.40). This is just a check for the correctness of derived formulae.
The real multiplet consisting of the scalars C, M_1, D, the pseudoscalar M_2, the vector V_μ and the Majorana spinors ϕ, λ is not an irreducible representation of the supersymmetry algebra as follows from the general theory about representations of the $N = 1$ supersymmetry algebra and the following considerations.

11.3.3 Sub-multiplets

It is not difficult to see that the fields

$$\tilde{D} = 2D + \tfrac{1}{2}\Box C \quad , \quad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu \qquad \text{and} \quad \chi = \lambda - \tfrac{i}{2}\partial\!\!\!/\phi$$

transform entirely among themselves and thus form a submultiplet dV with components

$$dV = (\chi, F_{\mu\nu}, \tilde{D}).$$

This multiplet is called *curl multiplet* and it is irreducible, contrary to V itself. The transformation laws are

$$\delta\chi = (\tilde{D} - \tfrac{i}{2}\gamma^{\mu\nu}F_{\mu\nu}\gamma_5)\beta, \qquad \delta F_{\mu\nu} = \bar{\beta}\gamma_5(\gamma_\mu\partial_\nu - \gamma_\nu\partial_\mu)\chi, \qquad \delta\tilde{D} = i\bar{\beta}\partial\!\!\!/\chi.$$

This is (almost) the transformation for the gauge multiplet we have studied in section (9.1.1). Hence we may take the corresponding calculations for the commutators of two supersymmetry transformations. The algebra closes, provided $F_{[\alpha\beta,\mu]} = 0$:

$$\begin{aligned}
[\delta_1, \delta_2]\tilde{D} &= 2i(\bar{\beta}_2\gamma^\beta\beta_1)(\partial_\beta\tilde{D} + \partial^\alpha\, {}^*F_{\alpha\beta}) = 2i(\bar{\beta}_2\gamma^\beta\beta_1)\partial_\beta\tilde{D} \\
[\delta_1, \delta_2]\chi &= 2i(\bar{\beta}_2\gamma^\beta\beta_1)\partial_\beta\chi \\
[\delta_1, \delta_2]F_{\mu\nu} &= 2i(\bar{\beta}_2\gamma^\beta\beta_1)(\partial_\nu F_{\mu\beta} + \partial_\mu F_{\beta\nu}) = 2i(\bar{\beta}_2\gamma^\beta\beta_1)\partial_\beta F_{\mu\nu}.
\end{aligned}$$

A different invariant submultiplet of V is the chiral multiplet

$$\partial V = (M_1, M_2;\ \chi = \lambda + \tfrac{i}{2}\partial\!\!\!/\phi;\ \partial_\mu V^\mu;\ \tilde{D} = 2D - \tfrac{1}{2}\Box C)$$

with the transformation laws

$$\begin{aligned}
&\delta M_1 = \bar{\beta}\chi \quad , \quad \delta M_2 = i\bar{\beta}\gamma_5\chi \\
&\delta(\partial_\mu V^\mu) = -\bar{\beta}\gamma_5\partial\!\!\!/\chi \\
&\delta\chi = (\tilde{D} + i\partial\!\!\!/M_1 + \gamma_5\partial\!\!\!/M_2 + i\gamma_5\partial_\mu V^\mu)\beta \\
&\delta\tilde{D} = i\bar{\beta}\partial\!\!\!/\chi
\end{aligned}$$

Both dV and ∂V are submultiplets of V, but their complements are not. Hence V is *reducible but not fully reducible* and this property is quit common in supersymmetry. To construct further multiplets it is useful to impose constraints on the superfield which are compatible with supersymmetry.

11.3.4 Covariant spinor derivatives

The problem we shall address here is to find conditions on the superfield Φ to reduce the number of degrees of freedom which they describe. The most interesting conditions involve covariant spinor derivatives which can be introduced in an elegant way. The associativity of group multiplication

$$(g_1 \circ g_2) \circ g_3 = g_1 \circ (g_2 \circ g_3) \tag{11.47}$$

has consequences for the infinitesimal left and right action of the group: Let

$$g_1 \circ g_2 \sim g_2 + O_1^L g_2 \quad \text{and} \quad g_2 \circ g_3 \sim g_2 + O_3^R g_2.$$

Then the associative law (11.47) implies

$$\begin{aligned} (g_2 + O_1^L g_2) \circ g_3 &= g_1 \circ (g_2 + O_3^R g_2) \Longrightarrow \\ g_2 + O_3^R g_2 + O_1^L g_2 + O_3^R O_1^L g_2 &= g_2 + O_1^L g_2 + O_3^R g_2 + O_1^L O_3^R g_2, \end{aligned}$$

or that left and right-'derivatives' commute,

$$[O_1^L, O_3^R] = 0. \tag{11.48}$$

So far we have introduced the left action of one element of the supergroup on another

$$\begin{aligned} U(a^\mu, \beta) U(x^\mu, \alpha) &= U(a^\mu + x^\mu - i\bar{\beta}\gamma^\mu \alpha, \beta + \alpha) \\ &\sim \Big(\mathbb{1} + ib^\mu r(P_\mu) + i\beta^\alpha r(Q_\alpha)\Big) U(x, \alpha) \end{aligned}$$

Now we introduce the right action which has the form

$$\begin{aligned} U(x^\mu, \alpha) U(a^\mu, \beta) &= U(x^\mu + a^\mu + i\bar{\beta}\gamma^\mu \alpha, \bar{\beta} + \alpha) \\ &\sim (\mathbb{1} + a^\mu D_\mu + \bar{\beta}^\alpha D_\alpha) U(x, \alpha) \end{aligned}$$

It is easy to calculate the derivative operators D :

$$D_\mu = \partial_\mu, \quad D_\alpha = \partial_{\bar{\alpha}} + i(\gamma^\mu \alpha)\partial_\mu. \tag{11.49}$$

In terms of Weyl spinors these differential operators read

$$\begin{array}{lll} \bar{\beta}^\alpha D_\alpha = \zeta^\alpha D_\alpha + \bar{\zeta}^{\dot{\alpha}} \bar{D}_{\dot{\alpha}} & \Longrightarrow & \\ D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_\mu & , & \bar{D}^{\dot{\alpha}} = \bar{\partial}^{\dot{\alpha}} + i\tilde{\sigma}^{\mu\dot{\alpha}\alpha} \theta_\alpha \partial_\mu \\ D^\alpha = -\partial^\alpha - i\bar{\theta}_{\dot{\alpha}} \tilde{\sigma}^{\mu\dot{\alpha}\alpha} \partial_\mu & , & \bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu. \end{array} \tag{11.50}$$

From our general discussion we know already that

$$[\bar{\beta}^\alpha r(Q_\alpha), \bar{\beta}'^\alpha D_\alpha] = 0$$

from which follows that the $r(Q_\alpha)$ must anticommute with the D_α

$$\{D, r(Q)\} = \{\bar{D}, r(Q)\} = \{D, r(\bar{Q})\} = \{\bar{D}, r(\bar{Q})\} = 0. \tag{11.51}$$

Since the operators D are invariant under supersymmetry transformations they are called *covariant derivatives*. The (anti)commutators of the D's with each other are found to be

$$\{D,D\}=\{\bar{D},\bar{D}\}=[D,\partial_\mu]=[\bar{D},\partial_\mu]=[\partial_\mu,\partial_\nu]=0 \tag{11.52}$$

and

$$\{D_\alpha,\bar{D}_{\dot\beta}\}=-2i(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu. \tag{11.53}$$

It follows that the (fermionic) derivative operators D commute with the covariant derivatives and satisfy the same anti-commutation relations as the supercharges. I leave the proof of these simple (anti)commutators to you. It is really very simple.

11.3.5 Projection operators

In subsequent sections we will consider superfields which satisfy certain constraints. The corresponding fields can be obtained from the most general superfield by projections. It is convenient to study the properties of these projection operators before applying them to superfields.
We begin by proving the following set of relations which all follow from the anticommutation relations derived above. We shall study properties of the covariant derivatives and of

$$D^2=D^\alpha D_\alpha \quad\text{and}\quad \bar{D}^2=\bar{D}_{\dot\alpha}\bar{D}^{\dot\alpha}. \tag{11.54}$$

Note that

$$D^\alpha D^2=\bar{D}\bar{D}^2=0.$$

The formulae will shall need later are:

$$\begin{aligned}
[D_\alpha,\bar{D}^2] &= -4i\sigma^\mu_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\partial_\mu && (11.55)\\
[D^\alpha,\bar{D}^2] &= 4i\bar{D}_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\alpha}\partial_\mu && (11.56)\\
[\bar{D}_{\dot\alpha},D^2] &= 4iD^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu && (11.57)\\
[\bar{D}^{\dot\alpha},\bar{D}^2] &= -4i\tilde\sigma^{\mu\dot\alpha\alpha}D_\alpha\partial_\mu && (11.58)\\
[D^2,\bar{D}^2] &= -8i(D\sigma^\mu\bar{D})\partial_\mu+16\Box && (11.59)\\
[\bar{D}^2,D^2] &= -8i(\bar{D}\tilde\sigma^\mu D)\partial_\mu+16\Box && (11.60)\\
\bar{D}\tilde\sigma^\mu D &= -D\sigma^\mu\bar{D}-4i\partial^\mu && (11.61)\\
D^\alpha\bar{D}^2D_\alpha &= \bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha} && (11.62)
\end{aligned}$$

We prove only some of these identities starting with the anticommutation relation (11.53). We multiply this with $\bar{D}^{\dot\alpha}$ from the right and summing over $\dot\alpha$ one obtains

$$D_\alpha\bar{D}^2+\bar{D}_{\dot\alpha}D_\alpha\bar{D}^{\dot\alpha}=-2i\sigma^\mu_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}\partial_\mu \tag{11.63}$$

Alternatively, we may multiply (11.53) with $\varepsilon^{\dot\alpha\dot\beta}$ and find

$$(D_\alpha \bar D_{\dot\alpha} + \bar D_{\dot\alpha} D_\alpha)\varepsilon^{\dot\alpha\dot\beta} = -(D_\alpha \bar D^{\dot\beta} + \bar D^{\dot\beta} D_\alpha) = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \varepsilon^{\dot\alpha\dot\beta}.$$

From this relation we conclude

$$\bar D_{\dot\beta} D_\alpha \bar D^{\dot\beta} = -\bar D^2 D_\alpha + 2i\sigma^\mu_{\alpha\dot\alpha}\bar D^{\dot\alpha}\partial_\mu$$

and this maybe used for the second term on the left in (11.63). This then yields the formula (11.55). The formula (11.56) follows from this one as follows

$$[D^\alpha, \bar D^2] = \varepsilon^{\alpha\beta}[D_\beta, \bar D^2] = -4i\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\alpha}\bar D_{\dot\beta}\partial_\mu = 4i\bar D_{\dot\beta}\sigma^{\mu\dot\beta\alpha}\partial_\mu,$$

where we used the identities we recall:

$$\tilde\sigma^{\mu\dot\alpha\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\tilde\sigma^\mu_{\beta\dot\beta} \quad , \quad \sigma^\mu_{\alpha\dot\alpha} = \varepsilon_{\alpha\beta}\varepsilon_{\dot\alpha\dot\beta}\tilde\sigma^{\mu\dot\beta\beta}. \tag{11.64}$$

Let us also prove the identity (11.59):

$$\begin{aligned}
[D^2, \bar D^2] &= D^\alpha[D_\alpha, \bar D^2] + [D^\alpha, \bar D^2]D_\alpha = -4iD^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar D^{\dot\alpha}\partial_\mu + 4i\bar D_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\alpha}D_\alpha\partial_\mu \\
&= -4i(D\sigma^\mu\bar D)\partial_\mu + 4i\tilde\sigma^{\mu\dot\alpha\alpha}(\{D_\alpha, \bar D_{\dot\alpha}\} - D_\alpha\bar D_{\dot\alpha})\partial_\mu \\
&= -8i(D\sigma^\mu\bar D)\partial_\mu + 8\tilde\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\alpha}\partial_\nu\partial_\mu = -8i(D\sigma^\mu\bar D)\partial_\mu + 16\Box
\end{aligned}$$

where in the last step we have used that

$$\tilde\sigma^{\mu\dot\alpha\alpha}\sigma^\nu_{\alpha\dot\alpha} = \mathrm{Tr}\,(\tilde\sigma^\mu\sigma^\nu) = 2\eta^{\mu\nu}.$$

Now we are ready to introduce various projection operators. These are

$$\Pi_+ = -\frac{1}{16\Box}\bar D^2 D^2, \quad \Pi_- = -\frac{1}{16\Box}D^2\bar D^2, \quad \Pi_T = \frac{1}{8\Box}\bar D_{\dot\alpha}D^2\bar D^{\dot\alpha} = \frac{1}{8\Box}D^\alpha\bar D^2 D_\alpha. \tag{11.65}$$

First we prove that the sum of these operators is the identity

$$\Pi_+ + \Pi_- + \Pi_T = \mathbb{1}. \tag{11.66}$$

This is shown as follows:

$$\begin{aligned}
\Pi_+ + \Pi_- + \Pi_T &= \frac{1}{16\Box}(\bar D_{\dot\alpha}D^2\bar D^{\dot\alpha} + D^\alpha\bar D^2 D_\alpha - \bar D^2 D^2 - D^2\bar D^2) \\
&= \frac{1}{16\Box}\{(\bar D_{\dot\alpha}D^2 - D^2\bar D_{\dot\alpha})\bar D^{\dot\alpha} + (D^\alpha\bar D^2 - \bar D^2 D^\alpha)D_\alpha\} \\
&\overset{(11.56,11.57)}{=} \frac{i}{4\Box}(D^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar D^{\dot\alpha} + \bar D_{\dot\alpha}\tilde\sigma^{\mu\dot\alpha\alpha}\partial_\mu D_\alpha) \\
&= \frac{i}{4\Box}(D\sigma^\mu\bar D + \bar D\tilde\sigma^\mu D)\partial_\mu \overset{(11.61)}{=} \frac{i}{4\Box}(-4i\partial^\mu)\partial_\mu = \mathbb{1}
\end{aligned}$$

Now we show that these operators are idempotent:

$$\begin{aligned}
\Pi_+\Pi_+ &= \frac{1}{16\Box}\bar D^2 D^2\frac{1}{16\Box}\bar D^2 D^2 = (\frac{1}{16\Box})^2\bar D^2 D^2\bar D^2 D^2 \\
&\overset{(11.59)}{=} (\frac{1}{16\Box})^2\bar D^2 D^2\{D^2\bar D^2 + 8iD\sigma^\mu\bar D\partial_\mu - 16\Box\} = \frac{1}{16\Box}\bar D^2 D^2 = \Pi_+.
\end{aligned}$$

In the second to last step we used that $D^3 = D^4 = 0$. Similarly one proves the same property for the other two operators, such that

$$\Pi_+^2 = \Pi_+ \quad , \quad \Pi_-^2 = \Pi_- \quad \text{and} \quad \Pi_T^2 = \Pi_T. \tag{11.67}$$

Finally, since $D^3 = \bar{D}^3 = 0$ these operators project onto orthogonal subspaces:

$$\begin{aligned}
\Pi_+\Pi_- &= (\frac{1}{16\Box})^2 \bar{D}^2 D^2 D^2 \bar{D}^2 = 0 \\
\Pi_-\Pi_+ &= (\frac{1}{16\Box})^2 D^2 \bar{D}^2 \bar{D}^2 D^2 = 0 \\
\Pi_+\Pi_T &= -\frac{1}{8\cdot 16\Box^2} \bar{D}^2 D^2 D^\alpha \bar{D}^2 D_\alpha = 0
\end{aligned}$$

From the very definition of the projection operators we conclude that Π_+ projects onto the kernel of $\bar{D}$ and Π_- projects on the kernel of Π_-:

$$\bar{D}_{\dot\alpha}\Pi_+ = 0 \quad \text{and} \quad D^\alpha \Pi_- = 0. \tag{11.68}$$

Fields $\Phi = \Pi_+\Phi$ and $\Phi^\dagger = \Pi_-\Phi^\dagger$ are called *(anti)chiral superfields.*

11.4 Chiral superfields

Like all covariant derivatives, D and $\bar{D}$ can be used to impose covariant conditions on superfields. The most prominent such conditions are those for a *chiral superfield* Φ which we already discussed in the last subsection,

$$\bar{D}_{\dot\alpha}\Phi = 0 \tag{11.69}$$

and an *anti-chiral superfield* $\bar{\Phi}$ with

$$D_\alpha \Phi^\dagger = 0. \tag{11.70}$$

Since $\{D, \bar{D}\} \sim \partial_\mu$ a superfield cannot be both chiral and anti-chiral except if it is a constant field. These conditions are first-order differential equations and can easily be solved:

$$\Phi(x,\theta,\bar\theta) = \exp(i\theta\not\partial\bar\theta)\Phi(x,\theta) \quad \text{and} \quad \Phi^\dagger(x,\theta,\bar\theta) = \exp(-i\theta\not\partial\bar\theta)\Phi^\dagger(x,\bar\theta), \qquad \not\partial = \sigma^\mu\partial_\mu \tag{11.71}$$

That these are the general solutions follows at once from

$$\bar{D}_{\dot\alpha}\exp(i\theta\not\partial\bar\theta) = \exp(i\theta\not\partial\bar\theta)\bar\partial_{\dot\alpha} \quad \text{and} \quad D_\alpha \exp(-i\theta\not\partial\bar\theta) = \exp(-i\theta\not\partial\bar\theta)\partial_\alpha. \tag{11.72}$$

The exponentials in (11.71) just shift the x-coordinate of the superfield and the explicit form of a chiral and anti-chiral superfield reads

$$\Phi = \Phi(x^\mu + i\theta\sigma^\mu\bar\theta, \theta) \quad \text{and} \quad \Phi^\dagger = \Phi(x^\mu - i\theta\sigma^\mu\bar\theta, \bar\theta). \tag{11.73}$$

The Taylor expansion of the fields $\Phi(x,\theta)$ and $\bar\Phi(x,\bar\theta)$ in (11.71) is particularly simple

$$\Phi(x,\theta) = A + \sqrt{2}\theta\psi + \theta^2 F \quad \text{and} \quad \Phi^\dagger(x,\bar\theta) = A^\dagger + \sqrt{2}\bar\psi\bar\theta + \bar\theta^2 F^\dagger. \tag{11.74}$$

Actually when dealing with chiral superfields it is more convenient to pass to a different parametrization of superfields.

11.4.1 Alternative parametrizations of superfields

First we introduce different parametrizations for the elements of the supergroup as compared to (11.8), namely

$$\begin{aligned}U_1(a,\theta,\bar\theta) &= \exp\big(-i(a,P)+i\theta Q\big)\exp\big(i\bar\theta\bar Q\big)\\ U_2(a,\theta,\bar\theta) &= \exp\big(-i(a,P)+i\bar\theta\bar Q\big)\exp\big(i\theta Q\big),\end{aligned} \tag{11.75}$$

where we made use of the conventions in (11.8). Using the group multiplication law (11.11) we can relate U and U_1, U_2:

$$\begin{aligned}U_1(x,\theta,\bar\theta) &= U(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta) \Longleftrightarrow U(x,\theta,\bar\theta) = U_1(x^\mu + i\theta\sigma^\mu\bar\theta,\theta,\bar\theta)\\ U_2(x,\theta,\bar\theta) &= U(x^\mu + i\theta\sigma^\mu\bar\theta,\theta,\bar\theta) \Longleftrightarrow U(x,\theta,\bar\theta) = U_2(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta)\end{aligned} \tag{11.76}$$

and correspondingly the associated superfields (11.15)

$$\begin{aligned}\Phi_1(x,\theta,\bar\theta) &= \Phi(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta) \quad , \quad \Phi(x,\theta,\bar\theta) = \Phi_1(x^\mu + i\theta\sigma^\mu\bar\theta,\theta,\bar\theta)\\ \Phi_2(x,\theta,\bar\theta) &= \Phi(x^\mu + i\theta\sigma^\mu\bar\theta,\theta,\bar\theta) \quad , \quad \Phi(x,\theta,\bar\theta) = \Phi_2(x^\mu - i\theta\sigma^\mu\bar\theta,\theta,\bar\theta)\end{aligned} \tag{11.77}$$

Let us calculate how $U(a,\zeta,\bar\zeta)$ acts on Φ_1 and Φ_2, repectively:

$$\begin{aligned}&U(a,\zeta,\bar\zeta)\,\Phi_1(x,\theta,\bar\theta)\,U^{-1}(a,\zeta,\bar\zeta) = U(a,\zeta,\bar\zeta)\,\Phi(x-i\theta\sigma\bar\theta,\theta,\bar\theta)\,U^{-1}(a,\zeta,\bar\zeta)\\ &\overset{(11.11)}{=} \Phi(a+x-i\theta\sigma\bar\theta - i\zeta\sigma\bar\theta - i\bar\zeta\tilde\sigma\theta,\theta+\zeta,\bar\theta+\bar\zeta)\\ &\overset{(11.77)}{=} \Phi_1(a+x+2i\theta\sigma\bar\zeta + i\zeta\sigma\bar\zeta,\theta+\zeta,\bar\theta+\bar\zeta)\end{aligned}$$

and similarly

$$U(a,\zeta,\bar\zeta)\,\Phi_2(x,\theta,\bar\theta)\,U^{-1}(a,\zeta,\bar\zeta) = \Phi_2(a+x-2i\zeta\sigma\bar\theta - i\zeta\sigma\bar\zeta,\theta+\zeta,\bar\theta+\bar\zeta).$$

The infinitesimal transformation of Φ_1 and Φ_2 read

$$\begin{aligned}\delta\Phi_1 &= a^\mu\partial_\mu\Phi_2 + \Big(\bar\zeta_{\dot\alpha}(\bar\partial^{\dot\alpha} - 2i\tilde\sigma^{\mu\dot\alpha\alpha}\theta_\alpha\partial_\mu) + \zeta^\alpha\partial_\alpha\Big)\Phi_2\\ \delta\Phi_2 &= a^\mu\partial_\mu\Phi_2 + \Big(\zeta^\alpha(\partial_\alpha - 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu) + \bar\zeta_{\dot\alpha}\partial^{\dot\alpha}\Big)\Phi_2\end{aligned} \tag{11.78}$$

from which we read off how the supercharges act on these superfields:

$$\begin{aligned}&\Phi_1: \quad r_1(Q_\alpha) = -i\partial_\alpha, \quad r_1(\bar Q^{\dot\alpha}) = -i\bar\partial^{\dot\alpha} - 2(\tilde\sigma^\mu\theta)^{\dot\alpha}\partial_\mu, \quad r_1(\bar Q_{\dot\alpha}) = i\bar\partial_{\dot\alpha} + 2(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\\ &\Phi_2: \quad r_2(Q_\alpha) = -i\partial_\alpha - 2(\sigma^\mu\bar\theta)_\alpha\partial_\mu, \quad r_2(\bar Q^{\dot\alpha}) = -i\bar\partial^{\dot\alpha}, \quad r_2(\bar Q_{\dot\alpha}) = i\bar\partial_{\dot\alpha}.\end{aligned} \tag{11.79}$$

Of course, these transformed supercharges fulfil the correct anticommutation relations

$$\{r(Q_\alpha), r(\bar Q_{\dot\alpha})\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu.$$

Along with these transformed supercharges we obtain transformed covariant derivatives:

$$\begin{aligned}D^{(1)}_\alpha &= \partial_\alpha + 2i(\sigma^\mu\bar\theta)_\alpha\partial_\mu \quad , \quad \bar D^{(1)}_{\dot\alpha} = -\bar\partial_{\dot\alpha}\\ D^{(2)}_\alpha &= \partial_\alpha \quad , \quad \bar D^{(2)}_{\dot\alpha} = -\bar\partial_{\dot\alpha} - 2i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu.\end{aligned} \tag{11.80}$$

As one may guess from the relations (11.72) the operators in the 3 parametrisations for the superfield are related as

$$O^{(1)} = e^{-i\theta\partial\!\!\!/\bar\theta}\,O\,e^{i\theta\partial\!\!\!/\bar\theta} \quad\text{and}\quad O^{(2)} = e^{i\theta\partial\!\!\!/\bar\theta}\,O\,e^{-i\theta\partial\!\!\!/\bar\theta}.$$

In the parametrization (1) a chiral superfield fulfils the simple condition

$$D_{\dot\alpha}\Phi_1 = -\partial_{\dot\alpha}\Phi_1 = 0$$

and hence has the simple expansion (11.74), namely

$$\begin{aligned}\Phi_1(x,\theta) &= A(x) + \sqrt{2}\theta\psi(x) + \theta^2 F(x) \Longrightarrow\\ \Phi(x,\theta,\bar\theta) &= A(x+i\theta\sigma\bar\theta) + \sqrt{2}\theta\psi(x+i\theta\sigma\bar\theta) + \theta^2 F(x+\theta\sigma\bar\theta).\end{aligned}\tag{11.81}$$

The last expansion can be simplified:

$$\begin{aligned}\Phi(x,\theta,\bar\theta) &= A(x) + i(\theta\sigma^\mu\bar\theta)\partial_\mu A(x) - \tfrac{1}{2}(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta)\partial_\mu\partial_\nu A(x)\\ &\quad+\sqrt{2}\theta\psi(x) + i\sqrt{2}(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\psi(x)) + \theta^2 F(x)\\ &\overset{(4.88)}{=} \Big(A + i(\theta\sigma^\mu\bar\theta)\partial_\mu A - \tfrac{1}{4}\theta^2\bar\theta^2\Box A + \sqrt{2}\theta\psi + i\sqrt{2}(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\psi) + \theta^2 F\Big)(x).\end{aligned}$$

The second to last term can be rewritten by using

$$(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\psi) = \theta^\beta\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\psi_\beta = \tfrac{1}{2}\varepsilon^{\alpha\beta}\theta^2\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu\psi_\beta = -\tfrac{1}{2}\theta^2(\partial_\mu\psi\sigma^\mu\bar\theta)$$

so that the chiral superfield has the following representation

$$\Phi = \Big(A + \sqrt{2}\theta\psi + \theta^2 F + i(\theta\sigma^\mu\bar\theta)\partial_\mu A + \frac{i}{\sqrt{2}}\theta^2(\bar\theta\tilde\sigma^\mu\partial_\mu\psi) - \tfrac{1}{4}\theta^2\bar\theta^2\Box A\Big)(x).\tag{11.82}$$

Now we may compare this expansion with the expansion (11.25) for a general superfield to make the following identifications:

$$\begin{array}{lllll} C\to A, & M\to F, & N\to 0, & D\to -\tfrac{1}{4}\Box A, & V_\mu \to i\partial_\mu A\\ \phi_\alpha\to\sqrt{2}\,\psi_\alpha, & \bar\chi\to 0, & \bar\lambda^{\dot\alpha}\to \dfrac{i}{\sqrt{2}}(\tilde\sigma^\mu\partial_\mu\psi)^{\dot\alpha}, & \psi(x)\to 0 & \end{array}$$

We may use this identification in the supersymmetry transformations (11.29-11.37) for the general $N=1$ superfield and end up with

$$\delta A = \sqrt{2}\,\zeta\psi,\quad \delta\psi_\alpha = \sqrt{2}\,\zeta_\alpha F + i\sqrt{2}\,(\sigma^\mu\bar\zeta)_\alpha\partial_\mu A,\quad \delta F = i\sqrt{2}\,\bar\zeta\tilde\sigma^\mu\partial_\mu\psi.\tag{11.83}$$

It is evident from these transformations that the fields (A,ψ,F) constitute an irreducible representation of the supersymmetry algebra. The supersymmetry algebra is realized linearly and offshell. If we would eliminate F, then supersymmetry would be realized nonlinearly and on-shell, as we have discussed previously. The transformations (11.83) are just the supersymmetry transformation of the fields in the Wess-Zumino model (in the chiral basis). For later use we note, that for a chiral superfield the F-term transforms into a total derivative. The commutator of two transformations are

$$[\delta_1, \delta_2] \ldots = 2i(\bar{\alpha}_2 \gamma^\mu \alpha_1)\partial_\mu \ldots ,$$

as expected. The α_i are Majorana spinors parameter with $(1/2, 0)$ component ζ_i.
We could repeat the same reasoning for antichiral fields fulfilling $D_\alpha \Phi^\dagger = 0$. We would find

$$\Phi^\dagger = A^*(z) + \sqrt{2}\bar{\theta}\bar{\psi}(z) + \bar{\theta}^2 F^*(z), \qquad z = x - i\theta\sigma^\mu\bar{\theta} \tag{11.84}$$

instead of

$$\Phi = A(y) + \sqrt{2}\theta\psi(y) + \theta^2 F(y), \qquad y = x + i\theta\sigma^\mu\bar{\theta}, \tag{11.85}$$

as we have found for a chiral field.

11.5 Invariant action for scalar superfields

There cannot exist a supersymmetric Lagrangian density since $\delta\mathcal{L} = 0$ would imply that

$$[\delta_1, \delta_2]\mathcal{L} \sim \partial\mathcal{L} = 0,$$

or that $\mathcal{L}$ must be constant. Even if $\mathcal{L}$ is not supersymmetric, the action is still supersymmetric if $\delta\mathcal{L}$ is a derivative which would not contribute to δS. In general $\mathcal{L}$ can be written as a sum of terms, each of which is some component of a superfield that is constructed out of elementary superfields and their covariant derivatives. The transformation rules (11.29-11.37) show that for a general $N = 1$ superfield the only component whose variation is a derivative is the D-component, since only the last two terms of the supercharges

$$r(Q_\alpha) = -i\partial_\alpha - (\sigma^\mu\bar{\theta})_\alpha\partial_\mu \quad \text{and} \quad r(\bar{Q}_{\dot{\alpha}}) = i\bar{\partial}_{\dot{\alpha}} + (\theta\sigma^\mu)_\alpha\partial_\mu$$

contribute to the variation of the D-terms. For D to be a scalar, the superfield itself must be a scalar. Thus for a general scalar superfield only the integral of the D-term is a good candidate for an invariant action

$$S = \int d^4x \Phi|_D. \tag{11.86}$$

But no satisfactory action can be gotten this way without special conditions on the superfield. For a general superfield Φ the only sort of kinematic action S_0 that is bilinear in Φ and $\Phi^\dagger$ and involves no more than two derivatives is of the form

$$S_0 \sim \int d^4x [\Phi^\dagger\Phi]_D. \tag{11.87}$$

Earlier we have seen, that for a (anti)chiral superfield the F-term transforms into a total derivative,

$$\delta F = i\sqrt{2}\bar{\zeta}\tilde{\sigma}^\mu\partial_\mu\psi,$$

so that this term is another good candidate for an invariant action

$$S = \int d^4x\,\Phi^{\text{chiral}}|_F \quad , \quad S = \int d^4x\,\Phi^{\dagger\,\text{antichiral}}|_F \tag{11.88}$$

For a renormalizable theory the chiral field in (11.88) will be the product of elementary chiral fields. Hence to make further progress we need to study the product of superfields.

11.5.1 Products of superfields

Let Φ_1 and Φ_2 be two superfields, that is two field on superspace which transform as in (11.16). Then $\Phi = \Phi_1\Phi_2$ transforms the same way as Φ_1 and Φ_2 do and hence is a superfield as well. This follows from

$$\begin{aligned} U\Phi(x,\alpha)U^{-1} \quad &= U\Phi_1(x,\alpha)U^{-1}U\Phi_2(x,\alpha)U^{-1} = \Phi_1(x',\alpha')\Phi_2(x',\alpha') = \Phi(x',\alpha'), \\ \text{where} \quad &U = U(b,\beta), \quad x' = x + b - i\bar\beta\gamma\alpha, \quad \alpha' = \alpha + \beta. \end{aligned}$$

Equivalently, the infinitesimal supersymmetry transformation of the product field is

$$\delta\Phi = i[(\bar\beta Q), \Phi_1\Phi_2] = \delta\Phi_1\Phi_2 + \Phi_1\delta\Phi_2.$$

Clearly, the product of chiral superfields is a chiral superfield and the product of anti-chiral superfields is a anti-chiral superfield. For example, let

$$\begin{aligned} \Phi_i &= A_i(y) + \sqrt{2}\theta\psi_i(y) + \theta^2 F_i(y), \quad y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta \\ \Phi_i^\dagger &= A_i^*(z) + \sqrt{2}\bar\theta\bar\psi_i(z) + \bar\theta^2 F_i^*(z), \quad z^\mu = x^\mu - i\theta\sigma^\mu\bar\theta \end{aligned} \tag{11.89}$$

be a collection of chiral and antichiral superfields. Then

$$\begin{aligned} (\Phi_i\Phi_j)(x,\theta,\bar\theta) &= (A_iA_j)(y) + \sqrt{2}\theta(A_i\psi_j + A_j\psi_i)(y) + \theta^2(A_iF_j + A_jF_i - \psi_i\psi_j)(y), \\ (\Phi_i^\dagger\Phi_j^\dagger)(x,\theta,\bar\theta) &= (A_i^*A_j^*)(z) + \sqrt{2}\bar\theta(A_i^*\bar\psi_j + A_j^*\bar\psi_i)(z) + \bar\theta^2(A_i^*F_j^* + A_j^*F_i^* - \bar\psi_i\bar\psi_j)(z), \end{aligned} \tag{11.90}$$

where we made use of the Fierz identity (4.88). More interesting for our purpose is the product of a chiral and antichiral superfield:

$$\begin{aligned} \Phi_i^\dagger(z)\Phi_j(y) &= A_i^*A_j + \sqrt{2}A_i^*\theta\psi_j + \sqrt{2}\bar\theta\bar\psi_iA_j + \theta^2A_i^*F_j + \bar\theta^2F_i^*A_j \\ &\quad + 2(\bar\theta\bar\psi_i)(\theta\psi_j) + \sqrt{2}\bar\theta^2\theta\psi_jF_i^* + \sqrt{2}\theta^2\bar\theta\bar\psi_iF_j + \theta^2\bar\theta^2F_i^*F_j, \end{aligned} \tag{11.91}$$

where the argument of (A,ψ,F) is y and that of $(A^*,\bar\psi,F^*)$ is z. Not all terms in this expansion are or interest to us. Since $\Phi^\dagger\Phi$ ia a general superfield we concentrate on the D term which is a good candidate for the Lagrangian density of an invariant action. We use (see 11.81):

$$\begin{aligned} A(y) &= A(x) + i(\theta\sigma^\mu\bar\theta)\partial_\mu A(x) - \tfrac14\theta^2\bar\theta^2\Box A(x) \\ A^*(z) &= A^*(x) - i(\theta\sigma^\mu\bar\theta)\partial_\mu A^*(x) - \tfrac14\theta^2\bar\theta^2\Box A^*(x) \\ \theta\psi(y) &= \theta\psi(x) + i(\theta\sigma^\mu\bar\theta)(\theta\partial_\mu\psi(x)) \\ \bar\theta\bar\psi(z) &= \bar\theta\bar\psi(x) - i(\theta\sigma^\mu\bar\theta)(\bar\theta\partial_\mu\bar\psi(x)) \\ \theta^2F(y) &= \theta^2F(x) \\ \bar\theta^2F^*(z) &= \bar\theta^2F^*(x) \end{aligned} \tag{11.92}$$

To calculate the D-term we only need consider terms in (11.91) which contain as many powers of θ as of $\bar\theta$. Hence the only non-vanishing D-terms come from

$$\begin{aligned}
A_i^*(z)A_j(y)|_D &= \tfrac{1}{2}(\partial_\mu A_i^*\partial^\mu A_j - \tfrac{1}{2}A_i^*\Box A_j - \tfrac{1}{2}\Box A_i^* A_j) = \partial_\mu A_i^*\partial^\mu A_j - \tfrac{1}{4}\Box(A_i^*A_j)\\
(\bar\theta\bar\psi_i(z))(\theta\psi_j(y)) &= i\theta\sigma^\mu\bar\theta\{(\bar\theta\bar\psi_i)(\theta\partial_\mu\psi_j) - (\theta\psi_j)(\bar\theta\partial_\mu\bar\psi_i)\} = \tfrac{i}{4}(\partial_\mu\psi_j\sigma^\mu\bar\psi_i) - \tfrac{i}{4}(\psi_j\sigma^\mu\partial_\mu\bar\psi_i)\\
\theta^2\bar\theta^2 F_i(z)^*F_j(y) &= \theta^2\bar\theta^2 F_i(x)F_j^*(x).
\end{aligned}$$

and we end up with

$$\Phi_i^\dagger(z)\Phi_j(y)|_D = \partial_\mu A_i^*\partial^\mu A_j - \tfrac{1}{4}\Box(A_i^*A_j) + \tfrac{i}{2}(\partial_\mu\psi_j\sigma^\mu\bar\psi_i) - \tfrac{i}{2}(\psi_j\sigma^\mu\partial_\mu\bar\psi_i) + F_i(x)F_j^*(x) \tag{11.93}$$

For $i=j$ this can be rewritten as

$$\Phi_i^\dagger(z)\Phi_i(y)|_D = \partial_\mu A_i^*\partial^\mu A_i - \tfrac{i}{2}\bar\psi_i\partial\!\!\!/\psi_i + |F_i(x)|^2 + \text{surface term}, \tag{11.94}$$

where ψ_i is the Majorana spinor with $(1/2,0)$-componennt ψ_i. Summing over i this just becomes the Lagrangian density for the well-studied free *Wess-Zumino* model[2]. Let us finally calculate the D-terms of the product of 3 chiral or 3 antichiral fields

$$\begin{aligned}
(\Phi_i\Phi_j\Phi_k)(y) &= (A_iA_jA_k + \sqrt{2}(A_iA_j\theta\psi_k + \text{cycl}) + \theta^2((A_iA_jF_k - A_k\psi_i\psi_j) + \text{cycl}))(y)\\
(\Phi_i^\dagger\Phi_j^\dagger\Phi_k^\dagger)(z) &= (A_i^*A_j^*A_k^* + \sqrt{2}(A_i^*A_j^*\bar\theta\bar\psi_k + \text{cycl}) + \bar\theta^2((A_i^*A_j^*F_k^* - A_k^*\bar\psi_i\bar\psi_j) + \text{cycl}))(z)
\end{aligned}\tag{11.95}$$

The D-terms of these composite chiral fields will enter the Lagrangian densities for scalar superfields.

11.5.2 Invariant actions

The most general supersymmetric, renormalizable Lagrangian, involving only N scalar superfields is given by

$$\begin{aligned}
\mathcal{L} &= \Phi_i^\dagger\Phi_i|_D + W|_F + W^\dagger|_F, \quad \text{where}\\
W(\Phi) &= g_i\Phi_i + \tfrac{1}{2}m_{ij}\Phi_i\Phi_j + \tfrac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k
\end{aligned}\tag{11.96}$$

is called the superpotential. The mass matrix m_{ij} and the λ_{ijk} are symmetric in their indices. We have already argued that the corresponding action is invariant under supersymmetry transformations. Let us see what are the dimensions of the various fields and coupling constants. We shall see that the polynomial $W(\Phi)$ may not contain quartic or higher powers of the superfield in order for the corresponding model to be perturbatively renormalizable. From the explicit form of the supercharges we take

$$[\theta] = [\bar\theta] = L^{1/2}.$$

A spinorfield has dimensions $L^{-3/2}$ and hence

$$\begin{aligned}
[\Phi] = [\Phi^\dagger] = L^{-1} &\implies [\Phi^\dagger\Phi] = L^{-2}, \quad [\Phi^\dagger\Phi|_D] = L^{-4}\\
[\Phi^n] = L^{-n} &\implies [\Phi^n|_F] = [\Phi^{\dagger n}|_F] = L^{-n-1}.
\end{aligned}\tag{11.97}$$

[2]The different sign for the Dirac term originates in the different signs of $[\delta_1,\delta_2]$ here and in earlier chapters.

We read off the following dimensions of the coupling constant and component fields

$$[m] = L^{-2}, \quad [\lambda] = L^{0}, \quad [A] = L^{-1} \quad \text{and} \quad [F] = L^{-1}.$$

If we would allow for powers of Φ higher than the third in the above Lagrangian density, then the supersymmetry model would not be perturbatively renormalizable.
Now we collect our previous results to express $\mathcal{L}$ in (11.96) in component fields:

$$\begin{aligned}\mathcal{L} &= \partial_\mu A_i^* \partial^\mu A_i - \tfrac{i}{2}\bar\psi_i \partial\!\!\!/ \psi_i + |F_i(x)|^2 \\ &\quad + \Big(g_i F_i + m_{ij}(A_i F_j - \tfrac{1}{2}\psi_i\psi_j) + \lambda_{ijk}(A_i A_j F_k - A_k \psi_i \psi_j)\Big) + \Big(\text{h.c.}\Big)\end{aligned} \tag{11.98}$$

In the first line ψ is regarded as Majorana spinor and in the second line as Weyl spinor. This is just the action of the Wess-Zumino model which we have studied in detail in chapter 5. As we have seen there, we could eliminate the auxiliary fields F_i to arrive at the more familiar action for the on-shell model containing N complex fields F_i and N Majorana spinors ψ_i.

11.5.3 Superspace integration

There in an elegant way to extract the D and F terms from a superfield which is based on an integration calculus on superspace. We begin with the Berezin integral for a single Grassmann parameter θ:

$$\int d\theta\, \theta = 1 \quad , \quad \int d\theta = 0 \quad , \quad \int d\theta\, f(\theta) = f_1 \tag{11.99}$$

where we have used the fact that an arbitrary function of a single Grassmann parameter θ has the Taylor series expansion $f(\theta) = f_0 + \theta f_1$. We demand that the $d\theta$ anticommute,

$$\{d\theta, d\theta\} = \{d\theta, \theta\} = 0.$$

We note three facts which follow from the definitions in (11.99).

- The Berezin integration is translationally invariant:

$$\int d(\theta+\xi)\, f(\theta+\xi) = \int d\theta\, f(\theta) \quad , \quad \int d\theta\, \frac{d}{d\theta} f(\theta) = 0. \tag{11.100}$$

- The Berezin integration is equivalent to differentiation:

$$\frac{d}{d\theta} f(\theta) = f_1 = \int d\theta\, f(\theta) \tag{11.101}$$

- We can define a Grassmann delta function by

$$\int d\theta f(\theta)\delta(\theta) = f(0) \Longrightarrow \delta(\theta) \equiv \theta \tag{11.102}$$

These results are easily generalized to the case of the N=1 superspace coordinates θ^α, $\bar{\theta}_{\dot\alpha}$: All $\theta_\alpha, \theta_\alpha, d\bar\theta^{\dot\alpha}, \bar\theta^{\dot\alpha}$ anticommute so that for example

$$\int d\theta_1 \int d\theta_2 \theta_1\theta_2 = -\int d\theta_1 (\int d\theta_2\theta_2)\theta_1 = -\int d\theta_1\theta_1 = -1$$

and the integral of an arbitary function

$$f = f^{(0)} + \theta_1 f^{(1)} + \theta_2 f^{(2)} + \theta_1\theta_2 f^{(3)}$$

is equal to

$$\int d\theta_1 d\theta_2 f = -f^{(3)}.$$

The volume elements in superspace are given by

$$d^2\theta = -\tfrac{1}{4} d\theta^\alpha d\theta^\beta \varepsilon_{\alpha\beta} \quad , \quad d^2\bar\theta = -\tfrac{1}{4} d\bar\theta_{\dot\alpha} d\bar\theta_{\dot\beta}\varepsilon^{\dot\alpha\dot\beta} \quad , \quad d^4\theta = d^2\theta\, d^2\bar\theta. \tag{11.103}$$

Using this notation and the spinor summation convention, we have the following identities:

$$\int d^2\theta\, \theta\theta = 1 \quad , \quad \int d^2\bar\theta\, \bar\theta\bar\theta = 1. \tag{11.104}$$

We prove the second to last identity:

$$\int d^2\theta\, \theta\theta = -\tfrac{1}{4}\int d\theta^\alpha d\theta^\beta \varepsilon_{\alpha\beta}\, \theta\theta = \tfrac{1}{2}\int d\theta^\alpha d\theta^\beta \varepsilon_{\alpha\beta}\theta^1\theta^2 = -\int d\theta^1 d\theta^2 \theta^1\theta^2 = 1.$$

The delta-function is defined by

$$\int d^2\theta f(\theta)\delta^2(\theta) = f(0) \Longrightarrow \delta^2(\theta) = \theta\theta \quad \text{and similarly} \quad \delta^2(\bar\theta) = \bar\theta\bar\theta.$$

Now we consider a general superfield with component expansion given by (11.25). If we integrate Φ with $d^4\theta$ we just obtain the D-component of the superfield,

$$\begin{aligned}\int d^4\theta \Phi(x,\theta,\bar\theta) &= \int d^4\theta \Big(f(x) + \theta\phi(x) + \bar\theta\bar\chi(x) + \theta\theta m(x) + \bar\theta\bar\theta n(x) + (\theta\sigma^\mu\bar\theta)\, V_\mu(x) \\ &\qquad + (\theta\theta)(\bar\theta\bar\lambda(x)) + (\bar\theta\bar\theta)(\theta\psi(x)) + (\theta\theta)(\bar\theta\bar\theta)\, D(x)\Big) \\ &= \int d^2\theta d^2\bar\theta (\theta\theta)(\bar\theta\bar\theta) D(x) = d(x).\end{aligned}$$

Analogouly, for a chiral field (11.85) we may project onto the F component as follows:

$$\int d^4\theta \Phi(y,\theta)\delta(\bar\theta) = \int d^2\theta \Phi(x,\theta) = F(x).$$

This shows, that the action of a chiral field giving rise to a renormalizable model can be written as follows

$$S = \int d^4x\, d^4\theta\, \Phi_i^\dagger \Phi_i + \int d^4x\, d^4\theta \delta^2(\bar\theta) \left(g_i\Phi_i + \tfrac{1}{2} m_{ij}\Phi_i\Phi_j + \tfrac{1}{3}\lambda_{ijk}\Phi_i\Phi_j\Phi_k + \text{h.c.}\right) \tag{11.105}$$

11.6 Vector-superfields and susy-gauge transformations

After having constructed the supersymmetric Lagrangian describing spin-0 and spin-1/2 particles we want to construct models containing spin-1 particles. Ultimately we are interested in supersymmetric Yang-Mills theories in the superfield formulation. In component fields we have already investigated these theories earlier on.
We start with a real superfield V satisfying the reality condition

$$V(x,\theta,\bar{\theta}) = V^\dagger(x,\theta,\bar{\theta}) \tag{11.106}$$

Earlier we have shown that such a field has the expansion (11.42). It is convenient to make the replacements

$$\lambda \longrightarrow \lambda + \tfrac{i}{2}\partial\!\!\!/\phi \quad \text{and} \quad D \longrightarrow D - \tfrac{1}{4}\Box C$$

in that formula, such that

$$\begin{aligned} V(x,\alpha) \;&=\; C + \bar{\alpha}\phi + \tfrac{1}{2}\bar{\alpha}\alpha M_1 + \tfrac{i}{2}\bar{\alpha}\gamma_5\alpha M_2 + \tfrac{1}{2}(\bar{\alpha}\gamma^\mu\gamma_5\alpha)V_\mu \\ &\quad +(\bar{\alpha}\alpha)\bar{\alpha}(\lambda + \tfrac{i}{2}\partial\!\!\!/\phi) + \tfrac{1}{2}(\bar{\alpha}\alpha)^2(D - \tfrac{1}{4}\Box C) \end{aligned} \tag{11.107}$$

with real C, D, M_1, M_2 and V_μ and Majorana spinors ϕ and λ. The component V_μ lends its name to the entire multiplet. Again the D term of V transforms into a spacetime derivative, see (11.43),

$$\delta D = \tfrac{i}{2}\bar{\beta}\partial\!\!\!/\lambda$$

which makes it a good candidate for the Lagrangian density giving rise to an invariant action. Examples of vector-superfields are

$$V = \Phi^\dagger\Phi, \quad V = \Phi + \Phi^\dagger,$$

where Φ is a chiral superfield. For Φ in (11.82) we find

$$\begin{aligned} \Phi + \Phi^\dagger \;&=\; A_1 + \sqrt{2}\bar{\alpha}\psi + \tfrac{1}{2}\bar{\alpha}\alpha F_1 + \tfrac{i}{2}\bar{\alpha}\gamma_5\alpha F_2 + \tfrac{1}{2}(\bar{\alpha}\gamma^\mu\gamma_5\alpha)\partial_\mu A_2 \\ &\quad + \frac{i}{\sqrt{2}}(\bar{\alpha}\alpha)(\bar{\alpha}\partial\!\!\!/\psi) - \tfrac{1}{8}(\bar{\alpha}\alpha)^2\Box A_1 \end{aligned} \tag{11.108}$$

where we introduced the real fields

$$A_1 = A + A^*, \quad A_2 = i(A - A^*), \quad F_1 = F + F^* \quad \text{and} \quad F_2 = i(F - F^*).$$

If we now transform the real superfield as

$$V(x,\alpha) \longrightarrow V'(x,\alpha) = V(x,\alpha) + \Phi(x,\alpha) + \Phi^\dagger(x,\alpha) \tag{11.109}$$

then the component field V_μ is gauge tansformed with gauge parameter A_2:

$$V'_\mu = V_\mu + \partial_\mu A_2. \tag{11.110}$$

Following Wess and Zumino [55] one therefore calls the transformation (11.109) the supersymmetric extension of a gauge transformation. Under this transformation the components of V transform as

$$\begin{aligned} &C' = C + A_1, \quad M_1' = M_1 + F_1, \quad M_2' = M_2 + F_2, \quad D' = D \\ &V_\mu' = V_\mu + \partial_\mu A_2, \quad \phi' = \phi + \sqrt{2}\psi, \quad \lambda' = \lambda \end{aligned} \tag{11.111}$$

We see that the λ and D components of V are super-gauge invariant, that is they are invariant under the transformation (11.109). As we already mentioned above, the field V_μ transforms as an abelian gauge potential so that the corresponding field strength

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$$

is also super-gauge invariant. The important conclusion is the following: the D term of a real superfield transforms into a spacetime derivative under supersymmetry transformations and at the same time is super-gauge invariant. Thus this term is a good candidate for a supersymmetric Lagrangian which is super-gauge invariant.

11.6.1 The Wess-Zumino gauge

To achieve the socalled Wess-Zumino gauge one chooses the chiral superfield in (11.109) such that

$$C' = M_1' = M_2' = \phi' = 0.$$

This gauge can always be attained as is easily seen from the transformation rules (11.111) for the component fields. This is not a complete gauge fixing, since we do not fix A_2 wich causes the gauge transformation of V_μ. In this gauge the real superfield simplifies considerably,

$$\begin{aligned} V_{WZ}(x,\alpha) &= \tfrac{1}{2}(\bar\alpha\gamma^\mu\gamma_5\alpha)V_\mu + (\bar\alpha\alpha)(\bar\alpha\lambda) + \tfrac{1}{2}(\bar\alpha\alpha)^2 D \\ &= (\theta\sigma^\mu\bar\theta)V_\mu + (\theta\theta)(\bar\theta\bar\lambda) + (\bar\theta\bar\theta)(\theta\lambda) + (\theta\theta)(\bar\theta\bar\theta)D \end{aligned} \tag{11.112}$$

and contains the gauge potentail V_μ, its supersymmetric partner λ and an auxiliary field D. After this partial gauge fixing the number of degrees of freedom reduces from $8+8$ to $4+4$. Recalling the supersymmetry transformations (11.43) of the components of a real superfield we see immediately that the Wess-Zumino gauge breaks supersymmetry. For example, $M_i = 0$ is not left invariant under susy transformations. When one chooses the Wess-Zumino gauge and performes a supersymmetry transformation then one needs a compensating gauge transformation to bring the superfield back into the Wess-Zumino gauge.
For constructing sensible actions out of V we need the exponential of V_{WZ}. Due to the anticommuting character of α the only non-vanishing power (bigger one) is two,

$$V_{WZ}^2 = \tfrac{1}{4}(\bar\alpha\gamma^\mu\gamma_5\alpha)(\bar\alpha\gamma^\nu\gamma_5\alpha)V_\mu V_\nu = \tfrac{1}{4}(\bar\alpha\alpha)^2 V_\mu V^\mu.$$

Then we have

$$\begin{aligned}\exp(V_{WZ}) &= 1+\tfrac{1}{2}(\bar\alpha\gamma^\mu\gamma_5\alpha)V_\mu+(\bar\alpha\alpha)(\bar\alpha\lambda)+\tfrac{1}{2}(\bar\alpha\alpha)^2(D+\tfrac{1}{4}V_\mu V^\mu)\\ &= 1+(\theta\sigma^\mu\bar\theta)V_\mu+(\theta\theta)(\bar\theta\bar\lambda)+(\bar\theta\bar\theta)(\theta\lambda)+(\theta\theta)(\bar\theta\bar\theta)(D+\tfrac{1}{4}V_\mu V^\mu)\end{aligned} \tag{11.113}$$

We continue by defining the analog of a field strength for a general real superfield by

$$W_\alpha=-\tfrac{1}{4}(\bar D\bar D)D_\alpha V \quad\text{and}\quad \bar W_{\dot\alpha}=-\tfrac{1}{4}(DD)\bar D_{\dot\alpha}V, \tag{11.114}$$

where we switched to the Weyl basis. Since $D^3=\bar D^3=0$ these fields are chiral or antichiral. More precisely, W_α is a chiral $(1/2,0)$ field and $\bar W_{\dot\alpha}$ is a antichiral $(0,1/2)$ field. Less obvious is the susy-gauge invariance of these field strengths. Since $D_\alpha\Phi^\dagger=0$ we have

$$W'_\alpha\equiv-\tfrac{1}{4}(\bar D\bar D)D_\alpha V'=-\tfrac{1}{4}(\bar D\bar D)D_\alpha V-\tfrac{1}{4}(\bar D\bar D)D_\alpha\Phi.$$

Since Φ is chiral, $\bar D_{\dot\alpha}\Phi=0$, the last term vanishes,

$$\bar D_{\dot\alpha}\bar D^{\dot\alpha}D_\alpha\Phi=\bar D_{\dot\alpha}\{\bar D^{\dot\alpha},D_\alpha\}\Phi=-2i\varepsilon^{\dot\alpha\dot\beta}(\sigma^\mu)_{\alpha\dot\beta}\bar D_{\dot\alpha}\partial_\mu\Phi=-2i\varepsilon^{\dot\alpha\dot\beta}(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu\bar D_{\dot\alpha}\Phi=0.$$

We used the anticommutation relation (11.53,11.52). This proves that W_α is susy-gauge invariant. The invariance of $\bar W_{\dot\alpha}$ is proven very similarly.
The explicit expression for these field strength is rather involved. I refer to [49] for a detailed derviation. One finds

$$\begin{aligned}W_\alpha &= \lambda_\alpha(y)+2D(y)\theta_\alpha+\tfrac{i}{2}(\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y)-i(\theta\theta)(\sigma^\mu\partial_\mu\bar\lambda(y))_\alpha\\ \bar W^{\dot\alpha} &= \bar\lambda^{\dot\alpha}(z)+2D(z)\bar\theta^\alpha+\tfrac{i}{2}(\tilde\sigma^{\mu\nu}\bar\theta)^{\dot\alpha}F_{\mu\nu}(z)-i(\bar\theta\bar\theta)(\tilde\sigma^\mu\partial_\mu\lambda(z))^{\dot\alpha},\end{aligned} \tag{11.115}$$

where the $\sigma^{\mu\nu}$ and $\tilde\sigma^{\mu\nu}$ have been defined in (8.2). Note that these (anti)chiral superfields only contain the gauge invariant component fields D,λ and $F_{\mu\nu}$. Furthermore, with (11.62) if follows immediately, that

$$\bar D^\alpha W_\alpha=\bar D_{\dot\alpha}\bar W^{\dot\alpha} \tag{11.116}$$

holds true.

11.6.2 Invariant actions

We have seen that the field strengths W_α and $\bar W^{\dot\alpha}$ are (anti) chiral and gauge invariant spinorial superfields which are linear in the field strength tensor $F_{\mu\nu}$. Then $W^\alpha W_\alpha$ and $\bar W_{\dot\alpha}\bar W^{\dot\alpha}$ are (anti)chiral and gauge invariant scalar superfields which are quadratic in $F_{\mu\nu}$. Hence, their F terms are good candidates for gauge and Lorentz invariant Lagrangian densities for a supersymmetric gauge theory. First observe that

$$\begin{aligned}W^\alpha W_\alpha &= \theta\theta(4D^2-2i\lambda\sigma^\mu\partial_\mu\bar\lambda)+2i\theta\sigma^{\mu\nu}\theta F_{\mu\nu}D-\tfrac{1}{4}(\sigma^{\mu\nu}\theta)^\sigma(\sigma^{\alpha\beta}\theta)_\sigma F_{\mu\nu}F_{\alpha\beta}+\dots\\ \bar W_{\dot\alpha}\bar W^{\dot\alpha} &= \bar\theta\bar\theta(4D^2-2i\bar\lambda\tilde\sigma^\mu\partial_\mu\lambda)+2i\bar\theta\tilde\sigma^{\mu\nu}\bar\theta F_{\mu\nu}D-\tfrac{1}{4}(\tilde\sigma^{\mu\nu}\bar\theta)_{\dot\sigma}(\tilde\sigma^{\alpha\beta}\bar\theta)^{\dot\sigma}F_{\mu\nu}F_{\alpha\beta}+\dots,\end{aligned}$$

where … stands for terms which do not contribute to the F-terms. Now we use that

$$\begin{aligned}
\theta\sigma^{\mu\nu}\theta+\bar\theta\tilde\sigma^{\mu\nu}\bar\theta &= \bar\alpha\gamma^{\mu\nu}\alpha=0\\
(\sigma^{\mu\nu}\theta)(\sigma^{\alpha\beta}\theta)+(\tilde\sigma^{\mu\nu}\bar\theta)(\tilde\sigma^{\alpha\beta}\bar\theta) &= \alpha^\dagger\gamma^{\mu\nu}\gamma^0\gamma^{\alpha\beta}\alpha=-\bar\alpha\gamma^{\mu\nu}\gamma^{\alpha\beta}\alpha\\
=\theta(\eta^{\mu\alpha}\eta^{\nu\beta}-\eta^{\mu\beta}\eta^{\nu\alpha}+i\epsilon^{\mu\nu\alpha\beta})\theta &+ \bar\theta(\eta^{\mu\alpha}\eta^{\nu\beta}-\eta^{\mu\beta}\eta^{\nu\alpha}-i\epsilon^{\mu\nu\alpha\beta})\bar\theta
\end{aligned}$$

For the sum of the F-terms we obtain

$$\begin{aligned}
W^\alpha W_\alpha|_F+\bar W_{\dot\alpha}\bar W^{\dot\alpha}|_F &= 8D^2-2i\lambda\sigma^\mu\partial_\mu\bar\lambda-2i\bar\lambda\tilde\sigma^\mu\partial_\mu\lambda-F_{\mu\nu}F^{\mu\nu}\\
&= 8D^2-2i\bar\lambda\gamma^\mu\partial_\mu\lambda-F_{\mu\nu}F^{\mu\nu} \qquad (11.117)
\end{aligned}$$

This is the action for a pure supersymmetric abelian gauge theory with an uncharged gaugino spinor λ. The field $D(x)$ is an auxiliary field which can be eliminated with the help of the equations of motion, see section 8.
We may rewrite the action belonging to (11.114) as superspace integral as follows

$$\begin{aligned}
S &= \tfrac14\int d^4x\int d^4\theta\Big(W^\alpha W_\alpha\delta^2(\bar\theta)+\bar W_{\dot\alpha}\bar W^{\dot\alpha}\delta^2(\theta)\Big)\\
&= \int d^4x(2D^2-\tfrac{i}{2}\bar\lambda\gamma^\mu\partial_\mu\lambda-\tfrac14F_{\mu\nu}F^{\mu\nu}). \qquad (11.118)
\end{aligned}$$

11.6.3 Minimal coupling

We study the simple gauge group $U(1)$. The gauge transformation of a scalar superfield under global $U(1)$ transformations is

$$\Phi_i'=e^{iq_i\Lambda}\Phi_i, \qquad (11.119)$$

where q_i is the $U(1)$-charge of the superfield Φ_i. Clearly, the kinetic term $\Phi_i^\dagger\Phi_i|_D$ is invariant under these global phase transformations. However, a general superpotential $W(\Phi)$ in (11.96) will not be invariant. For W to be invariant we must demand

$$\begin{aligned}
g_i&=0 &&\text{if}\quad q_i\neq0\\
m_{ij}&=0 &&\text{if}\quad q_i+q_j\neq0 \qquad (11.120)\\
\lambda_{ijk}&=0 &&\text{if}\quad q_i+q_j+q_k\neq0.
\end{aligned}$$

Next we wish to make this global symmetry local in which case the gauge parameter λ in (11.119) will depend on spacetime. But then the transformed Φ will in general not be a chiral superfield. For Φ' to be chiral if Φ is chiral we must demand that λ is a chiral superfield, so that

$$\Phi_i'=e^{iq_i\Lambda}\Phi_i,\quad \bar D_{\dot\alpha}\Lambda=0\quad\Longrightarrow\quad \Phi_i'^{\dagger}=\Phi_i^\dagger e^{-iq_i\Lambda^\dagger},\quad D^\alpha\Lambda^\dagger=0. \qquad (11.121)$$

The kinetic term is not invariant under these transformations of the chiral and antichiral superfields. But we have already argued that the transformations

$$V(x,\theta,\bar\theta)\longrightarrow V'(x,\theta,\bar\theta)-i(\Lambda(x,\theta,\bar\theta)-\Lambda^\dagger(x,\theta,\bar\theta))$$

are generalizations of the gauge transformation and are called susy gauge transformations, see (11.109) with $\lambda = i\Phi$. One sees at once that

$$\mathcal{L}_{kin} = \Phi_i^\dagger e^{q_i V}\Phi_i|_D \tag{11.122}$$

is gauge invariant. The coupling (11.122) of the matter superfield Φ to the gauge potential V is the supersymmetric extension of the principle of minimal coupling in ordinary gauge theories as we shall demonstrate now. This is most easily done in the Wess-Zumino gauge. With (11.113) we obtain

$$\begin{aligned}\Phi^\dagger e^{qV}\Phi|_D &= \Phi^\dagger\Phi|_D + q(\theta\sigma^\mu\bar\theta V_\mu \Phi^\dagger\Phi)|_D + q(\theta\theta\bar\theta\bar\lambda(x)\Phi^\dagger\Phi)|_D\\ &+ q(\bar\theta\bar\theta\theta\lambda(x)\Phi^\dagger\Phi)|_D + q\{\theta\theta\bar\theta\bar\theta(D(x) + \tfrac{q}{4}V_\mu(x)V^\mu(x))\Phi^\dagger\Phi\}|_D,\end{aligned} \tag{11.123}$$

where $\Phi = \Phi(y,\theta)$ and $\Phi^\dagger = \Phi^\dagger(z,\bar\theta)$ are the chiral and antichiral fields in (11.89). The first term on the right is the kinetic term without coupling to the gauge field and has been calculated in (11.93). To calculate the remaining terms we use (11.91) and the Fierz identies (4.88). The second term can be rewritten as

$$\begin{aligned}q(\theta\sigma^\mu\bar\theta V_\mu\Phi^\dagger\Phi)|_D &= q(\theta\sigma^\mu\bar\theta V_\mu)(iA^*\bar\theta\sigma^\nu\theta\partial_\nu A - i\theta\sigma^\nu\bar\theta\partial_\nu A^*A + 2\bar\theta\bar\psi\theta\psi)|_D\\ &= \tfrac{1}{2}q\{i(A^*\partial_\mu A - \partial_\mu A^*A) - \bar\psi\tilde\sigma^\mu\psi\}V_\mu.\end{aligned}$$

The third term on the right becomes

$$q(\theta\theta\,\bar\theta\bar\lambda(x)\,\Phi^\dagger\Phi)|_D = q\sqrt{2}\theta\theta\,\bar\theta\bar\lambda\,\bar\theta\bar\psi A|_D = -\frac{q}{\sqrt{2}}\,(\bar\lambda\bar\psi)A$$

and analogously the fourth term

$$q(\bar\theta\bar\theta\,\theta\lambda(x)\,\Phi^\dagger\Phi)|_D = q\sqrt{2}\bar\theta\bar\theta\,\theta\lambda\,\theta\psi A^*|_D = -\frac{q}{\sqrt{2}}\,(\lambda\psi)A^*.$$

Finally, the last term in (11.123) is simply

$$q\{\theta\theta\bar\theta\bar\theta(D + \tfrac{q}{4}V_\mu V^\mu)\Phi^\dagger\Phi\}|_D = q(D + \tfrac{q}{4}V_\mu V^\mu)A^*A.$$

Adding up the various terms we obtain the following gauge invariant kinetic term (in the WZ-gauge):

$$\begin{aligned}\Phi^\dagger e^{qV}\Phi|_D &= \partial_\mu A^*\partial^\mu A - \tfrac{1}{4}\Box(A^*A) + \tfrac{i}{2}q(A^*\partial_\mu A - \partial_\mu A^*A)V_\mu + \frac{q^2}{4}V_\mu V^\mu A^*A\\ &+ \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi - \tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi - \tfrac{1}{2}q\bar\psi\tilde\sigma^\mu\psi V_\mu + |F|^2 - \frac{q}{\sqrt{2}}(\bar\lambda\bar\psi A + \lambda\psi A^*) + qD|A|^2\end{aligned}$$

After introducing the covariant derivatives

$$D_\mu A = (\partial_\mu - \tfrac{i}{2}qV_\mu)A \quad\text{and}\quad D_\mu\psi = (\partial_\mu - \tfrac{i}{2}qV_\mu)\psi \tag{11.124}$$

the gauge invariant kinetic term reads

$$\Phi^\dagger e^{qV}\Phi|_D = (D_\mu A)^*D^\mu A - i\psi\sigma^\mu D_\mu^*\bar\psi + |F|^2 + q|A|^2D - \frac{q}{\sqrt{2}}(\bar\lambda\bar\psi A + \lambda\psi A^*) \tag{11.125}$$

The minimal coupling not only couples the charged field A and its superpartner ψ to the gauge field, it also couples these fields to the gaugino λ which is the superpartner of V_μ.

11.6.4 Super-QED

Here we extend ordinary electrodynamics to a supersymmetric theory. We need at least a photon field and an electron field. Off shell the photon is accompanied by the photino and an auxiliary field and each chirality of the electron by a scalar and an auxiliary field. Hence, besides the gauge multiplett we need two scalar superfields Φ_1 and Φ_2 which transform under $U(1)$ gauge transformations as

$$\Phi_1' = e^{ie\Lambda}\Phi_1 \quad \text{and} \quad \Phi_2' = e^{-ie\Lambda}\Phi_2. \tag{11.126}$$

Later on the two spin-$\frac{1}{2}$ components of these chiral superfields will be identified with the electron. Gauge invariance requires the condition (11.120) on the constants, which in the present case read

$$g_1 = g_2 = 0, \quad m_{11} = m_{22} = 0, \quad \lambda_{ijk} = 0 \quad \text{and} \quad m_{12} = m_{21} \equiv m \neq 0.$$

By using this contraints we obtain the following supersymmetric and $U(1)$-invariant action

$$\begin{aligned} S_{SED} &= \int d^4x \int d^4\theta \Big(\tfrac{1}{4} W^\alpha W_\alpha \delta^2(\bar\theta) + \bar W_{\dot\alpha} \bar W^{\dot\alpha} \delta^2(\theta) + \Phi_1^\dagger e^{eV}\Phi_1 + \Phi_2^\dagger e^{-eV}\Phi_2 \\ &\qquad - m\Phi_1\Phi_2\delta^2(\bar\theta) - m\Phi_1^\dagger\Phi_2^\dagger\delta^2(\theta)\Big) \end{aligned} \tag{11.127}$$

Note that each term is separately supersymmetric and gauge invariant. Using the previous results (11.118,11.125,11.93) we may rewrite this action in terms of the component fields and obtain

$$\begin{aligned} S_{SED} &= \int d^4x \Big(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{i}{2}\bar\lambda\gamma^\mu\partial_\mu\lambda + 2D^2 + |F_1|^2 + |F_2|^2 + |D_\mu A_1|^2 + |D_\mu A_2|^2 \\ &\qquad - i\psi_1\sigma^\mu D_\mu^*\bar\psi_1 - i\psi_2\sigma^\mu D_\mu^*\bar\psi_2 + eD\{|A_1|^2 - |A_2|^2\} \\ &\qquad - 2^{-1/2}e\{\bar\lambda(\bar\psi_1 A_1 - \bar\psi_2 A_2) + \lambda(\psi_1 A_1^* - \psi_2 A_2^*)\} \\ &\qquad - m\{A_1F_2 + A_1^*F_2^* + A_2F_1 + A_2^*F_1^* - \psi_1\psi_2 - \bar\psi_1\bar\psi_2\}\Big) \end{aligned} \tag{11.128}$$

Note that the covariant derivative act differently on the different fields, since the fields in Φ_2 have opposite charge to those in Φ_1. We may eliminate the auxiliary fields by their purely algebraic equations of motions

$$\begin{aligned} F_1: &\quad F_1^* - mA_2 = 0 \\ F_2: &\quad F_2^* - mA_1 = 0 \\ D: &\quad 4D + e(|A_1|^2 - |A_2|^2) = 0. \end{aligned} \tag{11.129}$$

After eliminating the auxiliary fields in S we obtain

$$\begin{aligned} S_{SED} &= \int d^4x \Big(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{i}{2}\bar\lambda\gamma^\mu\partial_\mu\lambda + |D_\mu A_1|^2 + |D_\mu A_2|^2 - m^2(|A_1|^2 + |A_2|^2) \\ &\qquad - i\psi_1\sigma^\mu D_\mu^*\bar\psi_1 - i\psi_2\sigma^\mu D_\mu^*\bar\psi_2 + m(\psi_1\psi_2 + \bar\psi_1\bar\psi_2) \\ &\qquad - \frac{e}{\sqrt{2}}\{\bar\lambda(\bar\psi_1 A_1 - \bar\psi_2 A_2) + \lambda(\psi_1 A_1^* - \psi_2 A_2^*)\} - \frac{e^2}{8}(|A_1|^2 - |A_2|^2)^2\Big). \end{aligned} \tag{11.130}$$

To compare this model with QED it is convenient to introduce the Dirac spinor

$$\psi = \begin{pmatrix} \psi_{1\alpha} \\ \bar{\psi}_2^{\dot{\alpha}} \end{pmatrix} \Longrightarrow \psi^\dagger = (\,\bar{\psi}_{1\dot{\alpha}}, \psi_2^\alpha\,) \quad \text{and} \quad \bar{\psi} = (\,\psi_2^\alpha, \bar{\psi}_{1\dot{\alpha}}\,)\,.$$

Up to surface terms in S the terms containing derivatives of the ψ_i can be rewritten as

$$\begin{aligned}&\psi_1\sigma^\mu D_\mu^*\bar{\psi}_1 + \psi\sigma^\mu D_\mu^*\bar{\psi}_2 = -D_\mu^*\bar{\psi}_{1\dot{\alpha}}\tilde{\sigma}^{\mu\dot{\alpha}\alpha}\psi_{1\alpha} + \psi_2^\alpha\sigma^\mu_{\alpha\dot{\alpha}}D_\mu^*\bar{\psi}_2^{\dot{\alpha}}\\ &= \bar{\psi}_1\tilde{\sigma}^\mu D_\mu\psi_1 + \psi_2\sigma^\mu D_\mu^*\bar{\psi}_2 = \bar{\psi}\not{D}\psi,\end{aligned}$$

where we have used that ψ_1 and ψ_2 have opposite charge,

$$D_\mu\psi_1 = (\partial_\mu - i\tfrac{e}{2}V_\mu)\psi_1 \quad \text{and} \quad D_\mu^*\bar{\psi}_2 = (\partial_\mu - i\tfrac{e}{2}V_\mu)\bar{\psi}_2.$$

We conclude that the second line in (11.130) may be written as

$$-\bar{\psi}(i\not{D} - m)\psi \tag{11.131}$$

which is just the Lagrangian for a massive electron field of mass m and charge $-e$, minimally coupled to the abelian gauge potential V_μ.

11.6.5 Fayet-Ilioopulos term

We could add the supersymmetric and gauge invariant Fayet-Iliopoulos term [56]

$$S_{FI} = \kappa\int d^4x d^4\,\theta V = \kappa\int D \tag{11.132}$$

to the action S_{SED} in (11.127) with the result that it induces spontaneous supersymmetry breaking. The whole construction has been generalized to the non-Abelian case in [57]. Adding S_{FI} to S_{SED} changes the third equation for the auxiliary field D in (11.129) into

$$4D + \kappa + e(|A_1|^2 + |A_2|^2) = 0.$$

After eliminating the auxiliary fields we again obtain the action (11.130), wherein the last term is replaced by

$$-\frac{e^2}{8}(|A_1|^2 - |A_2|^2)^2 \longrightarrow -\frac{1}{8}(e|A_1|^2 - e|A_2|^2 + \kappa)^2.$$

Let us have a closer look at the scalar potential of this socalled *Fayet-Iliopoulos* model. It is

$$\begin{aligned}V(A_1, A_2) &= m^2(|A_1|^2 + |A_2|^2) + \frac{1}{8}(e|A_1|^2 - e|A_2|^2 + \kappa)^2)\\ &= \frac{\kappa^2}{8} + (m^2 + \tfrac{1}{4}e\kappa)|A_1|^2 + (m^2 - \tfrac{1}{4}e\kappa)|A_2|^2 + \frac{e^2}{8}(|A_1|^2 - |A_2|^2)^2.\end{aligned}$$

Now we must distinguish between two cases.

• $4m^2 > e\kappa$:
In this case all coefficients of the quadratic and quartic terms are positive and the absolute minimum of V is attained at

$$A_1 = A_2 = 0 \Rightarrow V_{\min} = \frac{\kappa}{8} > 0, \quad D = -\frac{\kappa}{4},\ F_i = 0,$$

indicating that supersymmetry is broken. Both scalar fields have real masses and vanishing vacuum expectation values which shows that (at the classical level) the $U(1)$-gauge symmetry is unbroken. The model contains the following particles

$$\begin{array}{ll} A_1: \quad M_1^2 = m^2 + \frac{1}{4}e\kappa, & A_2: \quad M_2^2 = m^2 - \frac{1}{4}e\kappa \\ \psi = (\psi_1, \psi_2): \quad M = m & \\ \lambda, V_\mu: \quad \text{massless.} & \end{array}$$

The mass splitting inside the matter multiplett establishes that supersymmetry has been broken. Note that only the mass of the scalars is splitted by the supersymmetry breaking. The massless photino field λ is just the Goldstino field appearing in the process of supersymmetry breaking
• $4m^2 < e\kappa$:
Now the minimum of the potential is at a finite value of the matter fields,

$$|A_1| = 0 \quad \text{and} \quad |A_2|^2 \equiv v^2 = \frac{e\kappa - 4m^2}{e^2} \Longrightarrow V(0, v) = \frac{m^2}{e^2}(e\kappa - 2m^2) > 0.$$

This indicates that both supersymmetry and gauge symmetry are spontaneously broken. To study the mass spectrum we expand the Lagrangian about the minimum. For that we shift

$$A_2(x) \longrightarrow v + A_2(x).$$

We insert this into the Lagrangian and expand up to second order... (comes later).

11.7 Non-Abelian Gauge Theories

To be written

11.8 Spontaneous Symmetrybreaking

To be written

Kapitel 12

KK-Reduction of Wess-Zumino model to 2 dimensions

As an exercise in getting familiar with the Kaluza-Klein reduction, I shall show how the 4 dimensionsional Wess-Zumino-model is reduced to an two-dimensional model with extended supersymmetry.

12.1 Reduction to the $N = 2$ Lorentzian model

In 4 dimension a Majorana spinor has 4 real components which are two times the 2 real components of Majorana spinor in 2 dimensions (which exist for both signatures $(+,-)$ and $(+,+)$).
We start with the Dirac matrices γ^μ in 2-dimensional Minkowski space and give an explicit realization for the matrices $\Gamma^m, m = 0, \ldots, 3$ in 4 dimensions: We make the ansatz

$$\Gamma_\mu = \Delta \otimes \gamma_\mu, \quad \Gamma_{1+a} = \Delta_a \otimes \gamma_*, \quad \mu = 0, 1, \quad a = 1, 2, \qquad \gamma_* = -\gamma_0\gamma_1$$

The condition on the Δ-factors such that

$$\{\Gamma_m, \Gamma_n\} = 2\eta_{mn}, \qquad \eta = \mathrm{diag}(1, -1, -1, -1) \tag{12.1}$$

is easily seen to read

$$\Delta^2 = \mathbb{1}_2, \quad [\Delta, \Delta_a] = 0, \qquad \{\Delta_a, \Delta_b\} = -2\delta_{ab}\mathbb{1}_2.$$

Since Γ^0 must be hermitean and the $\Gamma^{m>0}$ antihermitean and since $\Gamma_5 = -i\gamma_0\gamma_1\gamma_2\gamma_3$ is hermitean it also follows that

$$\Delta^\dagger = \Delta, \quad \Delta_a^\dagger = -\Delta_a.$$

Since Δ commutes with all matrices we may choose it to be the identity,

$$\Delta = \mathbb{1}_2.$$

Note that the hermitean $i\Delta_a$ generate the Euclidean Clifford algebra in 2 dimensions and that the $[\Delta_a, \Delta_b]$ generate the group $spin(2)$. Earlier we have shown that in 2 dimensions there is a Majorana representation. Hence we may choose Δ_a to be imaginary and hence symmetric. To construct this Majorana representation we take

$$\Delta_1 = i\tau_1 \quad \text{and} \quad \Delta_2 = i\tau_3,$$

With our earlier convention the hermitean Γ_5 takes the form

$$\Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = -i\Delta_1\Delta_2 \otimes \gamma_0\gamma_1 = -\tau_2 \otimes \gamma_*. \tag{12.2}$$

We choose a Majorana representation for the 2-dimensional γ_μ such that they are imaginary (see section 3.5) and $\gamma_* = -\gamma_0\gamma_1$ is real. Since the Δ_a are imaginary it follows that the Γ_m are purely imaginary as well and that $\mathcal{C}_4 = -\Gamma_0 = -\mathbb{1}_2 \otimes \gamma_0$. Hence we are lead to take the following relation between the conjugation matrices in 4 and 2 dimensions,

$$\mathcal{C}_4 = -\Gamma_0 = \mathbb{1}_2 \otimes \mathcal{C}_2, \tag{12.3}$$

where $\mathcal{C}_2 = -\gamma_0$ is the antisymmetric charge conjugation matrix in 2 dimensions. We have

$$\mathcal{C}_4^{-1}\Gamma_m\mathcal{C}_4 = -\Gamma_m^T, \qquad \mathcal{C}_2^{-1}\gamma_\mu\mathcal{C}_2 = -\gamma_\mu^T. \tag{12.4}$$

Now its easy to reduce a Majorana spinor, since the Majorana condition reads

$$\Psi = \xi \otimes \chi = \Psi_c = \Psi^* \Longleftrightarrow \xi \in \mathbb{R}^2,\ \chi = \chi_c = \chi^*, \tag{12.5}$$

and an arbitrary Majorana spinor in 4 space-time dimensions has the expansion

$$\Psi = \sum_{r=1}^{2} e_r \otimes \chi_r, \tag{12.6}$$

where the χ_r are Majorana spinors in 2 dimensions and the e_r form a (real) base in $\mathbb{R}^2$.

12.1.1 Reduction of the $4d$ action

In 4 and 2 dimensions a spinor field has the dimension

$$[\Psi] = L^{-3/2} \quad \text{and} \quad [\chi] = L^{-1/2},$$

respectively. We start with the expansion (12.6) for a Majorana spinor and its adjoint in 4 dimensions. We rescale the spinors such that the χ_r in

$$\Psi = \frac{1}{\sqrt{V_2}}\sum_{r=1}^{2} e_r \otimes \chi_r, \quad \bar{\Psi} = \frac{1}{\sqrt{V_2}}\sum_{r=1}^{2} e_r^T \otimes \bar{\chi}_r$$

have the dimension of a spinorfield in 2-dimensional Minkowski spacetime. The spinor should be independent of the internal coordinates which are the coordinates x^2 and x^3. We obtain for the fermionic bilinears

$$\begin{aligned}\int d^4x\,\bar\Psi\Gamma^m\partial_m\Psi &= \sum_r\int d^2x\,\bar\chi_r\partial\!\!\!/\chi_r\\ \int d^4x\,\bar\Psi\Psi &= \sum_r\int d^2x\,\bar\chi_r\chi_r\\ \int d^4x\,\bar\Psi\Gamma_5\Psi &= -\int d^2x\,(\tau_2)_{rs}\,\bar\chi_r\gamma_*\chi_s\end{aligned}$$

Now we are ready to reduce the Wess-Zumino model with Lagrangian

$$\begin{aligned}\mathcal{L}_4 &= \tfrac12\partial_m A\partial^m A+\tfrac12\partial_m B\partial^m B-\tfrac12 m^2(A^2+B^2)+\tfrac{i}{2}\bar\Psi\Gamma^m\partial_m\Psi-\tfrac12 m\bar\Psi\Psi\\ &\quad -mgA(A^2+B^2)-\tfrac12 g^2(A^2+B^2)^2-g\bar\Psi(A-i\Gamma_5 B)\Psi \qquad (12.7)\end{aligned}$$

to two dimensions. For that we observe that a scalar field in $d=4$ has dimension L^{-1}, whereas in $d=2$ it has no dimension. Hence we make the replacements

$$A\longrightarrow\frac{1}{\sqrt{V_2}}A\quad\text{and}\quad B\longrightarrow\frac{1}{\sqrt{V_2}}B.$$

If we finally relate the coupling constants in 4 and 2 dimensions as

$$g_2=\frac{g_4}{\sqrt{V_2}},$$

then the action of the 2 dimensional model, given by

$$S_4=\int d^4x\mathcal{L}_4=S_2=\int d^2x\mathcal{L}_2$$

reads

$$\begin{aligned}\mathcal{L}_2 &= \tfrac12\partial_\mu A\partial^\mu A+\tfrac12\partial_\mu B\partial^\mu B-\tfrac12 m^2(A^2+B^2)+\tfrac{i}{2}\bar\chi\partial\!\!\!/\chi-\tfrac12 m\bar\chi\chi\\ &\quad -mgA(A^2+B^2)-\tfrac12 g^2(A^2+B^2)^2-g\bar\chi(A+i\tau_2\otimes\gamma_* B)\chi. \qquad (12.8)\end{aligned}$$

Here χ is a Majorana spinor with the two components χ_1 and χ_2. These two flavours maybe combined to a Dirac spinor. The first factor in $\tau_2\otimes\gamma_*$ affects the flavour indices. The Hamiltonian density is

$$\begin{aligned}\mathcal{H}_2 &= \tfrac12\Big(\pi_A^2+\pi_B^2+(\nabla A)^2+(\nabla B)^2+m^2A^2+m^2B^2\Big)-\tfrac{i}{2}\chi^\dagger\gamma^0\gamma^i\partial_i\chi+\tfrac12 m\bar\chi\chi\\ &\quad +mgA(A^2+B^2)+\tfrac12 g^2(A^2+B^2)^2+g\bar\chi(A+i\tau_2\otimes\gamma_* B)\chi. \qquad (12.9)\end{aligned}$$

12.1.2 R-symmetry

In 4 dimensions a scalar- and spinorfield transform under Lorentz transformations as

$$A(x)\longrightarrow A(\Lambda_4^{-1}x)\quad\text{and}\quad \Psi(x)\longrightarrow S_4\Psi(\Lambda_4^{-1}x),\quad \bar\Psi(x)\longrightarrow \bar\Psi(\Lambda_4^{-1}x)S_4^{-1},$$

where

$$S_4=\exp\left(\tfrac{1}{2}\omega_{mn}\Gamma^{mn}\right)\quad\text{and}\quad (\Lambda_4)^m_{\ n}=(e^\omega)^m_{\ n},\qquad \Gamma_{mn}=\tfrac{1}{2}[\Gamma_m,\Gamma_n]$$

are the spinrotation and Lorentz transformation with parameter ω_{mn} in 4-dimensional spacetime. They are related via

$$S_4^{-1}\Gamma^m S_4=(\Lambda_4)^m_{\ n}\Gamma^n.$$

The Lagrangian density is as scalar field. When we reduce the theory to $\mathbb{R}^2$ we must require that the fields do not depend on the internal coordinates. Clearly this condition is not compatible with the 4 dimensional Lorentz invariance. Only those Lorentz transformations survive which do not mix the coordinates on $\mathbb{R}^4$ with those in the internal space, hence

$$SO(1,3)\longrightarrow SO(1,1)\times SO(2)\quad\text{or}\quad \Lambda_4\longrightarrow\begin{pmatrix}\Lambda & 0\\ 0 & R\end{pmatrix},$$

where Λ is a 2-dimensional Lorentz transformation and $R\in O(2)$[1]. With our choice for the Γ_m the generators of the corresponding spin transformations in 4 dimensions read

$$\Gamma_{\mu\nu}=\mathbb{1}_2\otimes\gamma_{\mu\nu},\quad \Gamma_{1+a,1+b}=\Delta_{ab}\otimes\mathbb{1}_2\quad\text{and}\quad \Gamma_{\mu,1+a}=2\Delta_a\otimes\gamma_\mu\gamma_*\tag{12.10}$$

The Δ_{ab} generate the spin(2) subalgebra of $spin(1,3)$ and the $\gamma_{\mu\nu}$ the spin(1,1) subalgebra. Since the $\Gamma_{\mu\nu}$ act trivially on the first factor in the decomposition $\Psi=\xi\otimes\chi$, and the $\Gamma_{1+a,1+b}$ trivially on the second one, the spin rotations compatible with the dimensional reduction decompose as follows:

$$S_4=S_2\otimes S,\quad S\Psi=\sum_r S_2e_r\otimes S\chi_r,\qquad S_2^{-1}\Delta^a S_2=R^a_{\ b}\Delta^b,\quad S^{-1}\gamma^\mu S=\Lambda^\mu_{\ \nu}\gamma^\nu,$$

where S is a spin rotation acting on spinors in 2-dimensional Minkowski spacetime. To summarize, the remaining Lorentz transformations reduce to Lorentz transformations on the two dimensional spacetime,

$$S:\ \chi_a(x)\longrightarrow S\chi_a(\Lambda^{-1}x)\quad,\quad A(x)\to A(\Lambda^{-1}x),\quad B(x)\to B(\Lambda^{-1}x),$$

and to a symmetry which does not act on the coordinates of $\mathbb{R}^2$:

$$S_2:e_r\longrightarrow S_2e_r\quad,\quad A(x)\to A(x),\quad B(x)\to B(x)$$

[1] actually, only $\det\Lambda\cdot\det R=1$ is required.

We see that the spin rotations S_2 reduce to *internal symmetries* which rotate the Majorana fields without changing the coordinates (on $\mathbb{R}^2$). With our choice for the Δ_a we have $\Delta_{12} = i\tau_2$ such that

$$S_2 = e^{\omega\Delta_{12}} = e^{i\omega\tau_2} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \otimes \mathbb{1}_2 \equiv R,$$

where we have set $\omega^{23} = \omega$. Hence, spinors transform under the R transformations as

$$\Psi \longrightarrow \sum_r Re_r \otimes \chi_r = \sum_{rs} R_{sr} e_s \otimes \chi_r = \sum_s e_s \otimes (R\chi)_s.$$

Hence we may interpret the R-symmetry to act on the Majorana spinors. We conclude, that the reduced model has the following $U(1)$ invariance,

$$\chi \longrightarrow R\chi = e^{i\alpha\tau_2}\chi.$$

12.1.3 Chiral rotations

To investigate the behaviour under chiral rotations we rewrite the Lagrangian density in terms of the Dirac spinor

$$\psi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2).$$

With the identities

$$\bar\psi\psi = \tfrac{1}{2}\bar\chi\chi, \quad \bar\psi\partial\!\!\!/\psi = \tfrac{1}{2}\bar\chi\partial\!\!\!/\chi + \tfrac{i}{2}\partial_\mu(\bar\chi_1\gamma^\mu\chi_2), \quad \bar\psi\gamma_*\psi = -\tfrac{1}{2}\bar\chi\tau_2\otimes\gamma_*\chi,$$

where we made use of (13.3), the density for the massless model takes the following simple form,

$$\mathcal{L}_2 = \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{1}{2}\partial_\mu B\partial^\mu B + i\bar\psi\partial\!\!\!/\psi - \tfrac{1}{2}g^2(A^2+B^2)^2 - 2g\bar\psi(A - i\gamma_* B)\psi.$$

It is hermitean and possesses the following $U(1)$ chiral symmetry,

$$\psi \longrightarrow e^{i\alpha\gamma_*}\psi, \qquad \bar\psi \longrightarrow \bar\psi e^{i\alpha\gamma_*} \quad \text{and} \quad \begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}.$$

12.1.4 Supersymmetry transformations

To find the susy transformations we dimensionally reduce (6.12) by expanding

$$\Psi = \sum e_r \otimes \chi_r \quad \text{and} \quad \alpha = \sum e_r \otimes \beta_r$$

with two dimensional real supersymmetry parameters β_r. We easily obtain

$$\begin{aligned} \delta A &= \sum \bar\beta_r\chi_r, \qquad \delta B = -i\sum(\tau_2)_{rs}\,\bar\beta_r\gamma_*\chi_s \\ \delta\chi_r &= -(i\partial\!\!\!/ A + g(A^2 - B^2))\beta_r - (\tau_2)_{rs}\gamma^\mu\gamma_*\beta_s\partial_\mu B + 2igAB\,(\tau_2)_{rs}\gamma_*\beta_s. \end{aligned}$$

Again we introduce the Dirac field and $\alpha = (\beta_1 + i\beta_2)/\sqrt{2}$ and obtain

$$\begin{aligned}\delta A &= \bar{\alpha}\psi + \bar{\alpha}_c\psi_c \quad , \quad \delta B = i(\bar{\alpha}\gamma_*\psi - \bar{\alpha}_c\gamma_*\psi_c)\\ \delta\psi &= -\{i\partial\!\!\!/(A + i\gamma_* B) + g(A + i\gamma_* B)^2\}\alpha.\end{aligned}$$

To reduce the susy algebra of the supercharges, we insert

$$Q = \sum e_r \otimes q_r, \quad \alpha = \sum e_r \otimes \beta_r$$

into the 4-dimensional algebra

$$[\bar{Q}\alpha_1, \bar{\alpha}_2 Q] = -2i(\bar{\alpha}_2\Gamma^m\alpha_1)\partial_m \longrightarrow -2i(\bar{\alpha}_2\gamma^\mu\alpha_1)\partial_\mu, \qquad \mu = 0, 1. \tag{12.11}$$

Then the susy algebra reduces to the following $N = 2$ algebra for the Majorana supercharges q_1 and q_2:

$$\{q_{i\alpha}, \bar{q}_j^{\,\beta}\} = 2i\delta_{ij}(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu$$

Rewritten in terms of the Dirac-charges

$$Q_\alpha = \frac{1}{\sqrt{2}}(q_1 + iq_2)$$

this reads

$$\{Q_\alpha, \bar{Q}^\beta\} = 2i(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu.$$

12.2 Reduction to the $2d$ Euclidean model

We reduce the model on 4-dimensional Minkowki spacetime (Γ_m) to a model on 2-dimensional Euclidean space with Euclidean γ-matrices γ_μ and $\gamma_* = -i\gamma_0\gamma_1$. We make the ansatz

$$\Gamma_0 = \Delta_0 \otimes \mathbb{1}_2, \quad \Gamma_1 = i\Delta_1 \otimes \mathbb{1}_2, \quad \Gamma_{2+\mu} = i\Delta_* \otimes \gamma_\mu, \quad \mu = 0, 1, \quad \Delta_* = -i\Delta_0\Delta_1,$$

where the hermitean γ_μ and hermitean Δ_a generate the 2-dimensional Euclidean Clifford algebras,

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}_2 \quad \text{and} \quad \{\Delta_a, \Delta_b\} = 2\delta_{ab}\mathbb{1}_2.$$

With our earlier convention the hermitean Γ_5 reads

$$\Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = -\Delta_0\Delta_1 \otimes \gamma_0\gamma_1 = \Delta_* \otimes \gamma_*. \tag{12.12}$$

A 4-dimensional Majorana spinor has the expansion

$$\Psi = \frac{1}{\sqrt{2}}\sum_{r=1}^{2} e_r \otimes \chi_r, \tag{12.13}$$

where the χ_r are spinors in 2 Euclidean dimensions and the e_r form a (real) base in $\mathbb{R}^2$.

12.2.1 The Majorana conditions

We write the 4-dimensial charge conjugation matrix as

$$\mathcal{C}_4 = a \otimes \mathcal{C}_2, \qquad \mathcal{C}^{-1}\Gamma_m\mathcal{C} = -\Gamma_m^T.$$

Because of the particular form of Γ_0 and Γ_1 the last condition implies for $m = 0, 1$

$$a^{-1}\Delta_a a = -\Delta_a^T \Longrightarrow a^{-1}\Delta_* a = -\Delta_*^T.$$

With the last identity the factor $\mathcal{C}_2$ must then fulfil

$$\mathcal{C}_2^{-1}\gamma_\mu\mathcal{C}_2 = \gamma_\mu^T,$$

which just means, that it is a charge conjugation matrix in an Euclidean spacetime with signature $(+,+)$. In such a spacetime there are *real and symmetric* γ_μ giving rise to an imaginary and antisymmetric γ_*. In this representation we have $\mathcal{C}_2 = \mathbb{1}_2$. To obtain purely imaginary Γ_m in which case $\mathcal{C}_4 = -\Gamma_0$ and the 4-dimensional Majorana spinors become real, we could take

$$\Delta_0 = \tau_2 \quad \text{and} \quad \Delta_1 = \tau_3 \Longrightarrow \Delta_* = \tau_1. \tag{12.14}$$

To summarize we may choose a representation such that all Γ_m are imaginary and

$$\mathcal{C}_4 = -\Gamma_0 = -\tau_2 \otimes \mathbb{1}_2. \tag{12.15}$$

It follows, that Majorana spinors Ψ in $4d$ Minkowski spacetime and Majora spinors χ in $2d$ Euclidean space are both real. The last statement comes from

$$\chi = \chi_c = \mathcal{C}_2(\chi^\dagger)^T = \chi^*.$$

The choosen representation is very convenient, since in the expansion (12.13) all spinors happen to be real Majoran spinors. For the 2-dimensional Majorana spinors we have the identities listed above and below (13.4).

12.2.2 Reduction of $4d$ Wess-Zumino Lagrangian

Again we start with the expansion (12.13) for a Majorana spinor and its Dirac conjugate

$$\bar\Psi = \frac{1}{\sqrt{V_2}}\sum_{r=1}^{2} e_r^T\Delta_0 \otimes \chi_r^\dagger$$

The spinor should be independent of the internal coordinates which are the coordinates x^0 and x^1. We obtain for the fermionic bilinears

$$\begin{aligned}
\int d^4x\, \bar\Psi\Gamma^m\partial_m\Psi &= -\int d^2x\, (\Delta_1)_{rs}\chi_r^\dagger \partial\!\!\!/ \chi_s \\
\int d^4x\, \bar\Psi\Psi &= \int d^2x\, (\Delta_0)_{rs}\chi_r^\dagger\chi_s \\
\int d^4x\, \bar\Psi\Gamma_5\Psi &= -i\int d^2x\, (\Delta_1)_{rs}\, \chi_r^\dagger\gamma_*\chi_s
\end{aligned}$$

Now we are ready to reduce the Wess-Zumino model with Lagrangian (12.7) to two Euclidean dimensions,

$$-\mathcal{L}_2 = \mathcal{L}_B + \mathcal{L}_F \tag{12.16}$$

with

$$\mathcal{L}_B = \tfrac{1}{2}\Big(\partial_\mu A\partial^\mu A + \partial_\mu B\partial^\mu B + m^2(A^2+B^2) + 2mgA(A^2+B^2) + g^2(A^2+B^2)^2\Big) \tag{12.17}$$

and

$$\mathcal{L}_F = \tfrac{i}{2}(\Delta_1)_{rs}\chi_r^\dagger \not{\partial}\chi_s + \tfrac{1}{2}m(\Delta_0)_{rs}\chi_r^\dagger\chi_s + gA(\Delta_0)_{rs}\chi_r^\dagger\chi_s - gB(\Delta_1)_{rs}\chi_r^\dagger\gamma_*\chi_s, \tag{12.18}$$

where we choose the representation $\Delta_0 = \tau_2$ and $\Delta_1 = \tau_1$. With the hermiticity properties worked out in section (13.2), above (13.20) one shows that this gives rise to a hermitean action. For that one uses that Δ_0 is imaginary and Δ_1 real.
To reduce the susy transformations (6.12) we expand the Majorana susy parameter in 4 dimensions similarly as the Majorana spinor field Ψ as follows,

$$\alpha = \sum e_r \otimes \beta_r \quad \text{and} \quad \bar{\alpha} = \sum e_r^T \Delta_0 \otimes \beta_r^\dagger$$

In terms of these Majorana parameters in 2 Euclidean dimensions we obtain

$$\begin{aligned} \delta A &= \beta^\dagger \tau_2 \chi \quad , \quad \delta B = \beta^\dagger \tau_3 \gamma_* \chi \\ \delta\chi &= \{i\gamma_*\not{\partial}B - mA - g(A^2-B^2)\}\beta + i\{i\not{\partial}A - (mB + 2gAB)\gamma_*\}\tau_1\beta. \end{aligned} \tag{12.19}$$

The summations convention used is evident. Later we shall combine the two flavours χ_1 and χ_2 to a Dirac spinor.

12.2.3 R-symmetry

We reduce the theory to $\mathbb{R}^2$ by requireing that the fields do not depend on the internal coordinates which we choose to be x^0 and x^1.

$$SO(1,3) \longrightarrow SO(1,1)\times SO(2) \quad \text{or} \quad \Lambda_4 \longrightarrow \begin{pmatrix} \Lambda & 0 \\ 0 & R \end{pmatrix},$$

where Λ is now considered as internal symmetry and and $R \in O(2)$ as Lorentz transformation in the Euclidean 2-space. With our choice for the Γ_m the generators of the corresponding spin transformations in 4 dimensions read

$$\Gamma_{01} = -\Delta_* \otimes \mathbb{1}_2, \quad \Gamma_{23} = -\mathbb{1}_2 \otimes \gamma_{01} = -i\mathbb{1}_2 \otimes \gamma_*. \tag{12.20}$$

They yield the noncompact internal transformations $R = \exp(\omega_{01}\Delta_*)\otimes\mathbb{1}_2$ and the compact spinor transformations $S_2 = \mathbb{1}_2 \otimes \exp(-i\omega_{23}\gamma_*)$. The internal symmetry is ($\omega = \omega_{01}$)

$$\chi \longrightarrow e^{\omega\Delta_*}\chi \quad , \quad \chi^\dagger \longrightarrow \chi^\dagger e^{\omega\Delta_*} \quad \text{where} \quad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}. \tag{12.21}$$

The invariance of the action under this R-symmetry is evident, since both Δ_0 and Δ_1 anticommute with Δ_*. With the choice (12.14) the R-symmetry is given by

$$e^{\omega\Delta_*} = e^{\omega\tau_1} = \begin{pmatrix} \cosh\omega & \sinh\omega \\ \sinh\omega & \cosh\omega \end{pmatrix}.$$

12.2.4 Chiral symmetry

The model (12.18) seems not to be chirally invariant under a compact transformation. That is not true! It is actually invariant under the rotations

$$\chi \longrightarrow e^{i\alpha\Delta_*\otimes\gamma_*}\chi \quad \text{and} \quad \begin{pmatrix} A \\ B \end{pmatrix} \longrightarrow \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

To prove the invariance one uses

$$\begin{aligned} e^{-i\alpha\Delta_*\otimes\gamma_*}(\Delta_0\otimes \mathbb{1}_2)e^{i\alpha\Delta_*\otimes\gamma_*} &= \Delta_0\otimes\mathbb{1}_2\cos 2\alpha + \Delta_1\otimes\gamma_*\sin 2\alpha \\ e^{-i\alpha\Delta_*\otimes\gamma_*}(\Delta_1\otimes \gamma_*)e^{i\alpha\Delta_*\otimes\gamma_*} &= \Delta_1\otimes\gamma_*\cos 2\alpha - \Delta_0\otimes\mathbb{1}_2\sin 2\alpha. \end{aligned}$$

For the choice $\Delta_* = \tau_1$ the transformation reads

$$\begin{pmatrix} \chi_1 \\ i\gamma_*\chi_2 \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \chi_1 \\ i\gamma_*\chi_2 \end{pmatrix}$$

or equivalently

$$\begin{aligned} \chi_1+\chi_2 &\longrightarrow e^{i\alpha\gamma_*}(\chi_1+\chi_2) \\ \chi_1-\chi_2 &\longrightarrow e^{-i\alpha\gamma_*}(\chi_1-\chi_2). \end{aligned}$$

It follows that

$$\psi = \frac{1}{\sqrt{2}}(\chi_1 + i\chi_2) \quad \text{and} \quad \psi_c = \frac{1}{\sqrt{2}}(\chi_1 - i\chi_2) \tag{12.22}$$

transforms as follows under chiral rotations,

$$\begin{pmatrix} \psi \\ \psi_c \end{pmatrix} \longrightarrow \begin{pmatrix} \cos\alpha & -\gamma_*\sin\alpha \\ \sin\alpha\gamma_* & \cos\alpha \end{pmatrix} \begin{pmatrix} \psi \\ \psi_c \end{pmatrix}.$$

12.2.5 Rewriting the $2d$ model in a Dirac basis

The chiral symmetry and supersymmetry is more transparent if we use the Dirac spinors (12.22). With the help of the relations above (13.20) one shows, that the bilinears are related as follows:

$$\begin{aligned} \psi^\dagger\psi = -\psi_c^\dagger\psi_c = -\tfrac{1}{2}\chi^\dagger\tau_2\chi \quad &, \quad \psi^\dagger\psi_c = -\psi_c^\dagger\psi = -\tfrac{i}{2}\chi^\dagger\tau_1\chi \\ \psi^\dagger\gamma_*\psi_c = \tfrac{1}{2}\chi^\dagger\tau_3\gamma_*\chi - \tfrac{i}{2}\chi^\dagger\gamma_*\chi \quad &, \quad \psi_c^\dagger\gamma_*\psi = \tfrac{1}{2}\chi^\dagger\tau_3\gamma_*\chi + \tfrac{i}{2}\chi^\dagger\gamma_*\chi \\ \psi^\dagger\partial\!\!\!/\psi_c = \tfrac{1}{2}\chi^\dagger\tau_3\partial\!\!\!/\chi - \tfrac{i}{2}\chi^\dagger\tau_1\partial\!\!\!/\chi \quad &, \quad \psi_c^\dagger\partial\!\!\!/\psi = \tfrac{1}{2}\chi^\dagger\tau_3\partial\!\!\!/\chi + \tfrac{i}{2}\chi^\dagger\tau_1\partial\!\!\!/\chi. \end{aligned} \tag{12.23}$$

Hence we may rewrite the fermionic Lagrangian as

$$\mathcal{L}_F = \tfrac{i}{2}(\psi^\dagger\partial\!\!\!/\psi_c + \psi_c^\dagger\partial\!\!\!/\psi) - m\psi^\dagger\psi - 2gA\psi^\dagger\psi - gB(\psi^\dagger\gamma_*\psi_c + \psi_c^\dagger\gamma_*\psi). \tag{12.24}$$

To rewrite the susy transformations (12.19) in terms of Dirac spinors we introduce the complex supersymmetry parameter

$$\alpha = \frac{1}{\sqrt{2}}(\beta_1 + i\beta_2) \quad \text{and} \quad \alpha_c = \frac{1}{\sqrt{2}}(\beta_1 - i\beta_2),$$

and which should be distinguished from the 4-dimensional parameters, for which we used the same symbol. With the identities

$$\begin{aligned} 2\alpha^\dagger\psi = \beta^\dagger\chi - \beta^\dagger\tau_2\chi \quad &, \quad 2\alpha_c^\dagger\psi_c = \beta^\dagger\chi + \beta^\dagger\tau_2\chi \\ 2\alpha^\dagger\gamma_*\psi_c = \beta^\dagger\tau_3\gamma_*\chi - i\beta^\dagger\tau_1\gamma_*\chi \quad &, \quad 2\alpha_c^\dagger\psi = \beta^\dagger\tau_3\gamma_*\chi + i\beta^\dagger\tau_1\gamma_*\chi \end{aligned}$$

the susy transformations become

$$\begin{aligned} \delta A &= \alpha_c^\dagger\psi_c - \alpha^\dagger\psi \quad , \quad \delta B = \alpha^\dagger\gamma_*\psi_c + \alpha_c^\dagger\gamma_*\psi \\ \delta\psi &= \{i\gamma_*\not{\partial}B - mA - g(A^2 - B^2)\}\alpha - \{i\not{\partial}A - (mB + 2gAB)\gamma_*\}\alpha_c \\ \delta\psi_c &= \{i\gamma_*\not{\partial}B - mA - g(A^2 - B^2)\}\alpha_c + \{i\not{\partial}A - (mB + 2gAB)\gamma_*\}\alpha \end{aligned}$$

To construct the susy algebra, we again decompose the 4-supercharges as

$$Q = \sum e_r \otimes q_r, \quad \bar{Q} = \sum e_r^T\Delta_0 \otimes q_r^\dagger$$

with Majorana supercharges q_1, q_2 in 2 Euclidean dimensions. We correspondingly expand the 4-dimensional supersymmetry parametes as

$$\alpha = \sum e_r \otimes \beta_r \quad \text{and} \quad \tilde{\alpha} = \sum e_r \otimes \tilde{\beta}_r$$

such that

$$\bar{Q}\tilde{\alpha} = q^\dagger\tau_2\tilde{\beta} \quad \text{and} \quad \bar{\alpha}Q = \beta^\dagger\tau_2 q.$$

Now we introduce the two dimensional Dirac spinors

$$\epsilon = \frac{1}{\sqrt{2}}(\beta_1 + i\beta_2), \quad \tilde{\epsilon} = \frac{1}{\sqrt{2}}(\tilde{\beta}_1 + i\tilde{\beta}_2) \quad \text{and} \quad q = \frac{1}{\sqrt{2}}(q_1 + iq_2)$$

such that

$$\bar{Q}\tilde{\alpha} = q_c^\dagger\tilde{\epsilon}_c - q^\dagger\tilde{\epsilon} \quad \text{and} \quad \bar{\alpha}Q = \epsilon_c^\dagger q_c - \epsilon^\dagger q.$$

Inserting this into

$$[\bar{Q}\tilde{\alpha}, \bar{\alpha}Q] = -2i(\bar{\alpha}\Gamma^m\tilde{\alpha})\partial_m \longrightarrow -2(\bar{\alpha}\Delta_* \otimes \gamma^\mu\alpha)\partial_\mu = 2i\beta^\dagger\tau_3\gamma^\mu\tilde{\beta}\partial_\mu.$$

With

$$\epsilon^\dagger\gamma^\mu\tilde{\epsilon}_c = \tfrac{1}{2}\beta^\dagger(\tau_3 - i\tau_1)\gamma^\mu\tilde{\epsilon} \quad \text{and} \quad \epsilon_c^\dagger\gamma^\mu\tilde{\epsilon} = \tfrac{1}{2}\beta^\dagger(\tau_3 + i\tau_1)\gamma^\mu\tilde{\epsilon}$$

we obtain

$$[q_c^\dagger\tilde{\epsilon}_c - q^\dagger\tilde{\epsilon}, \epsilon_c^\dagger q_c - \epsilon^\dagger q] = 2i(\epsilon^\dagger\gamma^\mu\tilde{\epsilon}_c + \epsilon_c^\dagger\gamma^\mu\tilde{\epsilon})\partial_\mu.$$

$$\{q_{i\alpha}, q_j^{\dagger\beta}\} = -2i(\tau_3)_{ij}(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu.$$

For the Dirac supercharges

$$Q = \frac{1}{\sqrt{2}}(q_1 + iq_2) \quad \text{and} \quad Q^\dagger = \frac{1}{\sqrt{2}}(q_1 - iq_2)$$

we obtain

$$\{Q_\alpha, Q^{\dagger\beta}\} = 2(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu \quad \text{and} \quad \{Q_\alpha, Q_\beta\} = -2(\gamma^\mu)_\alpha^{\ \beta}\partial_\mu.$$

(There must be an error here)!!!!!

Kapitel 13

WZ-models in 2 Dimensions

The purpose of this section is to present again the ideas of supersymmetry and to elaborate upon them using the simpler examples of two dimensional SUSY-models. Also we shall investigate the Euclidean models and the relation between the Minkowski and Euclidean formulation in some detail. On the way the Ward-identities are discussed and are checked explicitly for some simple models.
In two dimensions the Lorentz group is Abelian. It is the twofold covering of

$$SO(1,1) \quad \text{and} \quad SO(2)$$

in Minkowski spacetime and Euclidean spacetime, respectively. According to our general analysis in chapter 2 there are Majorana or Weyl spinors in two dimensions for any choice of signature. In addition, in two-dimensional Minkowki spacetime there are Majorana -Weyl fermions.

13.1 Models in $2d$-Minkowski space

When discussing supersymmetric models it is always useful to recall the relevant properties of spinors which are relevant for the model under consideration. Although we have discussed these properties for fermions in spaces of arbitrary dimensions and signatures, it is useful to recall and further extend these results for the probably simplest case, namely 2-dimensional Minkowski space.
The general Fierz identity (4.83) simplifies to

$$2\psi\bar{\chi} = -(\bar{\chi}\psi) - \gamma_\mu(\bar{\chi}\gamma^\mu\psi) + \tfrac{1}{2}\gamma_{\mu\nu}(\bar{\chi}\gamma^{\mu\nu}\psi) = -(\bar{\chi}\psi) - \gamma_\mu(\bar{\chi}\gamma^\mu\psi) - \gamma_*(\bar{\chi}\gamma_*\psi). \tag{13.1}$$

In two spacetime dimensions there are Majorana spinors. I the Majorana representation (4.39)

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_1, \quad \text{such that} \quad \gamma_* = -\gamma_0\gamma_1 = \sigma_3 \tag{13.2}$$

the antisymmetric charge conjugation matrix reads $\mathcal{C} = -\gamma^0$. For Majorana spinors the symmetry properties (4.75) read

$$\bar{\psi}\chi = \bar{\chi}\psi, \quad \bar{\psi}\gamma^\mu\chi = -\bar{\chi}\gamma^\mu\psi \quad \text{and} \quad \bar{\psi}\gamma_*\chi = -\bar{\chi}\gamma_*\psi. \tag{13.3}$$

Some further FIERZ identities are easily derived from these properties: For Majorana spinors we find

$$\begin{aligned}(\bar\alpha\psi)(\bar\alpha\psi) &= (\bar\psi\alpha)(\bar\alpha\psi) = \bar\psi(\alpha\bar\alpha)\psi = -\tfrac{1}{2}(\bar\alpha\alpha)(\bar\psi\psi)\\ (\bar\psi\gamma^\mu\alpha)\psi &= (\psi\bar\psi)\gamma^\mu\alpha = -\tfrac{1}{2}(\bar\psi\psi)\gamma^\mu\alpha.\end{aligned}$$

In addition

$$\bar\psi\chi,\ \bar\psi\gamma_*\chi \quad \text{are hermitean and} \qquad \bar\psi\gamma^\mu\chi \quad \text{is antihermitean.} \tag{13.4}$$

To construct the Wess-Zumino model in $1+1$ dimensions we need a *real superfield* $\phi(x,\alpha)$ which has the following expansion

$$\Phi(x,\alpha) = A(x) + \bar\alpha\psi(x) + \tfrac{1}{2}\bar\alpha\alpha F(x) \tag{13.5}$$

with real (pseudo)scalar fields A, F and Majorana spinor ψ. Its supersymmetry variation is generated by the supercharge,

$$\delta_\beta\Phi = i\bar\beta Q\Phi, \qquad Q = -i\frac{\partial}{\partial\bar\alpha} - (\gamma^\mu\alpha)\partial_\mu. \tag{13.6}$$

The supersymmetry variation of the component fields follows from

$$\begin{aligned}\delta_\beta\Phi &= (\bar\beta\frac{\partial}{\partial\bar\alpha} - i(\bar\beta\gamma^\mu\alpha)\partial_\mu)\Phi = \bar\beta\psi + \bar\beta\alpha F - i(\bar\beta\gamma^\mu\alpha)\partial_\mu A - i(\bar\beta\gamma^\mu\alpha)\bar\alpha\partial_\mu\psi\\ &= \delta A + \bar\alpha\delta\psi + \tfrac{1}{2}\bar\alpha\alpha\delta F.\end{aligned}$$

We read off the following transformations of the component fields

$$\delta A = \bar\beta\psi, \quad \delta\psi = (F + i\partial\!\!\!/ A)\beta \Rightarrow \delta\bar\psi = \bar\beta(F - i\partial\!\!\!/ A), \quad \delta F = i\bar\beta\partial\!\!\!/\psi. \tag{13.7}$$

Calculating the commutator of two susy transformations is considerably simpler as in four dimensions:

$$\begin{aligned}[\delta_1,\delta_2]A &= 2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu A\\ [\delta_1,\delta_2]\psi &= i(\beta_2\bar\beta_1 - \beta_1\bar\beta_2)\partial\!\!\!/\psi + i\gamma^\mu(\beta_2\bar\beta_1 - \beta_1\bar\beta_2)\partial_\mu\psi = 2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu\psi\\ [\delta_1,\delta_2]F &= 2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu F.\end{aligned}$$

The corresponding anticommutator of two supercharges are

$$\{Q_\alpha, \bar Q^\beta\} = 2(\gamma^\mu)_\alpha{}^\beta P_\mu. \tag{13.8}$$

Multiplying with γ_0 and summing over the spinor indices yields

$$\sum_{\alpha=1,2}\{Q_\alpha, Q_\alpha^\dagger\} = 4P_0 \geq 0.$$

The supercovariant derivatives read

$$D = \frac{\partial}{\partial\bar\alpha} + i(\gamma^\mu\alpha)\partial_\mu \quad \text{and} \quad \bar D = -\frac{\partial}{\partial\alpha} - i(\bar\alpha\gamma^\mu)\partial_\mu, \tag{13.9}$$

and they anticommute with the supercharges.
To construct an invariant action we take the $\bar\alpha\alpha$ term of $\bar D\Phi D\Phi$. We need the covariant derivatives of the superfield, which are found to be

$$D\Phi=\psi+\alpha F+i\partial\!\!\!/ A\alpha-\tfrac{i}{2}(\bar\alpha\alpha)\partial\!\!\!/\psi\quad\text{and}\quad \bar D\Phi=\bar\psi+\bar\alpha F-i\bar\alpha\partial\!\!\!/ A+\tfrac{i}{2}(\bar\alpha\alpha)\partial_\mu\bar\psi\gamma^\mu.$$

Hence we obtain

$$\begin{aligned}\tfrac12\bar D\Phi D\Phi &= \tfrac12\bar\psi\psi+(\bar\alpha\psi)F-i\bar\alpha\gamma^\mu\psi\partial_\mu A+\bar\alpha\alpha\mathcal{L}_0\\ \mathcal{L}_0 &= \tfrac12\partial_\mu A\partial^\mu A-\tfrac{i}{4}\bar\psi\partial\!\!\!/\psi+\tfrac{i}{4}\partial_\mu\bar\psi\gamma^\mu\psi+\tfrac12F^2.\end{aligned}$$

In (13.7) we have calulculated the variations of the components for an arbitrary real superfield (13.5). We read off, that

$$\delta_\beta\mathcal{L}_0=\partial_\mu(\bar\beta V_0^\mu),\qquad V_0^\mu=\tfrac12(i\gamma^\mu\psi F+\gamma^\mu\gamma^\nu\psi\partial_\nu A).$$

As expected, the Lagrangean density $\mathcal{L}_0$ transforms into a spacetime derivative, and the action for the free and massless theory,

$$S_0=\int d^2x\mathcal{L}_0,$$

is invariant under the supersymmetry transformations (13.7). The equations of motion are simply

$$\Box A=0,\quad \partial\!\!\!/\psi=0\quad\text{and}\quad F=0.\tag{13.10}$$

Since

$$\sum_\phi\frac{\partial\mathcal{L}_0}{\partial_\mu\phi}\delta_\beta\phi=\partial^\mu A\,\delta_\beta A+\tfrac{i}{2}\delta_\beta\bar\psi\gamma^\mu\psi=\bar\beta(\partial^\mu A\,\psi+\tfrac{i}{2}F\gamma^\mu\psi+\tfrac12\gamma^\nu\gamma^\mu\partial_\nu A\psi)$$

we obtain the following *conserved Noether current* for the free massless model

$$J_0^\mu=(\eta^{\mu\nu}+\gamma_*\epsilon^{\mu\nu})\partial_\nu A\,\psi,\qquad \epsilon_{01}=-\epsilon^{01}=1.\tag{13.11}$$

By using the KLEIN-GORDON equation for A and the free Dirac equation for ψ one sees at once that J^μ is a conserved spinorial supercurrent.
Now we can add an interaction of the form

$$S_1=\int d^2xd^2\alpha W(\Phi),\tag{13.12}$$

which has the component expansion

$$W(\Phi)=W(A)+\bar\alpha\psi\,W'(A)+\tfrac12\bar\alpha\alpha\mathcal{L}_1,\qquad \mathcal{L}_1=FW'(A)-\tfrac12W''(A)\bar\psi\psi.$$

The variation of $\mathcal{L}_1$ is found to be

$$\delta_\beta\mathcal{L}_1=\partial_\mu(\bar\beta V_1^\mu),\qquad V_1^\mu=iW'(A)\gamma^\mu\psi$$

such that the Noether current of the interacting model with action

$$S = \int \mathcal{L}, \qquad \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 \tag{13.13}$$

becomes

$$J^\mu = (\eta^{\mu\nu} + \gamma_* \epsilon^{\mu\nu})\partial_\nu A\,\psi - iW'(A)\gamma^\mu\psi.$$

Again F is a nonpropagating auxiliary field, which satisfies the algebraic equation of motion $F = -W'(A)$. After its elimination the Lagrangean density aquires the by now familiar form

$$\mathcal{L} = \tfrac{1}{2}(\partial A)^2 - \tfrac{i}{2}\bar\psi\partial\!\!\!/\psi - \tfrac{1}{2}W'^2(A) - \tfrac{1}{2}W''(A)\bar\psi\psi. \tag{13.14}$$

The Hamiltonian density is found to be

$$\mathcal{H} = \tfrac{1}{2}\pi_A^2 + \tfrac{1}{2}A'^2 + \tfrac{i}{2}\bar\psi\gamma^1\psi' + \tfrac{1}{2}W'(A) + \tfrac{1}{2}W''(A)\bar\psi\psi, \quad \pi_A = \dot A, \quad \pi_\psi = -\tfrac{i}{2}\bar\psi\gamma^0. \tag{13.15}$$

The (anti)commutation relations for the field read

$$[\pi_A(x), A(y)] = i\delta(x-y) \quad \text{and} \quad \{\psi_\alpha(x), \psi_\beta(y)\} = -(\gamma^0\mathcal{C})_{\alpha\beta}\delta(x-y).$$

One can check, that the equations of motion are correctly gotten from[1]

$$\dot O = i[O, H].$$

Consider the following superpotential

$$W(\Phi) = g\Phi + \tfrac{1}{2}m\Phi^2 + \tfrac{1}{3}\lambda\Phi^3$$

which leads to the Langrangian density

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1, \qquad \mathcal{L}_1 = -\tfrac{1}{2}(g + mA + \lambda A^2)^2 - \tfrac{m}{2}\bar\psi\psi - \lambda A\bar\psi\psi.$$

To simplify the following discussion let us assume that $m = 0$. Then the quartic potential has a minimum at $A = 0$ and at $A^2 = -g/\lambda$, provided that g/λ is negativ.

• For $g/\lambda > 0$ we have

$$\langle A\rangle = 0, \quad \langle F\rangle = -g, \quad V(\langle A\rangle) > 0 \quad \text{and} \quad m_A = \sqrt{2g\lambda}, \quad m_\psi = 0$$

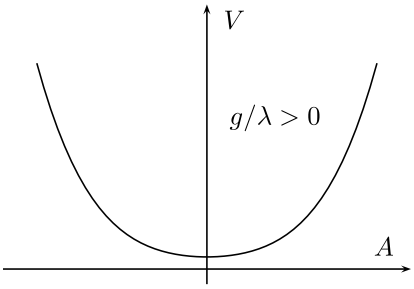
such that supersymmetry is broken at the classical level. The vanishing $\psi-$mass signals a Nambu-Goldstone fermion sometimes also called *goldstino*, associated with the spontaneous supersymmetry breaking at the tree level.

• For $g/\lambda < 0$ we have

$$\langle A\rangle = \sqrt{-g/\lambda}, \quad \langle F\rangle = 0, \quad V(\langle A\rangle) = 0 \quad \text{and} \quad m_A = m_\psi = 2\sqrt{-g\lambda}.$$

The fields A and ψ have equal masses, supersymmetry is exact at the tree level but the reflection symmetry $A \to -A$ is broken.

[1] we had to change the conventions used in section (6), since the kinetic term for the Dirac fermions have changed signs and since $\{\psi^\dagger, \psi\}$ must be positive. The matrix $-\gamma_0\mathcal{C}$ leads to $\{\psi^\dagger_\alpha(x), \psi_\beta(y)\} = \delta_{\alpha\beta}\delta(x-y)$.



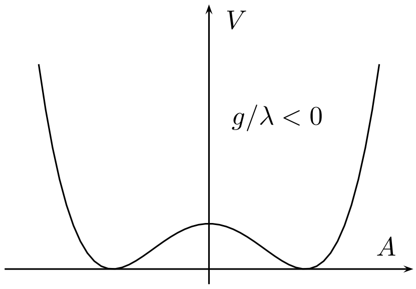


Abbildung 13.1: *Potentials for the Wess-Zumino model with and without susy breaking.*

13.1.1 Two-point functions

Here I follow the book of Bogoliubov and Shirkov, suitable generalized. The *positive frequency part* of the Pauli-Jordan function in d space-dimensions is

$$\Delta_+(\xi) = \frac{1}{i(2\pi)^{d-1}}\int e^{-ip\xi}\theta(p_0)\delta(p^2-m^2)dp = \frac{1}{i(2\pi)^{d-1}}\int \frac{1}{2\omega}e^{-i(\omega\xi^0+p_i\xi^i)}\,d\boldsymbol{p}.$$

Here dx, dp and denote integrals over time and space coordinates, whereas $d\boldsymbol{p}$ denotes the integral over the spatial components only. The corresponding Wightman function $i\Delta_+$ defines a positive inner product, as it must be,

$$\begin{aligned}\langle A(f)A(g)\rangle &= i\int dxdy\; f(x)\Delta_+(x,y)g(y)\\ &= \frac{1}{(2\pi)^{d-1}}\int dp\,dx\,dy\; f(x)e^{-ipx}\theta(p^0)\delta(p^2-m^2)e^{ipy}g(y)\\ &= \int dp\;\theta(p^0)\delta(p^2-m^2)\tilde f^*(p)g(p) = \int d\boldsymbol{p}\;\frac{1}{2\omega}\;\tilde f^*(\omega,\boldsymbol{p})g(\omega,\boldsymbol{p}),\end{aligned}$$

where we made use of the Fourier transform

$$\tilde f(p) = \frac{1}{\sqrt{2\pi}}\int e^{ipx}f(x).$$

Note that only the on-shell values of the test function enters. Let us find the explicit form of the Wightman function in two spacetime dimensions. First we change variables,

$$\omega = \sqrt{\boldsymbol{p}^2+m^2} = m\cosh\phi \quad\text{and}\quad \boldsymbol{p} = m\sinh\phi$$

which results in

$$\Delta_+(\xi) = \frac{1}{4\pi i}\int_{-\infty}^{\infty} e^{-im(\cosh\phi\xi^0+\sinh\phi\xi^1)}d\phi$$

Fortunately, this integral has been evaluated in the book cited. We take the result

$$\Delta_+(\xi) = \begin{cases}\frac{1}{4}\left(iN_0\left(m\sqrt{\xi^2}\right)-\epsilon(\xi^0)J_0\left(m\sqrt{\xi^2}\right)\right) & \text{for } \xi^2>0\\ \frac{1}{2\pi i}K_0\left(m\sqrt{-\xi^2}\right) & \text{for } \xi^2<0\end{cases} \tag{13.16}$$

The PAULI-JORDAN function becomes

$$\Delta(\xi) = \Delta_+(\xi) - \Delta_+(-\xi) = -\tfrac{1}{2}\epsilon(x^0)\,\theta(\xi^2)\;J_0(m\sqrt{\xi^2}),$$

and gives rise to the commutator

$$[A(x), A(y)] = i\Delta(\xi) = -\tfrac{i}{2}\epsilon(\xi^0)\theta(\xi^2)\;J_0(m\sqrt{\xi^2})$$

with support for timelike separated points, $\xi = x - y$. If we would quantize the free model on the cylinder

$$(x^0, x^1) \in \mathbb{R} \times S^1,$$

where the circumference of S^1 is L, then for spacelice x the function would has the form

$$\Delta_+(x) = \frac{1}{2\pi i}\sum_n K_0\left(m\sqrt{(Ln + x^1)^2 - x_0^2}\right). \tag{13.17}$$

Furthermore, the time derivative at $x^0 = 0$ is

$$(\partial_0\Delta_+)(0, x^1) = -\frac{1}{4\pi}\int e^{-ipx^1}dp = -\tfrac{1}{2}\delta(x^1) \Longrightarrow (\partial_0\Delta)(0, x^1) = -\delta(x^1).$$

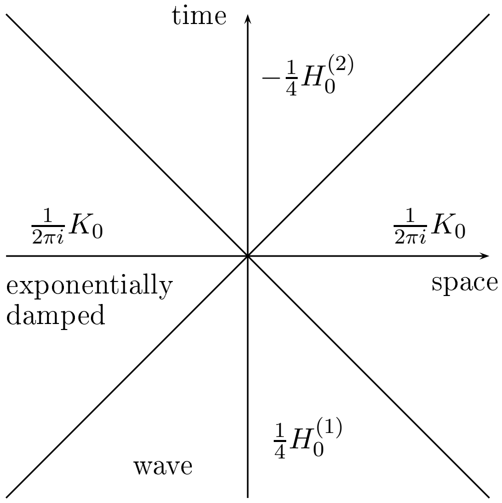


Abbildung 13.2: *The Wightman function Δ_+ in two dimensions.*

To obtain the WIGHTMAN funtion for the fermions we calculate

$$\begin{aligned} S &= (i\partial\!\!\!/ - m)(i\Delta_+) = \frac{1}{(2\pi)^{d-1}}\int (p\!\!\!/ - m)\theta(p^0)\delta(p^2 - m^2)e^{-ip\xi}dp \\ &= \frac{1}{(2\pi)^{d-1}}\int \frac{1}{2\omega}(p\!\!\!/ - m)e^{-ip\xi}d\boldsymbol{p}, \end{aligned}$$

where in the last formula $p_0 = \omega(\boldsymbol{p})$. Note that

$$S = (i\partial\!\!\!/ - m)(i\Delta_+) = \begin{pmatrix} -m & \partial_0 - \partial_1 \\ -\partial_0 - \partial_1 & -m \end{pmatrix} \begin{pmatrix} i\Delta_+ & 0 \\ 0 & i\Delta_+ \end{pmatrix}.$$

The inner product induced by the WIGHTMAN function is

$$\int dx\,dy\; g^*_\alpha(S\gamma^0)_{\alpha\beta}(x-y)f_\beta(y) = \frac{1}{(2\pi)^{d-1}}\int \frac{d\boldsymbol{p}}{2\omega}\tilde{g}^*(\omega,\boldsymbol{p})(\omega - h_f)\tilde{f}(\omega,\boldsymbol{p}),$$

where we introduced the hermitean Dirac Hamiltonian

$$h_f = \alpha^i p_i + m\gamma^0$$

which contains the hermitean matrices $\alpha^i = \gamma^0\gamma^i$. Since

$$h_f^2 = (\boldsymbol{p}^{\,2} + m^2)\mathbb{1} = \omega^2\mathbb{1},$$

the eigenvalues of h_f are $\pm\omega$. Hence the inner product of f with itself vanishes if and only if f is *on shell*. To proceed we need

$$(i\partial_1\Delta_+)(0,x^1) = -\frac{m}{2\pi}\sum K_1(mLn + nx^1)$$

which finally yiels

$$\langle\psi(0,x^1)\bar{\psi}(0,y^1)\rangle = S(0,\xi^1) = \tfrac{1}{2}\delta(\xi^1)\sigma_2 + \frac{m}{2\pi}(K_1(\xi^1)\sigma_1 - K_0(\xi^1)\sigma_0).$$

The two point function of two fermion fields are

$$\langle\psi_\alpha(0,x^1)\psi_\beta(0,y^1)\rangle = \tfrac{1}{2}\delta(\xi^1)\delta_{\alpha\beta} + \frac{m}{2\pi}(iK_1(\xi^1)\sigma_3 - K_0(\xi^1)\sigma_2)_{\alpha\beta}.$$

After smearing with real testfunctions $f:\mathbb{R}^2 \to \mathbb{R}^2$ we obtain

$$\langle\psi(0,f)\psi(0,f)\rangle = \tfrac{1}{2}\int dx^1 f_\alpha(x^1)f_\alpha(x^1).$$

13.2 Model in $2d$-Euclidean spacetime

Instead to analytically continue the Minkowski model to the Euclidean one, we construct the Euclidean model starting from the known facts about the Clifford algebra on E_2 with metric diag$(1,1)$. The Dirac conjugation is just

$$\bar{\psi} = \psi^\dagger. \tag{13.18}$$

and there is a charge conjugation matrix $\mathcal{C}$ satifying

$$\gamma_\mu^T = \mathcal{C}^{-1}\gamma_\mu\mathcal{C} \quad \text{and} \quad \mathcal{C}^T = \mathcal{C}. \tag{13.19}$$

For example, for the real representation

$$\gamma_0 = \sigma_1, \quad \gamma_1 = \sigma_3 \quad \text{we have} \quad \mathcal{C} = \mathbb{1}, \qquad \gamma_* = i\gamma_0\gamma_1 = \sigma_2$$

a Majorana spinor is just real. For Majorana spinors the symmetry properties are

$$\bar\psi\chi = -\bar\chi\psi, \quad \bar\psi\gamma_\mu\chi = -\bar\chi\gamma_\mu\psi \quad \text{and} \quad \bar\psi\gamma_*\chi = \bar\chi\gamma_*\psi, \quad \text{where} \quad \bar\psi \equiv \psi^\dagger,$$

and the hermiticity poperties become

$$\bar\psi\chi,\ \bar\psi\gamma_\mu\chi \quad \text{are antihermitean and} \quad \bar\psi\gamma_*\chi \quad \text{is hermitean.}$$

Finally, the FIERZ identity reads

$$\psi\bar\chi = -\tfrac{1}{2}\bar\chi\psi - \tfrac{1}{2}\gamma_\mu(\bar\chi\gamma_\mu\psi) - \tfrac{1}{2}\gamma_*(\bar\chi\gamma_*\psi). \tag{13.20}$$

Some further FIERZ identities are easily derived from these properties: For Majorana spinors we have

$$\begin{aligned}(\bar\alpha\psi)(\bar\alpha\psi) &= -(\bar\psi\alpha)(\bar\alpha\psi) = -\bar\psi(\alpha\bar\alpha)\psi = \tfrac{1}{2}(\bar\alpha\gamma_*\alpha)(\bar\psi\gamma_*\psi)\\ (\bar\psi\gamma^\mu\alpha)\psi &= (\psi\bar\psi)\gamma^\mu\alpha = -\tfrac{1}{2}(\bar\psi\gamma_*\psi)\gamma_*\gamma^\mu\alpha\end{aligned}$$

A *hermitean superfield* has the following component expansion,

$$\Phi(x,\alpha) = A(x) + \bar\alpha\gamma_*\psi(x) + \tfrac{1}{2}(\bar\alpha\gamma_*\alpha)F(x)$$

with real fields A, F and Majorana spinorfield ψ. As supercharges we may take

$$Q = -i\frac{\partial}{\partial\bar\alpha} - \gamma^\mu\alpha\partial_\mu = \quad \text{and} \quad \bar Q = -i\frac{\partial}{\partial\alpha} - \bar\alpha\gamma^\mu\partial_\mu = Q^T\sigma_1.$$

One finds the following nontrivial anticommutation relations

$$\{Q_\alpha, \bar Q_\beta\} = 2i(\gamma^\mu\partial_\mu)_{\alpha\beta}. \tag{13.21}$$

Thus we have recovered the correct anticommutation relations for the supercharges. The superderivatives are

$$D = \frac{\partial}{\partial\bar\alpha} + i(\gamma^\mu\alpha)\partial_\mu \quad , \quad \bar D = -\frac{\partial}{\partial\alpha} - i(\bar\alpha\gamma^\mu)\partial_\mu = -D^T\sigma_1.$$

Up to a sign they satisfy the same anticommutation relations as the supercharges. The supersymmetry variation of the superfield is

$$\begin{aligned}\bar\beta(\frac{\partial}{\partial\bar\alpha} - i\gamma^\mu\alpha\partial_\mu)\Phi &= \bar\beta(\gamma_*\psi + \gamma_*\alpha F - i\partial\!\!\!/ A\alpha + \tfrac{i}{2}(\bar\alpha\gamma_*\alpha)\partial\!\!\!/\psi)\\ &= \delta A(x) + \bar\alpha\gamma_*\delta\psi(x) + \tfrac{1}{2}(\bar\alpha\gamma_*\alpha)\delta F(x),\end{aligned}$$

from which we read off the following transformation rules for the component fields

$$\delta A = \bar\beta\gamma_*\psi, \quad \delta\psi = (F + i\gamma_*\partial\!\!\!/ A)\beta \Longrightarrow \delta\bar\psi = \bar\beta(F - i\partial\!\!\!/ A\gamma_*), \quad \delta F = i\bar\beta\partial\!\!\!/\psi. \tag{13.22}$$

Now we are ready to calculate the action of the free model. First we need the covariant derivatives of the superfield

$$\begin{aligned} D\Phi &= \gamma_*\psi + \gamma_*\alpha F + i\partial\!\!\!/ A\alpha - \tfrac{i}{2}(\bar\alpha\gamma_*\alpha)\partial\!\!\!/\psi \\ \bar D\Phi &= \bar\psi\gamma_* + \bar\alpha\gamma_* F - i\bar\alpha\partial\!\!\!/ A + \tfrac{i}{2}(\bar\alpha\gamma_*\alpha)\partial_\mu\bar\psi\gamma^\mu \end{aligned}$$

Since $\gamma_* D\Phi$ is a Majorana field we obtain

$$\tfrac{1}{2}\bar D\Phi D\Phi = -\tfrac{i}{4}(\bar\alpha\gamma_*\alpha)\partial_\mu(\bar\psi\gamma_*\gamma^\mu\psi) = 0,$$

and this can be checked by a direct calculation using the Fierz identities in Euclidean spacetime. The supersymmetric and Lorentz-invariant action is derived from the density

$$\begin{aligned} \tfrac{1}{2}\bar D\Phi\gamma_* D\Phi = &= \tfrac{1}{2}\bar\psi\gamma_*\psi + (\bar\alpha\gamma_*\psi)F - i\bar\alpha\gamma^\mu\psi\partial_\mu A - \bar\alpha\gamma_*\alpha\mathcal{L}_0 \\ \mathcal{L}_0 &= \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{i}{4}\bar\psi\partial\!\!\!/\psi - \tfrac{i}{4}\partial_\mu\bar\psi\gamma^\mu\psi - \tfrac{1}{2}F^2. \end{aligned}$$

The variation of the Lagrangean density $\mathcal{L}_0$ with respect to supersymmetry transformations is

$$\delta\mathcal{L}_0 = \partial_\mu(\bar\beta V_0^\mu), \quad V_0^\mu = \tfrac{1}{2}(\gamma_*\gamma^\mu\gamma^\nu\,\partial_\nu A - i\gamma^\mu F)\psi.$$

Note that $\mathcal{L}_0$ gives rise to an action which is *unbounded below* and above. Again we switch on an interaction by adding a superpotential $W(\Phi)$ with component expansion,

$$W(\Phi) = W(A) + \bar\alpha\gamma_*\psi W'(A) + \tfrac{1}{2}\bar\alpha\gamma_*\alpha(W'(A)F - \tfrac{1}{2}W''\psi^\dagger\gamma_*\psi),$$

and which gives rise to the Lagrangean density

$$\mathcal{L}_1 = \tfrac{1}{2}W''(A)\psi^\dagger\gamma_*\psi - W'(A)F.$$

The total Lagrangean is $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$ and from the superfield formulation we read off that

$$\delta\mathcal{L} = \partial_\mu(\bar\beta V^\mu), \quad V^\mu = V_0^\mu - iW'(A)\gamma^\mu\psi. \tag{13.23}$$

This can be checked directly by using the explicit form for the tranformations of the component fields A, ψ and F. The auxiliary field fulfils the algebraic equation

$$F = -W'(A).$$

Eliminating F we obtain

$$\mathcal{L} = \tfrac{1}{2}(\partial A)^2 + \tfrac{i}{2}\bar\psi\partial\!\!\!/\psi + \tfrac{1}{2}W'^2(A) + \tfrac{1}{2}W''(A)\,\bar\psi\gamma_*\psi. \tag{13.24}$$

This density is invariant under the on-shell supersymmetry transformations

$$\delta A = \bar\beta\gamma_*\psi \quad\text{and}\quad \delta\psi = (i\gamma_*\partial\!\!\!/ A - W'(A))\beta, \qquad \delta\bar\psi = \bar\beta(i\gamma_*\partial\!\!\!/ A - W'(A)). \tag{13.25}$$

Hence, on the classical level we have constructed an $N = 1$ supersymmetric and *stable* Euclidean Wess-Zumino model. Already for the simple Euclidean Wess-Zumino model we

have the strange result that off-shell the action is unbounded from below. Only after eliminating the auxiliary field does the theory become stable. In theories with extended supersymmetry the models are unstable even on-shell.
The equations of motion read

$$\begin{aligned} \triangle A &= (W'W'')(A) + \tfrac{1}{2}W'''(A)\bar{\psi}\gamma_*\psi \\ \tfrac{i}{2}\partial\!\!\!/\psi &= -\tfrac{1}{2}W''(A)\gamma_*\psi. \end{aligned}$$

The solutions are integral curves to the hamiltonian vectorfield generated by

$$\mathcal{H} = \pi_A\dot{A} + \pi_\psi\dot{\psi} - \mathcal{L} = \tfrac{1}{2}\pi_A^2 - \tfrac{1}{2}(A')^2 - \tfrac{i}{2}\bar{\psi}\gamma^1\partial_1\psi - \tfrac{1}{2}W'^2(A) - \tfrac{1}{2}W''\bar{\psi}\gamma_*\psi,$$

where

$$\pi_A = \dot{A} \quad \text{and} \quad \pi_\psi = \tfrac{i}{2}\bar{\psi}\gamma^0 = \tfrac{i}{2}\psi^T(\mathcal{C}^{-1})^T\gamma^0$$

are the momentum field conjugate to A and ψ. The (anti)commutation relations read

$$[A(x), \pi_A(y)] = i\delta(x-y) \quad \text{and} \quad \{\psi_\alpha(x), \pi_\psi^\beta(y)\} \sim \delta_\alpha^{\ \beta}\delta(x-y).$$

Indeed one finds

$$\begin{aligned} \dot{A}(x) &= i[H, A(x)] = \pi_A(x) \\ \dot{\pi}_A(x) &= i[H, \pi_A(x)] = -(\partial_x A)^2 + (W'W'')(A) + \tfrac{1}{2}W'''(A)\bar{\psi}\gamma_*\psi \end{aligned}$$

which imply the correct equations of motion for the scalar field. To continue we choose the following Majorana representation,

$$\gamma_0 = \sigma_1, \quad \gamma_1 = \sigma_3 \Longrightarrow \gamma_* = \sigma_2.$$

The charge conjugation matrix is the identity and

$$\pi_\psi = \tfrac{i}{2}(\psi_2, \psi_1).$$

The anticommutation relations read[2]

$$\{\psi_1(x), \psi_2(y)\} = \delta(x-y), \qquad \{\psi_1, \psi_1\} = \{\psi_2, \psi_2\} = 0.$$

The fermionic part of the Hamiltonian density reads

$$\mathcal{H}_F = -\tfrac{i}{2}(\psi_1\psi_1' - \psi_2\psi_2') + \tfrac{i}{2}W''(A)(\psi_1\psi_2 - \psi_2\psi_1).$$

With $[AB, C] = A\{B, C\} - \{A, C\}B$ we obtain

$$\begin{aligned} \dot{\psi}_1 &= i[H, \psi_1] = +\psi_2' - W''\psi_1 \\ \dot{\psi}_2 &= i[H, \psi_2] = -\psi_1' + W''\psi_2, \end{aligned}$$

which yields the above Dirac equation in the given representation.

[2] I choose the normalization such, that the I get the correct equations of state

13.2.1 Noether currents

With

$$\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}\delta\phi = \bar{\beta}(\partial^\mu A\gamma_* - \tfrac{i}{2}F\gamma^\mu - \tfrac{1}{2}\partial\!\!\!/ A\gamma_*\gamma^\mu)\psi,$$

the Noether current reads

$$\bar{\beta}J^\mu = \bar{\beta}(\partial^\mu\gamma_* + i\epsilon^{\mu\nu}\partial_\nu A + iW'(A)\gamma^\mu)\psi, \qquad \epsilon^{01} = 1,$$

and gives rise to the Noether charge

$$Q = \int dx\Big(\pi_A\gamma_* + iA' + iW'(A)\gamma^0\Big)\psi.$$

In our Majorana representation

$$\bar{\beta}Q = \int dx\bar{\beta}\Big(\pi_A\sigma_2 + iA' + iW'(A)\sigma_1\Big)\psi.$$

It is easy to calculate

$$\delta A = i[\bar{\beta}Q, A] = \bar{\beta}\gamma_*\psi.$$

To compute the commutator of the supercharge with the fermionic field, we use

$$\int dy[\bar{\beta}_\alpha A_{\alpha\beta}(y)\psi_\beta(y), \psi_\sigma(x)] = \int dy\bar{\beta}_\alpha A_{\alpha\beta}(y)\{\psi_\beta(y), \psi_\sigma(x)\} = \bar{\beta}_\alpha(A\sigma_1)_{\alpha\sigma}(x).$$

With this formula one finds

$$\delta\psi = i\bar{\beta}(\pi_A\sigma_2 + iA' + iW'\sigma_1)\sigma_1 = (i\sigma_2(\sigma_1\partial_0 + \sigma_3\partial_1)A - W')\beta.$$

Hence we recover the transformation law for the scalar field A and the Majorana spinor ψ.
In components (and the Majorana representation) we find

$$\begin{aligned} Q_1 &= i\int dx^1\,((W' - \pi_A)\,\psi_2 + A'\psi_1) \\ Q_2 &= i\int dx^1\,((W' + \pi_A)\,\psi_1 + A'\psi_2). \end{aligned}$$

13.3 Euclidean path integral for free model

Here we recall the quantisation of the free massive $N = 1$ model. The main emphasis is put on the correct quantisation of Majorana fermions in the path integral formulation. Since all functional integrals are Gaussian, the generating functional and hence all correlations functions, can be calculated explicitely.

First, to get the free massive model we choose a linear superpotential

$$W = \tfrac{m}{2}\Phi^2 = \tfrac{m}{2}A^2 + m(\bar{\alpha}\gamma_*\psi)A + \tfrac{m}{2}\bar{\alpha}\gamma_*\alpha(AF - \tfrac{1}{2}\bar{\psi}\gamma_*\psi),$$

giving rise to a quadratic Langrangian density,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{i}{4}\bar{\psi}\partial\!\!\!/\psi - \tfrac{i}{4}\partial_\mu\bar{\psi}\gamma^\mu\psi - \tfrac{1}{2}F^2 - mAF + \tfrac{m}{2}\bar{\psi}\gamma_*\psi.$$

We have seen that this density is invariant under the following susy transformations

$$\delta A = \bar{\beta}\gamma_*\psi \quad \text{and} \quad \delta\psi = (F + i\gamma_*\partial\!\!\!/ A)\psi,$$

and that the density is invariant up to a divergence,

$$\delta\mathcal{L} = \partial_\mu(\bar{\beta}V^\mu), \quad V^\mu = V_0^\mu - imA\gamma^\mu\psi.$$

After eliminating the auxiliary field via its equation of motion, $F = -mA$, we end up with the massive free theory for a real scalar field and a Majorana spinor,

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu A\partial^\mu A + \tfrac{m}{2}A^2 + \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\psi + \tfrac{m}{2}\bar{\psi}\gamma_*\psi. \tag{13.26}$$

Next we wish to study the Gaussion functional integral for the generating functional of the Euclidean correlation functions

$$Z(j,\eta) = \int \mathcal{D}A\mathcal{D}\psi\, e^{-S+i\int(jA+\bar{\eta}\psi)} \equiv Z_B(j)Z_F(\eta) \tag{13.27}$$

in detail. Note that the bosonic source term is imaginary, whereas the fermionic source term is hermitean. Clearly the functional factorizes into bosonic and fermionic parts. We shall work on the two-dimensional torus such that the relevant differential operators possess discrete spectra. Hence we identify x^0 with $x^0 + \beta$ and x^1 with $x^1 + L$. The functional integral over the scalar field is simply

$$Z_B(j) = \frac{1}{\det^{1/2}(-\triangle + m^2)}\ \exp\Big\{-(j, \frac{1}{-\triangle + m^2}\, j)\Big\}.$$

The integral over the Majorana fermions is more subtle and hence we shall be a bit more explicit. The eigenvalue equation for the Euclidean Dirac equation,

$$(i\partial\!\!\!/ + m\gamma_*)\psi = \lambda\psi,$$

reads in the Majorana representation

$$i\begin{pmatrix}\partial_1 & \partial_0 - m\\ \partial_0 + m & -\partial_1\end{pmatrix}\psi = \lambda\psi.$$

On the space of real Majorana spinors this euclidean Dirac equation has no solution. But we can solve

$$(i\partial\!\!\!/ + m\gamma_*)\chi_\lambda = i\lambda\phi_\lambda \quad \text{and} \quad (i\partial\!\!\!/ + m\gamma_*)\phi_\lambda = -i\lambda\chi_\lambda \Longrightarrow (-\triangle + m^2)\chi_\lambda = \lambda^2\chi_\lambda.$$

on real modes. The possible eigenvalues are

$$\lambda_{rs} = \pm\sqrt{p^2+m^2}, \qquad p_0 = \frac{2\pi r}{\beta}, \quad p_1 = \frac{2\pi s}{L}.$$

To construct these solutions one starts with a (necessarily) complex and normalized eigenfunction ψ_λ of the Dirac operator with eigenvalue λ. We assume that λ is positive. Then ψ^*_λ is an eigenfunction as well with eigenvalue $-\lambda$. Now we may take

$$\chi_\lambda = \frac{1}{\sqrt{2}}(\psi_\lambda + \psi^*_\lambda) \quad \text{and} \quad \phi_\lambda = \frac{1}{i\sqrt{2}}(\psi_\lambda - \psi^*_\lambda), \qquad \lambda > 0.$$

Since the ψ_λ form an orthonormal set of (complex) eigenfunctions, we have

$$(\chi_\lambda, \chi_\lambda) = (\phi_\lambda, \phi_\lambda) = 1 \quad \text{and} \quad (\chi_\lambda, \phi_\lambda) = 0.$$

Now we expand the Majorana field and source in the functional integral as

$$\psi = \sum_\lambda (b_\lambda \chi_\lambda + c_\lambda \phi_\lambda), \qquad \bar\eta = \sum_\lambda (u_\lambda \chi^T_\lambda + v_\lambda \phi^T_\lambda),$$

where the sum extends over all *positive* eigenvalues λ of the Dirac operator. It follows that

$$\tfrac{1}{2}(\bar\psi, (i\partial\!\!\!/ + m\gamma_*)\psi) = \tfrac{i}{2}\sum \lambda(b_\lambda c_\lambda - c_\lambda b_\lambda) \quad \text{and} \quad (\bar\eta, \psi) = \sum (u_\lambda b_\lambda + v_\lambda c_\lambda).$$

Next we take linear combinations of the real Grassmannian parameters and introduce the complex objects

$$a = \frac{1}{\sqrt{2}}(b + ic), \quad , \quad s = \frac{1}{\sqrt{2}}(u + iv)$$

such that

$$a^\dagger a = \tfrac{i}{2}(bc - cb), \qquad s^\dagger a + s a^\dagger = ub + vc.$$

Now we may rewrite the path integral in terms of complex Grassmann variables as follows,

$$Z_F(\eta) = \int da_\lambda da^\dagger_\lambda \, \exp\Big(\sum \{-\lambda a^\dagger_\lambda a_\lambda + i s^\dagger_\lambda a_\lambda + i s_\lambda a^\dagger_\lambda\}\Big).$$

The remaining manipulations are wellknown. First we shift variables

$$-\lambda a^\dagger a + i(s^\dagger a + s a^\dagger) = \lambda \tilde a^\dagger \tilde a + \frac{1}{\lambda} s^\dagger s, \quad \tilde a = a + \frac{i}{\lambda} s, \quad \tilde a^\dagger = a^\dagger - \frac{1}{\lambda} s^\dagger,$$

such that the path integral simplifies to

$$Z_F(\eta) = \exp\big(-\sum \frac{1}{\lambda} s^\dagger_\lambda s_\lambda\big) \int d\tilde a_\lambda d\tilde a^\dagger_\lambda \, \exp\big(-\sum \lambda \tilde a^\dagger_\lambda \tilde a_\lambda\big) = \big(\prod \lambda\big) \, \exp\big(-\frac{1}{\lambda} s^\dagger_\lambda s_\lambda\big),$$

where the product and sum extends over the positive eigenvalues of the Dirac operator. Notice the difference as compared to the path integral for Dirac fermions. The eigenvalues of $-\partial\!\!\!/^2$ are at least double degenerate (the γ_*-doubling). When we quantize Dirac fermions, then the two eigenvalues are both included in the above formula. But for Majorana fermions we combined two eigenfunction in the subspace with fixed λ^2 to one complex eigenfunction. Hence we only must include one of the eigenvalues of the pair of eigenvalues $\pm\lambda$ of $i\partial\!\!\!/$. Hence the product is

$$\prod \lambda = \det{}^{1/2}(i\partial\!\!\!/ + \gamma_* m) = \det{}^{1/2}(-\triangle + m^2).$$

In the last step we used $\partial\!\!\!/^2 = \triangle \mathbb{1}_2$. This way we arrive at the intermediate result

$$Z_F(\eta) = \det{}^{1/2}(-\triangle + m^2)\ \exp\big(-i\sum \frac{u_\lambda v_\lambda}{\lambda}\big).$$

To rewrite the last term we use the spectral decomposition of the Greensfunction of the Dirac operator,

$$S(x,y) \equiv \langle x|\frac{1}{i\partial\!\!\!/ + m\gamma_*}|y\rangle = i\sum_{\lambda>0}\frac{1}{\lambda}(\phi_\lambda(x)\chi_\lambda^T(y) - \chi_\lambda(x)\phi_\lambda^T(x)) = \sum_{\text{all}\lambda}\frac{1}{\lambda}\psi_\lambda\psi_\lambda^\dagger.$$

This Greensfunction is purely imaginary and antisymmetric

$$S_{\alpha\beta}(x,y) = -S_{\beta\alpha}(y,x).$$

Using the eigenmode expansion of the source and the orthogonality of the eigenmodes one proves that

$$(\bar\eta, \frac{1}{i\partial\!\!\!/ + m\gamma_*}\eta) = -2i\sum\frac{u_\lambda v_\lambda}{\lambda}.$$

This finally proves the following nice formula for the path integral

$$Z(j,\eta) = \exp\Big(-\tfrac{1}{2}\int j(x)G(x,y)j(y) + \tfrac{1}{2}\int \bar\eta(x)S(x,y)\eta(y)\Big). \tag{13.28}$$

Notice that the (UV-singular) functional determinants have exactly canceled in the final result. This (partial) cancellation of divergences is one of the most attractive features of supersymmetric theories.

13.4 Do we get a sensible susy field theory?

After having computed the generating functional it is a easy exercise to derive the two-point function and hence all n-point function for the free $N = 1$ model. The bosonic 2-point function is

$$\langle A(x)A(y)\rangle = \frac{1}{i^2}\frac{\delta^2}{\delta j(x)j(y)}\log Z(j,\eta)|_{j=0} = G(x,y). \tag{13.29}$$

It is positive in the following sense,

$$(f, Gf) \geq 0 \quad \text{and} \quad = 0 \quad \text{only for} \quad f = 0,$$

implying that the in the bosonic sector of the corresponding Minkowski theory the Hilbertspace has positive norm.

13.4.1 The requirements on the Euclidean Theory

Let us recall the relations between Minkowski and Euklidean 2-point functions: In Minkowski spacetime one defines the two-point WIGHTMAN *funtion*

$$\begin{aligned}\langle\Omega|\hat{A}(x)\hat{A}(y)|\Omega\rangle &= \langle\Omega|e^{ix^0H}\hat{A}(0,\boldsymbol{x})e^{-ix^0H}e^{iy^0H}\hat{A}(0,\boldsymbol{y})e^{-iy^0H}|\Omega\rangle\\ &= \langle\Omega|\hat{A}(0,\boldsymbol{x})e^{-i(x^0-y^0)H}\hat{A}(0,\boldsymbol{y})|\Omega\rangle = W(x,y)=W(\xi),\end{aligned}$$

where $\xi = x - y$. This Poincaré invariant distribution must be positive, that is for non-vanishing test functions f we have $(f, Wf) > 0$. From $H \geq 0$ it may be inferred that $W(\xi)$ admits an analytic continuation with respect to x^0. Writing $S(z,\boldsymbol{\xi})$ for the analytic function with $z = \xi^0_E + i\xi^0$ restricted to the halfplane $\xi^0_E > 0$,

$$S(z,\boldsymbol{\xi}\,) = \langle\Omega|\hat{A}(0,\boldsymbol{x}\,)e^{-zH}\hat{A}(0,\boldsymbol{y}\,)|\Omega\rangle, \qquad z = \xi^0_E + i\xi,$$

we have

$$W(\xi^0,\boldsymbol{\xi}\,) = \lim_{\xi^0_E\downarrow 0} S(\xi^0_E + i\xi^0,\boldsymbol{x}\,).$$

As a rule the boundary values only exist in the sense of distributions. Else there are no singularities for $\xi^0_E > 0$, and this is the domain where the Euclidean two-point function is evaluated:

$$S(\xi) = S(\xi^0_E,\boldsymbol{\xi}\,).$$

Now let us recall the basic axioms of Euclidean field theory such that it gives rise to a interpretable field theory in Minkowski space.
The starting point is the normalized generating functional

$$Z(f) = \int d\mu(A)\, e^{iA(f)}, \qquad d\mu(A) \sim \mathcal{D}Ae^{-S[A]}, \quad A(f) = \int f(x)A(x), \quad f\in\mathcal{S}. \tag{13.30}$$

We have changed the notation. Here and below f (and not j) denotes the external source. The space $\mathcal{S}$ is a suitable real test function space, usually one takes the Schwartz space[3]. Obvious properties are

$$\begin{aligned}&|Z(f)| \leq Z[0] = 1\\ &Z[\,f_1 + tf_2] \quad \text{is continuous function of} \quad t\in\mathbb{R} \quad \text{for all} \quad f_i\in\mathcal{S}\\ &a_{ij} = Z[f_i - f_j] \quad \text{is positive definite.}\end{aligned} \tag{13.31}$$

Next we introduce the space $\mathcal{S}^c$ of all complex $f = f_1 + if_2$, where $f_i \in \mathcal{S}$. Then we assume
Analyticity: *The Schwinger functional $Z[f]$ admits an extension to complex $f\in\mathcal{S}^c$ with the extension being analytic.*

[3]The Schwartz space consists of all $\mathbb{C}^\infty$-functions $f:\mathbb{R}^d\to\mathbb{R}$, such that f and all its derivatives decrease faster than any inverse power of $|x|$ as $|x|\to\infty$. The dual $\mathcal{S}'$ of $\mathcal{S}$ consists of all $\mathbb{R}$-linear functionals $\phi:\mathcal{S}\to\mathbb{R}$.

This means that for $f = \sum_1^n z_i f_i$ with $z_i \in \mathbb{C}$ and $f_i \in \mathcal{S}^c$ the mapping $(z_1, \dots, z_n) \to Z[f]$ is entirely analytic in $\mathbb{C}^n$. Then Z admits a power series expansion which represents the functional. The next property we require is

Regularity: *There exists constants c_1 and c_2 such that for all $f \in \mathcal{S}^c$*

$$|Z[f]| \leq \exp \int dx \Big(c_1 |f(x)| + c_2 |f(x)|^2. \tag{13.32}$$

Moreover, the two point function $\langle A(x)A(y)\rangle$ is assumed to be locally integrable.

This postulate may physically not be as well founded as the postulate of

Invariance: *The functional $Z(f)$ is invariant under all symmetries of the Euclidean space.*

We shall use the Hilbert space $\mathcal{E} = L^2(\mathcal{S}', d\mu(A))$ of square integrable functions $F : \mathcal{S}' \to \mathbb{C}$ with respect to the functional measure μ. The scalar product is given by

$$(F, G) = \int d\mu(A)\, \overline{F(A)}\, G(A). \tag{13.33}$$

Let us explain the rather important lat axiom in more detail: The action of the Euclidean group on the test functions is described by

$$(a, R) f(x) = f(R^{-1}(x - a)), \qquad a \in \mathbb{R}^d, \quad R \in O(d). \tag{13.34}$$

A unitary representation $U(a, R) : \mathcal{E} \to \mathcal{E}$ is then given by

$$U(a, R) F(A) = \sum_k c_k \exp\{iA((a, R) f_k))\}, \qquad F(A) = \sum_{k=1}^n c_k \exp(iA(f_k)). \tag{13.35}$$

Particular elements of the Euclidean group are

• *Time reversal:* This is the element $\theta = (0, R)$, where R is a time reflection. The corresponding unitary operator Θ is an involution, $\Theta^2 = \mathbb{1}$.

• *Time shifts:* These are the elements $(a, \mathbb{1})$, where $a = (t, \vec{0})$ shifts the time coordinate. We obtain a one-parameter unitary group $U(t) = \exp(itB)$. But B is not to be confused with the Hamiltonian. Since

$$\Theta U(t) = U(-t)\Theta \tag{13.36}$$

the spectrum of B is symmetric about zero.

The last property we require is

Reflection positivity: *The matrix with elements $a_{ij} = Z(f_i - \theta f_j)$, $i, j = 1, \dots, n$ with arbitrary n, is poitive for all test functions $f_i \in \mathcal{S}$ that vanish in the halfplane $x^0 < 0$, i.e. $f(x) = 0$ for negative x^0.*

On may rephrase this crucial property in terms of the Schwinger functions, that is the moments of the Schwinger functional, as follows:

For each finite sequence $(f_0, \dots, f_n)$ of test functions $f_0 \in \mathbb{C}$ and $f_k \in \mathcal{S}(\mathbb{R}^{kd})$ with

$$f_k(x_1, \dots, x_k) = 0 \quad \text{except for} \quad 0 \geq x_1^0 \geq \dots \geq x_k^0$$

on has

$$\sum_{n,m} \mathcal{S}^{n+m}(\overline{\theta f_n} \otimes f_m) \geq 0. \tag{13.37}$$

Test functions with this property form a subspace $\mathcal{S}_+$ of $\mathcal{S}$. There is a corresponding subspace $\mathcal{E}_+ \subset \mathcal{E}$, which we define as the closed complex-linear hull of all vectors of the form $F(A) = \exp(iA(f))$ with $f \in \mathcal{S}_+$. Reflection positivity implies that $(\Theta F, G) \geq 0$ for all $F, G \in \mathcal{E}_+$. Thus we introduce a second scalar product

$$\langle F, G\rangle \equiv (\Theta F, G) \tag{13.38}$$

under the provision that we factor out the null space

$$\mathcal{E}_0 = \{F \in \mathcal{E}_+ | \langle F, F\rangle = 0\}. \tag{13.39}$$

This then gives rise to the (physical) Hilbert space

$$\mathcal{H} = \overline{\mathcal{E}_+/\mathcal{E}_0}. \tag{13.40}$$

Two vectors F and G in $\mathcal{E}_+$ give rise to the same vector in $\mathcal{H}$ if they differ by some vector in $\mathcal{E}_0$. Thus one considers the equivalence class $F_\bullet$ associated with each $F \in \mathcal{E}_+$ and treat $F_\bullet$ as some vector in $\mathcal{H}$ so that the scalar product becomes

$$(F_\bullet, G_\bullet) = \langle F, G\rangle.$$

The Hamiltonian: It is remarkable that one may construct the Hamiltonian of the model within the Euclidian formulation of the model. First we observe that if a test function vanishes for negative euclidean times $f(x^0 < 0, \boldsymbol{x}) = 0$, so does the function

$$f_t(x^0, \boldsymbol{x}) = f(x^0 - t, \boldsymbol{x})$$

for all $t > 0$. This time shifts with positive t define a semigroup which maps $\mathcal{S}_+$ into itself. Hence

$$U(t) : \mathcal{E}_+ \longrightarrow \mathcal{E}_+.$$

One can show that $\mathcal{E}_+$ is left invariant under $U(t)$ such that the $U(t)$ maybe viewed as linear bounded operator $U_\bullet$ on the quotient space $\mathcal{E}_+/\mathcal{E}_0$ which is then extended to $\mathcal{H}$,

$$U(t)_\bullet F_\bullet = (U(t)F)_\bullet.$$

One can prove

$$0 \leq U_\bullet^\dagger(t) = U_\bullet(t) \leq 1 \Longrightarrow U_\bullet(t) = e^{-tH}, \qquad t \geq 0,\ H \geq 0. \tag{13.41}$$

The operator H is then just the selfadjoint and positive Hamiltonian of the theory under consideration.

13.4.2 The fermionic subspace

When we differentiate (13.27) with respect to the Grassman-valued fermionic sources we get

$$\frac{\delta^2}{\delta\bar\eta_\beta(y)\delta\eta_\alpha(x)} Z_F(\eta)\big|_{\eta=0} = Z_F\cdot\langle\psi_\alpha(x)\bar\psi_\beta(y)\rangle,$$

where in the Majorana representation there is no difference between $\bar\psi_\alpha$ and ψ_α. When we differentiate (13.28) we obtain

$$\frac{\delta^2}{\delta\bar\eta_\beta(y)\delta\eta_\alpha(x)} Z_F(\eta)\big|_{\eta=0} = Z_F S_{\alpha\beta}(x,y).$$

Not unexpectedly this proves that

$$\langle\psi_\alpha(x)\bar\psi_\beta(y)\rangle = \frac{\delta^2}{\delta\bar\eta_\beta(y)\delta\eta_\alpha(x)}\log Z_F(\eta)\big|_{\eta=0} = S_{\alpha\beta}(x,y). \tag{13.42}$$

This does not define a positive form on the set of functions $\mathbb{R}^2\to\mathbb{R}^2$,

$$\int f_\alpha(x) S_{\alpha\beta}(x,y) f_\alpha(y) = 0.$$

However, this is not a serious problem. Only the positivity of the state space of the Minkowski theory is really required.

13.4.3 Supersymmetric Ward identities:

In (13.26) we change variables $(A,\psi)\to(A',\psi')$, where the mapping is a supersymmetry transformation. We shall assume that the measure respects the supersymmetry. Then

$$\begin{aligned} Z(j,\eta) &= \int \mathcal{D}A\mathcal{D}\psi\;\exp(-S)\exp\Big(i\int jA'+\bar\eta\psi'\Big)\\ &= \int \mathcal{D}A\mathcal{D}\psi\;\exp(-S)\exp\Big(i\int j'A+\bar\eta'\psi\Big) = Z(j',\eta'). \end{aligned}$$

When calculate the transformations of the sources on needs some partial integrations and Fierz identities. The partial integrations are justified on the torus. It follows in particular, that

$$0 = \int \mathcal{D}A\mathcal{D}\psi \exp(-S)\exp\Big(i\int jA+\bar\eta\psi\Big)\cdot\Big(\int j\delta A+\bar\eta\delta\psi\Big). \tag{13.43}$$

where we have set $A' = A+\delta A$ and $\psi'=\psi+\delta\psi$. The Ward identities are gotten by differentiating (13.43) several times with respect to the sources j and η and setting them to zero after the differentiation has been performed. We give some examples:

Operator	Ward identity
$\frac{\delta}{\delta j(x)}$	$\langle\delta A(x)\rangle = 0$
$\frac{\delta^2}{\delta j(y)\delta j(x)}$	$\langle A(x)\delta A(y) - A(y)\delta A(x)\rangle = 0$
$\frac{\delta}{\delta\bar\eta_\alpha(x)}$	$\langle\delta\psi_\alpha(x)\rangle = 0$
$\frac{\delta^2}{\delta\bar\eta_\beta(y)\delta\bar\eta_\alpha(x)}$	$\langle\psi_\alpha(x)\delta\psi_\beta(y) - \psi_\beta(y)\delta\psi_\alpha(x)\rangle = 0$
$\frac{\delta^2}{\delta\bar\eta_\beta(y)\delta j(x)}$	$\langle A(x)\delta\psi_\beta(y) + \psi_\beta(y)\delta A(x)\rangle = 0$

For the free model only the last Ward identity yields a nontrivial relations between the two-point functions,

$$(i\partial\!\!\!/ + \gamma_* m)_x\langle A(x)A(y)\rangle = \langle\psi(x)\bar\psi(y)\rangle$$

which is indeed fulfilled, since

$$(i\partial\!\!\!/ + \gamma_* m)_x G(x,y) = S(x,y)$$

holds true. Next we investigate the Ward identities when we perform two successive supersymmetry transformations with parameters α and β:

$$\begin{aligned} A &\longrightarrow A + \delta_\alpha A + \delta_\beta A + \delta_\alpha\delta_\beta A + O(\alpha^2,\beta^2) = A + \delta_{\alpha\beta}A + O(\alpha^2,\beta^2). \\ \psi &\longrightarrow \psi + \delta_\alpha\psi + \delta_\beta\psi + \delta_\alpha\delta_\beta\psi + O(\alpha^2,\beta^2) = \psi + \delta_{\alpha\beta}\psi + O(\alpha^2,\beta^2). \end{aligned}$$

Clearly, we get the identity

$$0 = \int \exp\Big(-S + i\int(jA + \bar\eta\psi)\Big)\ \bigg(i\int(j\,\delta_{\alpha\beta}A + \bar\eta\delta_{\alpha\beta}\psi) \\ -\tfrac{1}{2}\Big\{\int j(\delta_\alpha+\delta_\beta)A + \bar\eta(\delta_\alpha+\delta_\beta)\psi)\Big\}^2\bigg) + O(\alpha^2,\beta^2).$$

Now exchange α and β and subtract the corresponding equation which results in

$$\begin{aligned} 0 &= \int\exp\Big(-S + i\int(jA+\bar\eta\psi)\Big)\int\Big(j\,[\delta_\alpha,\delta_\beta]A + \bar\eta[\delta_\alpha,\delta_\beta]\psi\Big) \\ &= 2i\bar\beta\gamma^\mu\alpha\int\exp\Big(-S+i\int(jA+\bar\eta\psi)\Big)\int\Big(j\partial_\mu A + \bar\eta\partial_\mu\psi\Big). \end{aligned} \tag{13.44}$$

But this is just the Ward identity for the translations in Euclidean spacetime. For example, it follows that the two-point functions only depend on the difference $x-y$, and not on x and y separately. We see, that on the level of Ward identities the supersymmetry transformations indeed close on spacetime translations.

13.4.4 Changing the representation:

If we would have started with the following definition of the Dirac conjugate in Euclidean space,

$$\bar\psi = \psi^\dagger A,\quad A = \gamma_0\gamma_1$$

then Majorana spinors would have $\eta = \epsilon = 1$, so that

$$\mathcal{C}^T = -\mathcal{C} \quad \text{and} \quad \gamma_\mu^T = -\mathcal{C}^{-1}\gamma_\mu \mathcal{C}.$$

A possible representation is

$$\gamma_0 = \sigma_1, \quad \gamma_1 = \sigma_2 \Longrightarrow \mathcal{C} = \sigma_2, \quad \gamma_* = i\gamma_0\gamma_1 = \sigma_3, \quad A = i\sigma_3, \quad B = -\sigma_1.$$

Now the Majorana condition reads

$$\psi_c = B\psi^* = \psi \Longrightarrow \psi_2 = -\psi_1^*, \quad \text{or} \quad \psi = \begin{pmatrix} \psi_1 \\ -\psi_1^* \end{pmatrix}, \quad \bar{\psi} = (i\psi_1^*, i\psi_1).$$

For Majorana spinors the symmetry properties are

$$\bar{\psi}\chi = \bar{\chi}\psi, \quad \bar{\psi}\gamma_\mu\chi = -\bar{\chi}\gamma_\mu\psi \quad \text{and} \quad \bar{\psi}\gamma_*\chi = -\bar{\chi}\gamma_*\psi$$

and the hermiticity poperties become

$$\bar{\psi}\chi \quad \text{is hermitean and} \quad \bar{\psi}\gamma_*\chi, \ \bar{\psi}\gamma_*\chi \quad \text{are antihermitean.}$$

Finally, the FIERZ identity reads

$$\psi\bar{\chi} = -\tfrac{1}{2}\bar{\chi}\psi - \tfrac{1}{2}\gamma_\mu(\bar{\chi}\gamma_\mu\psi) - \tfrac{1}{2}\gamma_*(\bar{\chi}\gamma_*\psi).$$

For Majorana spinors further FIERZ identities are easily derived from these properties:

$$(\bar{\alpha}\psi)(\bar{\alpha}\psi) = (\bar{\psi}\alpha)(\bar{\alpha}\psi) = -\tfrac{1}{2}(\bar{\alpha}\alpha)(\bar{\psi}\psi) \quad , \quad (\bar{\psi}\gamma^\mu\alpha)\psi = -\tfrac{1}{2}(\bar{\psi}\psi)\gamma^\mu\alpha.$$

A hermitean superfield has the form

$$\Phi(x,\alpha) = A(x) + i\bar{\alpha}\psi(x) + \tfrac{i}{2}(\bar{\alpha}\alpha)F(x)$$

with real fields A, F and Majorana spinorfield ψ. As supercharges we may take

$$Q = -i\frac{\partial}{\partial\bar{\alpha}} - \gamma^\mu\alpha\partial_\mu \quad \text{and} \quad \bar{Q} = i\frac{\partial}{\partial\alpha} + \bar{\alpha}\gamma^\mu\partial_\mu = Q^T\sigma_1.$$

One finds the following nontrivial anticommutation relations

$$\{Q, \bar{Q}\} = -2i(\gamma^\mu\partial_\mu).$$

The superderivatives are

$$D = -i\frac{\partial}{\partial\bar{\alpha}} + (\gamma^\mu\alpha)\partial_\mu \quad , \quad \bar{D} = i\frac{\partial}{\partial\alpha} - (\bar{\alpha}\gamma^\mu)\partial_\mu.$$

From the supersymmetry variation of the superfield we read off the following transformation rules for the component fields

$$\delta A = i\bar{\beta}\psi, \quad \delta\psi = (F + \partial\!\!\!/ A)\beta \quad \delta F = i\bar{\beta}\partial\!\!\!/\psi.$$

The supersymmetric, lorentzinvariant and *unstable* Euclidean action is gotten from the density

$$\begin{aligned}\tfrac{i}{2}\bar{D}\Phi D\Phi = &= \tfrac{i}{4}\bar{\psi}\psi + \tfrac{i}{2}(\bar{\alpha}\psi)F - \tfrac{i}{2}\bar{\alpha}\gamma^\mu\psi\partial_\mu A + \tfrac{i}{2}\bar{\alpha}\alpha\mathcal{L}_0\\ \mathcal{L}_0 &= -\tfrac{1}{2}\partial_\mu A\partial^\mu A - \tfrac{i}{4}\bar{\psi}\partial\!\!\!/\psi + \tfrac{i}{4}\partial_\mu\bar{\psi}\gamma^\mu\psi + \tfrac{1}{2}F^2.\end{aligned}$$

Again we switch on an interaction by adding a superpotential

$$W(\Phi) = g\Phi + \tfrac{1}{3}\lambda\Phi^3,$$

which has the component expansion

$$W(\Phi) = W(A) + i\bar{\alpha}\psi W'(A) + \tfrac{i}{2}\bar{\alpha}\alpha\mathcal{L}_1 \quad \text{with} \quad \mathcal{L}_1 = gF + \lambda(A^2F - iA\bar{\psi}\psi).$$

The interacting theory has density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$.
The auxiliary field fulfils the algebraic equation

$$F = -g - \lambda A^2.$$

Eliminating it we obtain

$$-\mathcal{L} = \tfrac{1}{2}(\partial A)^2 + \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\psi + \tfrac{1}{2}(g + \lambda A^2)^2 + i\lambda A\bar{\psi}\psi.$$

Hence, on the classical level we have constructed a $N = 1$ supersymmetric and stable Wess-Zumino model. Recall that here $\bar{\psi}$ is not $\psi^\dagger$ but rather $\psi^\dagger A$. Again we conclude that in the off-shell formulation the Euclidean model is unstable. It becomes stable only after the elimination of the auxiliary field F.

Kapitel 14

Wess-Zumino-models in 3 Dimensions

The general Fierz identity (4.83) simplifies to

$$2\psi\bar{\chi} = -(\bar{\chi}\psi) - \gamma_\mu(\bar{\chi}\gamma^\mu\psi) \tag{14.1}$$

There is a charge conjugation matrix with

$$\eta = \epsilon = 1$$

and when we choose the Majorana representation

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_3, \quad \gamma^2 = i\sigma_1 \tag{14.2}$$

then a antisymmetric charge conjugation matrix reads $\mathcal{C} = -\gamma^0$ and in this representation Majorana spinors are real. For Majorana spinors the general symmetry property

$$\bar{\psi}_C\gamma^{(n)}\chi_C = \epsilon(-)^{n(n+1)/2}\eta^t\bar{\chi}\gamma^{(n)}\psi$$

reads

$$\bar{\psi}\chi = \bar{\chi}\psi, \quad \bar{\psi}\gamma^\mu\chi = -\bar{\chi}\gamma^\mu\psi. \tag{14.3}$$

Some further Fierz identities are easily derived from these properties: For Majorana spinors we find

$$\begin{array}{rcl}(\bar{\alpha}\psi)(\bar{\alpha}\psi) &=& (\bar{\psi}\alpha)(\bar{\alpha}\psi) = \bar{\psi}(\alpha\bar{\alpha})\psi = -\frac{1}{2}(\bar{\alpha}\alpha)(\bar{\psi}\psi)\\ (\bar{\psi}\gamma^\mu\alpha)\psi &=& (\psi\bar{\psi})\gamma^\mu\alpha = -\frac{1}{2}(\bar{\psi}\psi)\gamma^\mu\alpha.\end{array}$$

In addition

$$\bar{\psi}\chi \quad \text{is hermitean and} \qquad \bar{\psi}\gamma^\mu\chi \quad \text{antihermitean.} \tag{14.4}$$

14.1 $N = 1$ Models

To construct the Wess-Zumino model in $2+1$ dimensions we need a *real superfield* $\phi(x, \alpha)$ which has the following expansion

$$\Phi(x,\alpha) = A(x) + \bar\alpha\psi(x) + \tfrac{1}{2}\bar\alpha\alpha F(x) \tag{14.5}$$

with real (pseudo)scalar fields A, F and Majorana spinor ψ. Its supersymmetry variation is generated by the supercharge,

$$\delta_\beta\Phi = i\bar\beta Q\Phi, \qquad Q = -i\frac{\partial}{\partial\bar\alpha} - (\gamma^\mu\alpha)\partial_\mu. \tag{14.6}$$

The supersymmetry variation of the component fields follows from

$$\begin{aligned}\delta_\beta\Phi &= (\bar\beta\frac{\partial}{\partial\bar\alpha} - i(\bar\beta\gamma^\mu\alpha)\partial_\mu)\Phi = \bar\beta\psi + \bar\beta\alpha F - i(\bar\beta\gamma^\mu\alpha)\partial_\mu A - i(\bar\beta\gamma^\mu\alpha)\bar\alpha\partial_\mu\psi \\ &= \delta A + \bar\alpha\delta\psi + \tfrac{1}{2}\bar\alpha\alpha\delta F.\end{aligned}$$

We read off the following transformations of the component fields

$$\delta A = \bar\beta\psi, \quad \delta\psi = (F + i\partial\!\!\!/ A)\beta \Rightarrow \delta\bar\psi = \bar\beta(F - i\partial\!\!\!/ A), \quad \delta F = i\bar\beta\partial\!\!\!/\psi. \tag{14.7}$$

Calculating the commutator of two susy transformations is considerably simpler as in four dimensions:

$$\begin{aligned}[\delta_1,\delta_2]A &= 2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu A \\ [\delta_1,\delta_2]\psi &= i(\beta_2\bar\beta_1 - \beta_1\bar\beta_2)\partial\!\!\!/\psi + i\gamma^\mu(\beta_2\bar\beta_1 - \beta_1\bar\beta_2)\partial_\mu\psi = 2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu\psi \\ [\delta_1,\delta_2]F &= 2i(\bar\beta_2\gamma^\mu\beta_1)\partial_\mu F.\end{aligned}$$

The corresponding anticommutator of two supercharges are

$$\{Q_\alpha, \bar Q^\beta\} = 2(\gamma^\mu)_\alpha^{\ \beta} P_\mu. \tag{14.8}$$

Multiplying with γ_0 and summing over the spinor indices yields

$$\sum_{\alpha=1,2}\{Q_\alpha, Q_\alpha^\dagger\} = 4P_0 \geq 0.$$

The supercovariant derivatives read

$$D = \frac{\partial}{\partial\bar\alpha} + i(\gamma^\mu\alpha)\partial_\mu \quad \text{and} \quad \bar D = -\frac{\partial}{\partial\alpha} - i(\bar\alpha\gamma^\mu)\partial_\mu, \tag{14.9}$$

and they anticommute with the supercharges.
To construct an invariant action we take the $\bar\alpha\alpha$ term of $\bar D\Phi D\Phi$. We need the covariant derivatives of the superfield, which are found to be

$$D\Phi = \psi + \alpha F + i\partial\!\!\!/ A\alpha - \tfrac{i}{2}(\bar\alpha\alpha)\partial\!\!\!/\psi \quad \text{and} \quad \bar D\Phi = \bar\psi + \bar\alpha F - i\bar\alpha\partial\!\!\!/ A + \tfrac{i}{2}(\bar\alpha\alpha)\partial_\mu\bar\psi\gamma^\mu.$$

Hence we obtain

$$\begin{aligned}\tfrac{1}{2}\bar{D}\Phi D\Phi &= \tfrac{1}{2}\bar{\psi}\psi + (\bar{\alpha}\psi)F - i\bar{\alpha}\gamma^\mu\psi\partial_\mu A + \bar{\alpha}\alpha\mathcal{L}_0 \\ \mathcal{L}_0 &= \tfrac{1}{2}\partial_\mu A\partial^\mu A - \tfrac{i}{4}\bar{\psi}\partial\!\!\!/\psi + \tfrac{i}{4}\partial_\mu\bar{\psi}\gamma^\mu\psi + \tfrac{1}{2}F^2.\end{aligned}$$

In (13.7) we have calulculated the variations of the components for an arbitrary real superfield (13.5). We read off, that

$$\delta_\beta\mathcal{L}_0 = \partial_\mu(\bar{\beta}V_0^\mu), \qquad V_0^\mu = \tfrac{1}{2}(i\gamma^\mu\psi F + \gamma^\mu\gamma^\nu\psi\partial_\nu A).$$

As expected, the Langrangian density $\mathcal{L}_0$ transforms into a spacetime derivative, and the action for the free and massless theory,

$$S_0 = \int d^2x\mathcal{L}_0,$$

is invariant under the supersymmetry transformations (13.7). The equations of motion are simply

$$\Box A = 0, \quad \partial\!\!\!/\psi = 0 \quad \text{and} \quad F = 0. \tag{14.10}$$

Since

$$\sum_\phi \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu\phi)}\delta_\beta\phi = \partial^\mu A\,\delta_\beta A + \tfrac{i}{2}\delta_\beta\bar{\psi}\gamma^\mu\psi = \bar{\beta}(\partial^\mu A\,\psi + \tfrac{i}{2}F\gamma^\mu\psi + \tfrac{1}{2}\gamma^\nu\gamma^\mu\partial_\nu A\psi)$$

we obtain the following *conserved Noether current* for the free massless model

$$J_0^\mu = (\eta^{\mu\nu} - \gamma^{\mu\nu})\psi\,\partial_\nu A. \tag{14.11}$$

By using the Klein-Gordon equation for A and the free Dirac equation for ψ one sees at once that J^μ is a conserved spinorial supercurrent.
Now we can add an interaction of the form

$$S_1 = \int d^2x d^2\alpha W(\Phi), \tag{14.12}$$

which has the component expansion

$$W(\Phi) = W(A) + \bar{\alpha}\psi\,W'(A) + \tfrac{1}{2}\bar{\alpha}\alpha\mathcal{L}_1, \qquad \mathcal{L}_1 = FW'(A) - \tfrac{1}{2}W''(A)\bar{\psi}\psi.$$

The variation of $\mathcal{L}_1$ is found to be

$$\delta_\beta\mathcal{L}_1 = \partial_\mu(\bar{\beta}V_1^\mu), \qquad V_1^\mu = iW'(A)\gamma^\mu\psi$$

such that the Noether current of the interacting model with action Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 = \tfrac{1}{2}\partial_\mu A\partial^\mu A - \tfrac{i}{4}\bar{\psi}\partial\!\!\!/\psi + \tfrac{i}{4}\partial_\mu\bar{\psi}\gamma^\mu\psi + \tfrac{1}{2}F^2 + FW'(A) - \tfrac{1}{2}W''(A)\bar{\psi}\psi \tag{14.13}$$

becomes

$$J^\mu = (\eta^{\mu\nu} - \gamma^{\mu\nu})\partial_\nu A\,\psi - iW'(A)\gamma^\mu\psi.$$

Again F is a nonpropagating auxiliary field, which satisfies the algebraic equation of motion $F = -W'(A)$. After its elimination the Lagrangian density aquires the familiar form

$$\mathcal{L} = \tfrac{1}{2}(\partial A)^2 - \tfrac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}W'^2(A) - \tfrac{1}{2}W''(A)\bar{\psi}\psi. \tag{14.14}$$

14.2 Reduction to $N = 2$-models

We construct the $N = 2$ model by dimensional reduction of the $4d$ model. For that purpose we write

$$\Gamma_\mu = \Delta \otimes \gamma_\mu, \quad \mu = 0, 1, 2 \quad \text{and} \quad \Gamma_3 = \tilde{\Delta} \otimes \mathbb{1}_2, \tag{14.15}$$

Note that we cannot choose $\Delta = \sigma_0$ since then we could not find a solution of $\{\Gamma_\mu, \Gamma_3\} = 0$. The Dirac algebra for Γ_M and γ_μ imply

$$\Delta^2 = \mathbb{1}_2, \quad \tilde{\Delta}^2 = -\mathbb{1}_2 \quad \text{and} \quad \{\Delta, \tilde{\Delta}\} = 0. \tag{14.16}$$

The product of the γ_μ is antihermitean and proportional to the identity and we choose

$$\gamma^0\gamma^1\gamma^2 = \gamma_0\gamma_1\gamma_2 = i\mathbb{1}_2. \tag{14.17}$$

With this choice

$$\Gamma_5 = -i\Gamma_0\Gamma_1\Gamma_2\Gamma_3 = -i\Delta\tilde{\Delta} \otimes \gamma_0\gamma_1\gamma_1 = \Delta\tilde{\Delta} \otimes \mathbb{1}_2. \tag{14.18}$$

Sometimes it is convenient to choose Majorana representations in both 4 and 3 dimensions, e.g.

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_3, \quad \gamma^2 = i\sigma_1, \quad \Delta \text{ real}, \quad \tilde{\Delta} \text{ imaginary}. \tag{14.19}$$

For the reduction we write the spinor in 4 dimensions as linear combination of spinors in 3 dimensions

$$\Psi = \frac{1}{\sqrt{L}} \sum_r e_r \otimes \chi_r \quad \text{with} \quad (e_r, e_s) = \delta_{rs}. \tag{14.20}$$

The Dirac-conjugate spinor has the form

$$\bar{\Psi} = \Psi^\dagger \Gamma^0 = \frac{1}{\sqrt{L}} \sum_r (e_r^T \otimes \chi_r^\dagger)(\Delta \otimes \gamma^0) = \frac{1}{\sqrt{L}} \sum e_r^T \Delta \otimes \bar{\chi}_r. \tag{14.21}$$

Because of (16.1) we choose a hermitean Δ and an antihermitean $\tilde{\Delta}$. In a Majorana representation the real Δ and the imaginary $\tilde{\Delta}$ are both symmetric[1]. Then it follows

$$\begin{aligned}
\int d^4x \bar{\Psi} \Gamma^M \partial_M \Psi &= \sum_{r,s} \int d^3x (e_r^T \Delta \otimes \bar{\chi}_r)(\Delta \otimes \gamma^\mu)(e_s \otimes \partial_\mu \chi_s) = \sum_r \int d^3x \bar{\chi}_r \partial\!\!\!/ \chi_r \\
\int d^4x \bar{\Psi} \Psi &= \sum_{r,s} \int d^3x (e_r^T \Delta \otimes \bar{\chi}_r)(e_s \otimes \chi_s) = \sum_{r,s} \int d^3x \, \Delta_{rs} \bar{\chi}_r \chi_s, \qquad (14.22) \\
\int d^4x \bar{\Psi} \Gamma_5 \Psi &= \sum_{r,s} \int d^3x (e_r^T \Delta \otimes \bar{\chi}_r)(\Delta\tilde{\Delta} \otimes \mathbb{1}_2)(e_s \otimes \chi_s) = \sum_{r,s} \int d^3x \, \tilde{\Delta}_{rs} \bar{\chi}_r \chi_s,
\end{aligned}$$

[1]For example, $\Delta = \sigma_3$ and $\tilde{\Delta} = i\sigma_1$.

where we have introduced the matrix elements $\Delta_{rs} = (e_r, \Delta e_s)$ and $\tilde{\Delta}_{rs} = (e_r, \tilde{\Delta} e_s)$. Finally we rescale the scalar fields according to their mass-dimensions in 4 and 3 dimensions:

$$A \longrightarrow \frac{1}{\sqrt{L}} A \quad \text{and} \quad B \longrightarrow \frac{1}{\sqrt{L}} B. \tag{14.23}$$

Now we are ready to dimensionally reduce the Wess-Zumino model with $4d$-Lagrangian

$$\begin{aligned} \mathcal{L}_4 &= \frac{1}{2}\partial_m A \partial^m A + \frac{1}{2}\partial_m B \partial^m B + \frac{i}{2}\bar{\Psi}\Gamma^m \partial_m \Psi + \frac{1}{2}(\mathcal{F}^2 + \mathcal{G}^2) \\ &+ m\Big[\mathcal{F}A + \mathcal{G}B - \frac{1}{2}\bar{\Psi}\Psi\Big] + g[\mathcal{F}(A^2 - B^2) + 2\mathcal{G}AB - \bar{\Psi}(A - i\Gamma_5 B)\Psi\Big] \end{aligned} \tag{14.24}$$

if we in addition relate the coupling constants in 4 and 2 dimensions according to

$$g_3 = \frac{g_4}{\sqrt{L}}.$$

The reduced in 3 dimensions is give by

$$S_4 = \int d^4x\, \mathcal{L}_4 = S_3 = \int d^3x\, \mathcal{L}_3$$

with 3-dimensional Lagrangian density

$$\begin{aligned} \mathcal{L}_3 &= \frac{1}{2}\partial_\mu A \partial^\mu A + \frac{1}{2}\partial_\mu B \partial^\mu B + \frac{i}{2}\bar{\chi}\partial\!\!\!/\chi + \frac{1}{2}(\mathcal{F}^2 + \mathcal{G}^2) \\ &+ m\Big[\mathcal{F}A + \mathcal{G}B - \frac{1}{2}\bar{\chi}\Delta\chi\Big] + g\Big[\mathcal{F}(A^2 - B^2) + 2\mathcal{G}AB - \bar{\chi}(A\Delta - i\tilde{\Delta}B)\chi\Big], \end{aligned} \tag{14.25}$$

where for example $\bar{\chi}\partial\!\!\!/\chi = \sum \bar{\chi}_r \partial\!\!\!/\chi_r$ and $\bar{\chi}\Delta\chi = \sum \bar{\chi}_r \Delta_{rs}\chi_s$ with $\Delta_{rs} = e_r^T \Delta e_s$.

Lorentz invariance: In the four dimensions the spin transformations are

$$\Psi \longrightarrow S_4\Psi, \quad S_4 = \exp\left(\frac{1}{4}\,\omega_{mn}\Gamma^{mn}\right), \quad \Gamma^{mn} = \frac{1}{2}(\Gamma^m\Gamma^n - \Gamma^n\Gamma^m). \tag{14.26}$$

Since $\Gamma^{\mu\nu} = \mathbb{1}_2 \otimes \gamma^{\mu\nu}$ they reduce to the Lorentz transformations

$$\chi \longrightarrow S_2\chi, \quad S_3 = \exp\left(\frac{1}{4}\,\omega_{\mu\nu}\gamma^{\mu\nu}\right)\chi, \tag{14.27}$$

Supersymmetry transformations: The off-shell supersymmetry transformations in 4 dimensions have the form

$$\begin{aligned} \delta_\alpha A &= \bar{\alpha}\Psi, \qquad \delta_\alpha B = i\bar{\alpha}\Gamma_5\Psi \\ \delta_\alpha \Psi &= -i\Gamma^m\partial_m(A + i\Gamma_5 B)\alpha + (\mathcal{F} + i\Gamma_5\mathcal{G})\alpha \\ \delta_\alpha \mathcal{F} &= -i\bar{\alpha}\Gamma^m\partial_m\Psi, \qquad \delta_\alpha\mathcal{G} = \bar{\alpha}\Gamma_5\Gamma^m\partial_m\psi. \end{aligned} \tag{14.28}$$

We decompose

$$\alpha = \sum e_r \otimes \alpha_r, \quad \delta\Psi \sum e_r \otimes \delta\chi_r$$

and similarly

$$\bar{\alpha} = \sum e_r^T \Delta \otimes \bar{\alpha}_r, \quad \delta\bar{\Psi} = \sum e_r^T \Delta \otimes \delta\bar{\chi}_r$$

and end up with the reduced transformation laws

$$\begin{aligned}
\delta_\alpha A &= \bar{\alpha}\Delta\chi \\
\delta_\alpha B &= i\bar{\alpha}\tilde{\Delta}\chi \\
\delta_\alpha \chi &= -i\partial\!\!\!/ A\Delta\alpha + \partial\!\!\!/ B\tilde{\Delta}\alpha \\
\delta_\alpha \mathcal{F} &= -i\bar{\alpha}\partial\!\!\!/\chi \\
\delta_\alpha \mathcal{G} &= \bar{\alpha}\tilde{\Delta}\Delta\partial\!\!\!/\chi
\end{aligned} \tag{14.29}$$

The transformations rules can be simplified a bit, if we replace χ by $\Delta\chi$ Let us have a closer look at the fermionic part of the Lagrangian,

$$\mathcal{L}_F = \frac{1}{2}\bar{\chi}\mathcal{D}\chi, \qquad \mathcal{D} = i\mathbb{1}_2 \otimes \partial\!\!\!/ - m\Delta \otimes \mathbb{1}_2 - gA\,\Delta \otimes \mathbb{1}_2 + igB\,\tilde{\Delta} \otimes \mathbb{1}_2. \tag{14.30}$$

Thus we are led to calculate the Pfaffian of

$$(\mathbb{1}_2 \otimes \gamma^0)\mathcal{D} = i\mathbb{1}_2 \otimes \gamma^0\partial\!\!\!/ - m\Delta \otimes \gamma^0 - gA\,\Delta \otimes \gamma^0 + igB\,\tilde{\Delta} \otimes \gamma^0. \tag{14.31}$$

This defines a purely imaginary 'matrix'. The first factors are symmetric and the second factors are antisymmetric, such that $\gamma^0\mathcal{D}$ is a antisymmetric and imaginary 'matrix'.

Kapitel 15

Susy sigma-Models in 2 dimensions

The non-linear sigma models have been introduced at the end of the sixties [59] to describe the infrared properties in $d > 2$ spacetime dimensions of systems with sponteneously broken symmetry. This lead to the study of models the fields of which take their values in coset spaces. Later on sigma models have been discussed in $d = 2$ dimensions to investigate the propagation of strings in curved target spaces [60]. In this case the target space may be a more general Riemannian manifold.
From the point of view of perturbation theory, $d = 2$ is particularly interesting, since non-linear sigma models are power-counting renormalizable. Some particular models based on coset spaces, as e.g. the Heisenberg model, have been proved to be really renormalizable in 2 dimensions [61].
A purely bosonic model contains a scalar field ϕ mapping ordinary Minkowski spacetime $\mathcal{M}$ into a target manifold X. If the target space X is a linear space, then a theory with the field ϕ is called a linear σ-model; if the target space is curved, it is a nonlinear σ-model. In susy models there are of course additional spinor fields.
In this section we study supersymmetric nonlinear σ-models with target space metric g_{ij}. As a preparation we collect some useful formulae for nonlinear σ-model calculations in curved space time.

15.1 Geometric preliminaries

We consider a (pseudo)Riemannian manifold X with line element

$$ds^2 = g_{ij} d\phi^i d\phi^j.$$

We recall the definition of the Christoffel symbols,

$$\begin{aligned} \Gamma^p_{ij} &= g^{pq}\Gamma_{qij}, \qquad \Gamma_{pij} = \tfrac{1}{2}(g_{pi,j} + g_{pj,i} - g_{ij,p}) = \Gamma_{pji} \\ \Rightarrow &\qquad \nabla_i V^i = \partial_i V^i + \Gamma^j{}_{ij} V^j = \frac{1}{\sqrt{g}}\,\partial_i(\sqrt{g}\,V^i). \end{aligned} \tag{15.1}$$

which are needed to calculate covariant derivatives,

$$\nabla_i V^j = \partial_i V^j + \Gamma^j{}_{ik} V^k \quad \text{and} \quad \nabla_i V_j = \partial_i V_j - \Gamma^k{}_{ij} V_k.$$

and the curvature tensor, Ricci tensor and Ricci scalar:

$$R^i_{\ jpq} = \Gamma^i_{\ qj,p} - \Gamma^i_{\ pj,q} + \Gamma^i_{\ pr}\Gamma^r_{\ qj} - \Gamma^i_{\ qr}\Gamma^r_{\ pj}, \qquad R_{jq} = R^i_{jiq} \quad \text{and} \quad R = g^{jq}R_{jq}.$$

By using that the connection is metric, $g_{pq;m} = 0$, it follows

$$\begin{aligned} R_{ijpq} &= \Gamma_{iqj,p} - \Gamma_{ipj,q} + \Gamma_{riq}\Gamma^r_{\ pj} - \Gamma_{rip}\Gamma^r_{\ qj} \\ &= -\tfrac{1}{2}(g_{ip,jq} + g_{jq,ip} - g_{iq,jp} - g_{jp,iq}) - \Gamma_{rip}\Gamma^r_{\ jq} + \Gamma_{riq}\Gamma^r_{\ jp} \end{aligned} \tag{15.2}$$

The symmetry properties of this tensor are

$$R_{ijpq} = -R_{ijqp}, \quad R_{ijpq} = -R_{jipq}, \quad R_{ijpq} = R_{pqij},$$

and the Bianchi identity reads

$$R_{ijpq} + R_{iqjp} + R_{ipqj} = 0.$$

In a conformally flat target space we may (locally) introduce isothermal coordinates

$$g_{ij} = e^{2\sigma}\delta_{ij} \tag{15.3}$$

such that the Christoffel symbols have the simple form

$$\Gamma^p_{\ ij} = \partial_j\sigma\delta_{pi} + \partial_i\sigma\delta_{pj} - \partial_p\sigma\delta_{ij} \tag{15.4}$$

and the components of the curvature tensor read

$$\begin{aligned} R^i_{\ jpq} &= \delta_{ip}\left(\partial_q\sigma\partial_j\sigma - \partial_j\partial_q\sigma - \frac{1}{2}\delta_{qj}(\partial\sigma)^2\right) - \delta_{iq}\left(\partial_p\sigma\partial_j\sigma - \partial_j\partial_p\sigma - \frac{1}{2}\delta_{pj}(\partial\sigma)^2\right) \\ &+ \delta_{jq}\left(\partial_p\sigma\partial_i\sigma - \partial_i\partial_p\sigma - \frac{1}{2}\delta_{ip}(\partial\sigma)^2\right) - \delta_{jp}\left(\partial_i\sigma\partial_q\sigma - \partial_i\partial_q\sigma - \frac{1}{2}\delta_{iq}(\partial\sigma)^2\right) \end{aligned} \tag{15.5}$$

In a 2-dimensional target space with conformally flat metric the curvature tensor takes th simpler form

$$R_{1212} = -e^{2\sigma}\partial^2\sigma, \quad R_{11} = R_{22} = -\partial^2\sigma, \quad R_{12} = 0, \quad R = -2e^{-2\sigma}\partial^2\sigma. \tag{15.6}$$

Next we consider a geodesic on a compact n-dimensional Riemannian manifold X, parametrized by $\phi^i(s)$ and satisfying the differential equation

$$\frac{d^2\phi^i}{ds^2} + \Gamma^i_{\ jk}(\phi(s))\,\frac{d\phi^j}{ds}\frac{d\phi^k}{ds} = 0, \qquad \phi(0) = \phi_0, \quad \dot{\phi}(0) = \xi, \tag{15.7}$$

where s is the arc length and $\Gamma^i_{\ jk}$ denotes the Christoffel symbol for the Levi-Civita connection. Any integral curve of (15.7) is completely determined by a point ϕ_0 and a direction defined by the tangent vector $\xi^i = \dot{\phi}^i(\phi_0)$. The power series solution of (15.7) is:

$$\phi^i(s) = \phi^i_0 + \sum_{a=1}^{\infty}\frac{1}{a!}(\frac{d^a\phi^i}{ds^a})_p s^a, \qquad (\frac{d\phi^i}{ds})(0) = \xi^i. \tag{15.8}$$

Inserting this expansion into the differential equation (15.7) yields

$$\sum_{a=2}^{\infty}\frac{s^{a-2}}{(a-2)!}\frac{d^a\phi^i}{ds^a}+\Gamma^i{}_{jk}\Big(\phi_0+\sum_{b=1}^{\infty}\frac{s^b}{b!}\frac{d^b\phi}{ds^b}\Big)\sum_{c,d=1}^{\infty}\frac{s^{c+d-2}}{(c-1)!\,(d-1)!}\,\frac{d^c\phi^j}{ds^c}\frac{d^d\phi^k}{ds^d}=0,$$

where all derivatives are to be evaluated at the point p. In the lowest orders we obtain

$$\begin{aligned}
s^0: &\qquad (\frac{d^2\phi^i}{ds^2})_p = -\Gamma^i{}_{jk}(p)\,\xi^j\xi^k\\
s^1: &\qquad (\frac{d^3\phi^i}{ds^3})_p = -\Gamma^i{}_{jk}(p)\cdot(\ddot{\phi}^j\xi^k+\xi^j\ddot{\phi}^k)_p+\Gamma^i{}_{jk,l}(p)\,\xi^l\xi^j\xi^j\\
&\qquad\qquad\qquad = (2\Gamma^i{}_{kj}\Gamma^j{}_{mn}-\Gamma^i{}_{km,n})_p\,\xi^m\xi^n\xi^k.
\end{aligned}$$

This infinite set of equations define the generalized Christoffel symbols

$$\phi^i=\phi^i_0+s\xi^i-\frac{s^2}{2}\Gamma^i{}_{jk}(p)\,\xi^i\xi^k-\sum_{n=3}^{\infty}\frac{1}{n!}\Gamma^i{}_{i_1i_2\dots i_n}(p)\,\xi^{i_1}\xi^{i_2}\cdots\xi^{i_n}s^n, \tag{15.9}$$

which are found to be

$$\begin{aligned}
\Gamma^i{}_{i_1\dots i_n} &= \tilde\nabla_{(i_1\dots}\ \tilde\nabla_{i_{n-2}}\Gamma^i{}_{i_{n-1}i_n)}, \quad \text{e.g.}\\
\Gamma^i{}_{i_1i_2i_3} &= \Gamma^i{}_{(i_2i_3,i_1)}-2\Gamma^i{}_{j(i_1}\Gamma^j{}_{i_2i_3)}=\tilde\nabla_{(i_1}\Gamma^i{}_{i_2i_3)}
\end{aligned} \tag{15.10}$$

where $\tilde\nabla$ indicates that the covariant derivative is to be taken with respect to the lower indices only. The domain of convergence depends on the metric and the values of ξ^i. However, for sufficiently small values of s it defines an integral curve of (15.7).

15.1.1 Background field expansion

Now we are ready to expand

$$\mathcal{L}=\tfrac12 g_{ij}(\phi)\partial_\mu\phi^i\partial^\mu\phi^j$$

about a background field ϕ_0:

$$\mathcal{L}=\tfrac12 g_{ij}(\phi^k_0+\xi^k-\tfrac12\Gamma^k_{mn}\xi^m\xi^n)\ \partial_\mu(\phi^i_0+\xi^i-\tfrac12\Gamma^i_{pq}\xi^p\xi^q)\partial^\mu(\phi^j_0+\xi^j-\tfrac12\Gamma^j_{rs}\xi^r\xi^s)+O(\xi^3).$$

When using

$$g_{mn,p}=\Gamma_{mnp}+\Gamma_{nmp}$$

one finds

$$\begin{aligned}
\mathcal{L}(\phi) \;=\;& \mathcal{L}(\phi_0)+g_{ij}\partial^\mu\phi^i_0\nabla_\mu\xi^j\\
&+\tfrac12 g_{ij}\partial_\mu\xi^i\partial^\mu\xi^j-\tfrac12 g_{ij}\Gamma^j{}_{mn,k}\partial_\mu\phi^i_0\partial^\mu\phi^k_0\xi^m\xi^n-\Gamma_{jmn}\partial_\mu\phi^j_0\partial^\mu\xi^m\xi^n\\
&+g_{ij,k}\partial_\mu\phi^i_0\partial^\mu\xi^j\xi^k+\tfrac14(g_{ij,kl}-g_{ij,s}\Gamma^l{}_{kl})\partial_\mu\phi^i_0\ \partial^\mu\phi^j_0\,\xi^k\xi^l,
\end{aligned}$$

where the metric, Christoffel symbols and their derivatives are to be evaluated at the background field ϕ_0. We have introduced the covariant derivative

$$\nabla_\mu \xi^i = \partial_\mu \xi^i + \Gamma^i{}_{jk} \partial_\mu \phi_0^j \, \xi^k \tag{15.11}$$

of the vectorfield ξ^i. To rearrange these rather complicated terms into covariant ones we use

$$g_{ij}\partial^\mu \xi^i \partial_\mu \xi^j = g_{ij}\nabla^\mu \xi^i \nabla_\mu \xi^j - 2\Gamma_{imn}\partial^\mu \phi_0^m \partial_\mu \xi^i \xi^n - \Gamma_{ipq}\Gamma^i{}_{mn}\partial_\mu \phi_0^p \, \partial^\mu \phi_0^m \xi^q \xi^n$$

and

$$(\tfrac{1}{2}g_{pm,qn} - \tfrac{1}{2}g_{pm,s}\Gamma^s{}_{qn} - g_{ps}\Gamma^s{}_{qn,m} - \Gamma^i{}_{pq}\Gamma_{imn})\partial_\mu \phi_0^p \partial^\mu \phi_0^m \xi^q \xi^n = R_{pqnm}\partial_\mu \phi_0^p \partial^\mu \phi_0^m \xi^q \xi^n .$$

This way we end up with the following covariant expansion in ξ:

$$\mathcal{L}(\phi) = \mathcal{L}(\phi_0) + g_{ij}\partial^\mu \phi_0^i \nabla_\mu \xi^j + \tfrac{1}{2}g_{ij}\nabla^\mu \xi^i \nabla_\mu \xi^j - \tfrac{1}{2}R_{pqmn}\partial_\mu \phi_0^p \, \partial^\mu \phi_0^m \, \xi^q \xi^n + O(\xi^3) \tag{15.12}$$

We read off the field equation for the background field

$$\nabla_\mu \partial^\mu \phi_0^i = 0 \tag{15.13}$$

and the fluctuation operator for the vector fluctuations

$$(S''\xi)_i = g_{ij}\nabla_\mu \nabla^\mu \xi^j - R_{p(in)m}\partial_\mu \phi_0^p \, \partial^\mu \phi_0^m \, \xi^n . \tag{15.14}$$

15.1.2 Riemann normal coordinates

Now let us simplify the calculation by choosing Riemann normal coordinates. In Riemann normal coordinates (RNC) the geodesics through p are straight lines defined by:

$$\bar{\phi}^i = \xi^i \, s. \tag{15.15}$$

This coordinates cover a neighbourhood of the point p. Substituting (15.15) into (15.8) one has

$$\phi^i = \phi_0^i + \bar{\phi}^i - \tfrac{1}{2}\Gamma^i{}_{jk}\bar{\phi}^i \bar{\phi}^j - \sum_{n=3}^\infty \frac{1}{n!}\,(\Gamma^i_{i_1 \dots i_n})_p \bar{\phi}^{i_1} \cdots \bar{\phi}^{i_n}. \tag{15.16}$$

The Jacobian

$$|\frac{\partial \phi^i}{\partial \bar{\phi}^j}| \neq 0$$

and thus the series (15.16) can be inverted. For this local system of coordinates the geodesic eqation can be written as

$$\frac{d^2}{ds^2}\bar{\phi}^i + \bar{\Gamma}^i{}_{jk}(\bar{\phi})\,\frac{d\bar{\phi}^j}{ds}\frac{d\bar{\phi}^k}{ds} = \bar{\Gamma}^i{}_{jk}(\bar{\phi})\,\frac{d\bar{\phi}^j}{ds}\frac{\bar{\phi}^k}{ds} = 0,$$

where $\bar\Gamma$ are the Christoffel symbols in Riemann normal coordinates. Since $\bar\phi = \xi s$, the Taylor expansion in s implies that all symmetric derivatives of the affine connection vanish at the origin in RNC:

$$\bar\Gamma^i{}_{(i_1 i_2\,,\,j_3\dots i_n)}(p) = 0.$$

In field theory we employ the background field expansion in terms of normal coordinates. If ϕ^i is close to ϕ_0^i, there is a unique geodesic which connects ϕ_0^i to ϕ^i and reaches ϕ^i in unit proper time. If ξ^i represents the tangent vector to the geodesic at ϕ_0^i we can parametrize ϕ in terms of ξ. Since ξ is a vector under coordinate changes the covariance of the expansion is ensured.

I particular, an arbitary second rank tensor field (e.g. g_{ij}) can be expanded according to

$$\bar T_{ij}(\bar\phi) = \bar T_{ij}(s\xi) = \sum_{n=0}^\infty \frac{s^n}{n!}\,\bar T_{ij\,,\,i_1\dots i_n}(p)\,\xi^{i_1}\cdots\xi^{i_n}. \tag{15.17}$$

The coefficients are tensors and can be expressed in terms of the components $R^i{}_{jpq}$ of the Riemann tensor and the covariant derivatives $\nabla_i T_{jk}$ and $\nabla_i R^j{}_{kpq}$. To find the explicit form for these coefficients we first express the symmetric derivatives of the Christoffel symbols,

$$\bar\Gamma^i_{j(j_1\,,\,j_2\dots j_n)},$$

in terms of the Riemannian curvature tensor and its covariant derivatives. For example,

$$\bar R^i{}_{(pq)j} = \tfrac12(\bar R^i{}_{pqj}+\bar R^i{}_{qpj}) = \tfrac12(\bar R^i{}_{pqj}+\bar R^i{}_{qpj}) + 3\bar\Gamma^i{}_{(pq\,,\,j)} = 3\bar\Gamma^i_{j(p,q)}$$

or

$$\bar\nabla_{(p}\bar R^i{}_{qr)j} = \partial_{(p}\bar R^i{}_{qr)j} = \bar\Gamma^i{}_{j(p,qr)} - \bar\Gamma^i{}_{(pq,r)j} = \bar\Gamma^i{}_{j(p,qr)} - \bar\Gamma^i{}_{(pr,q)r} - \tfrac12\bar\Gamma^i{}_{(jp,qr)} = -2\bar\Gamma^i{}_{(pq,r)j}.$$

For the identities containing higher order derivatives of the connection in RNC I refer to [58].

Now we may use these results to find the covariant form of the expansion coefficients in (15.17). For example, in RNC we have

$$\begin{aligned}\bar\nabla_{i_1}\bar T_{ij} &= \bar T_{ij,i_1}\\ \bar\nabla_{(i_1}\bar\nabla_{i_2)}\bar T_{ij} &= \bar T_{ij,(i_1i_2)} - \bar\Gamma^k{}_{i(i_2,i_1)}\bar T_{kj} - \bar\Gamma^k{}_{j(i_2,i_1)}\bar T_{ik}\\ &= \bar T_{ij,(i_1i_2)} - \tfrac13\bar R^k{}_{(i_1i_2)i}\bar T_{kj} - \tfrac13\bar R^k{}_{(i_1i_2)j}\bar T_{ik},\end{aligned}$$

and hence (for $s=1$)

$$\begin{aligned}T_{ij}(\bar\phi) &= \bar T_{ij} + \bar\nabla_{i_1}\bar T_{ij}\xi^{i_1}\\ &\quad+\tfrac12(\bar\nabla_{(i_1}\bar\nabla_{i_2)}\bar T_{ij} + \tfrac13\bar R^k{}_{(i_1i_2)i}\bar T_{kj} + \tfrac13\bar R^k{}_{(i_1i_2)j}\bar T_{ik})\xi^{i_1}\xi^{i_2} + \dots .\end{aligned} \tag{15.18}$$

All quantities on the right are to be evaluated at the point p with coordinates $\bar\phi_0$. Since this is an identity between tensors (recall that the ξ transform as a vector) it holds in any coordinate system. For example, we obtain

$$g_{ij}(\phi) = g_{ij}(\phi_0) + \tfrac13 R_{i(i_1i_2)j}(\phi_0)\,\xi^{i_1}\xi^{i_2} + O(\xi^3). \tag{15.19}$$

15.1.3 An alternative background field expansion

In the following exposition of an efficient method to arrive at a covariant background field expansion we follow Mukhi [62]. We begin by recalling that

$$\nabla_k g_{ij} = 0, \quad [\nabla_p, \nabla_q] f = 0, \qquad [\nabla_p, \nabla_q] V^i = R^i_{\ jpq} V^j \tag{15.20}$$

for a scalar field f and a vector field V^i. As earlier we define an interpolating field with

$$\phi^i(0) = \phi^i_0, \quad \frac{d\phi^i}{ds}|_{s=0} = \xi^i \quad \text{and} \quad \phi^i(1) = \phi^i \tag{15.21}$$

that satisfies the geodesic equation. Now we study the s-dependency of the Lagrangian along the geodesic

$$\mathcal{L}(s) = \tfrac{1}{2} g_{ij}(\phi(s)) \partial_\mu \phi^i(s)\, \partial^\mu \phi^j(s), \tag{15.22}$$

so that

$$\mathcal{L}(\phi) = L(1) = L(0) + \sum_{n=1}^\infty \frac{1}{n!} \frac{d^n L(s)}{ds^n}\Big|_{s=0} = L(0) + \sum_{n=1}^\infty \frac{1}{n!} (\nabla_s)^n L(s)\Big|_{s=0}, \tag{15.23}$$

where

$$\nabla_s = \dot\phi^i(s) \nabla_i \tag{15.24}$$

is the covariant derivative along the curve $\phi(s)$. From (15.20) it follows that

$$\begin{aligned} \nabla_s g_{ij} &= 0, \qquad \nabla_s \dot\phi^i(s) = 0 \\ \nabla_s \partial_\mu \phi^i &= \dot\phi^k (\partial_k \partial_\mu \phi^i + \Gamma^i_{\ jk} \partial_\mu \phi^j) = \partial_\mu \dot\phi^i + \Gamma^i_{\ jk} \partial_\mu \phi^j \dot\phi^k \equiv \nabla_\mu \nabla_s \phi^i \\ [\nabla_s, \nabla_\mu] V^i &= R^i_{\ jpq} V^j \dot\phi^p \partial_\mu \phi^q. \end{aligned} \tag{15.25}$$

Now we are ready to calculate the leading term in the series (15.22):

$$\begin{aligned} \mathcal{L}(\phi) &= \mathcal{L}(\phi_0) + g_{ij}(\phi_0) \nabla_\mu \xi^i \partial^\mu \phi^j_0 + \tfrac{1}{2} \Big(g_{ij} \nabla_\mu \xi^i \nabla^\mu \xi^j + g_{ij} \nabla_s \nabla_\mu \dot\phi^i \partial^\mu \phi^j_0 \Big) \\ &= \mathcal{L}(\phi_0) + g_{ij}(\phi_0) \nabla_\mu \xi^i \partial^\mu \phi^j_0 + \tfrac{1}{2} g_{ij} \nabla_\mu \xi^i \nabla^\mu \xi^j - \tfrac{1}{2} R_{jipq} \partial^\mu \phi^i_0 \partial_\mu \phi^p_0 \xi^j \xi^q. \end{aligned}$$

15.2 Susy sigma models

As in section (13.1) we start with a real superfield, but now with several components,

$$\Phi^i = A^i + \bar\alpha \psi^i + \tfrac{1}{2} \bar\alpha \alpha F^i.$$

The real A^i, F^i take their values in the curved target space. We calculate

$$\begin{aligned} \bar D \Phi^i D \Phi^j &= \bar\psi^i \psi^j + \bar\alpha (\psi^i F^j + \psi^j F^i) - i \bar\alpha \gamma^\mu \psi^j \partial_\mu A^i - i \bar\alpha \gamma^\mu \psi^i \partial_\mu A^j \\ &\quad + \bar\alpha \alpha (\partial_\mu A^i \partial^\mu A^j - \tfrac{i}{2} \bar\psi^i \partial\!\!\!/ \psi^j + \tfrac{i}{2} \partial_\mu \bar\psi^i \gamma^\mu \psi^j + F^i F^j). \end{aligned}$$

This bilinear form is contracted with the metric g_{ij} evaluated at Φ^k which has the following component expansion:

$$g_{ij}(\Phi^k) = g_{ij} + g_{ij,k}\bar{\alpha}\psi^k + \tfrac{1}{2}\bar{\alpha}\alpha(g_{ij,k}F^k - \tfrac{1}{2}g_{ij,kl}\bar{\psi}^k\psi^l),$$

where the argument of the metric and its derivatives on the right is the body A of the superfield.
After using the identities

$$g_{ij,k}\bar{\psi}^k\gamma^\mu\psi^j\partial_\mu A^i = \tfrac{1}{2}(g_{ij,k} - g_{jk,j})\bar{\psi}^k\gamma^\mu\psi^j\partial_\mu A^i = -\Gamma_{kij}\bar{\psi}^k\gamma^\mu\psi^i\partial_\mu A^i$$

and

$$(g_{ij,k} + g_{ji,k} - g_{ki,j})\bar{\psi}^k\psi^i F^j = (g_{ij,k} + g_{jk,i} - g_{ki,j})\bar{\psi}^k\psi^i F^j = 2\Gamma_{jki}\bar{\psi}^k\psi^i F^j$$

we find the following result in superspace

$$\tfrac{1}{2}g_{ij}(\Phi)\bar{D}\Phi^i\, D\Phi^j = \tfrac{1}{2}g_{ij}\bar{\psi}^i\psi^j + \bar{\alpha}(g_{ij}(\psi^i F^j - i\not{\partial}A^i\psi^j) + \tfrac{1}{2}g_{ij,k}\psi^k(\bar{\psi}^i\psi^j)) + \bar{\alpha}\alpha\mathcal{L}$$

giving rise to the Langrangian density

$$\begin{aligned}\mathcal{L} \;&=\; \tfrac{1}{2}g_{ij}\Big(\partial_\mu A^i\partial^\mu A^j - \tfrac{i}{2}\bar{\psi}^i\not{\partial}\psi^j + \tfrac{i}{2}\partial_\mu\bar{\psi}^i\gamma^\mu\psi^j + F^iF^j\Big)\\ &\quad -\tfrac{i}{2}\Gamma_{kij}\bar{\psi}^k\gamma^\mu\psi^j\partial_\mu A^i - \tfrac{1}{2}\Gamma_{jki}\bar{\psi}^k\psi^i F^j - \tfrac{1}{8}g_{ij,kl}(\bar{\psi}^k\psi^l)(\bar{\psi}^i\psi^j). \qquad (15.26)\end{aligned}$$

We may elminiate the auxiliary field by its algebraic equation of motion

$$F^i = \tfrac{1}{2}\Gamma^i{}_{pq}\bar{\psi}^p\psi^q$$

which leads to the following density for the dynamical fields

$$\tfrac{1}{2}g_{ij}(\Phi)\bar{D}\Phi^i\, D\Phi^j = \tfrac{1}{2}g_{ij}\bar{\psi}^i\psi^j + \tfrac{1}{2}\bar{\alpha}(g_{ip,q} + g_{iq,p} + g_{pq,i})\psi^i\,\bar{\psi}^p\psi^q - ig_{ij}\bar{\alpha}\gamma^\mu\psi^j\partial_\mu A^i + \bar{\alpha}\alpha\mathcal{L}$$

with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}g_{ij}(\partial_\mu A^i\partial^\mu A^j - i\bar{\psi}^i\gamma^\mu(D_\mu\psi)^j) - \tfrac{1}{8}(\Gamma_{rmn}\Gamma^r{}_{pq} + g_{pq,mn})(\bar{\psi}^m\psi^n)(\bar{\psi}^p\psi^q), \quad (15.27)$$

where we have introduced the covariant derivative

$$(D_\mu\psi)^j = \partial_\mu\psi^j + \Gamma^j{}_{pq}\partial_\mu A^p\psi^q,$$

which takes into account that ψ^j transforms in the tangent space of X. We have used, that

$$\partial_\mu g_{ij}\bar{\psi}^i\gamma^\mu\psi^j = 0,$$

on account of the antisymmetry of the fermionic bilinear, so that the terms containing derivatives of the Majorana spinors are equal, up to a total derivative. To make further progress we need the symmetry properties of the four-fermi term $(\bar{\psi}^m\psi^n)(\bar{\psi}^p\psi^q)$. Clearly, this term is symmetric in (m, n) and symmetric in (p, q). We can also prove the following property for the cyclic sum of three indices:

$$(\bar\psi^m\psi^n)(\bar\psi^p\psi^q) + (\bar\psi^m\psi^q)(\bar\psi^n\psi^p) + (\bar\psi^m\psi^p)(\bar\psi^q\psi^n) = 0. \tag{15.28}$$

This is shown with the help of the Fierz identity (13.1):

$$\begin{aligned} \ldots &= (\bar\psi^m\psi^n)(\bar\psi^p\psi^q) + (\bar\psi^n\psi^p)(\bar\psi^q\psi^m) + (\bar\psi^m\psi^p)(\bar\psi^q\psi^n) \\ &= (\bar\psi^m\psi^n)(\bar\psi^p\psi^q) - \tfrac{1}{2}\bar\psi^n(\bar\psi^q\psi^p + \gamma_\mu\bar\psi^q\gamma^\mu\psi^p + \gamma_*\bar\psi^q\gamma_*\psi^p)\psi^m \\ &\quad -\tfrac{1}{2}\bar\psi^m(\bar\psi^q\psi^p + \gamma_\mu\bar\psi^q\gamma^\mu\psi^p + \gamma_*\bar\psi^q\gamma_*\psi^p)\psi^n = 0. \end{aligned}$$

In the last step we used the symmetry properties of the bilinears in the Majorana fields. Now we are ready to prove that the last term in (15.27) is proportional to

$$\begin{aligned} R_{mpnq}\bar\psi^m\psi^n\bar\psi^p\psi^q &= -\tfrac{1}{2}(g_{mn,pq} + g_{pq,mn} - g_{mq,pn} - g_{pn,mq})\,\bar\psi^m\psi^n\bar\psi^p\psi^q \\ &\quad -(\Gamma_{kmn}\Gamma^k_{\ pq} - \Gamma_{kmq}\Gamma^k_{\ pn})\,\bar\psi^m\psi^n\bar\psi^p\psi^q. \end{aligned}$$

Using the symmetry and cyclicity properties of the four fermi term we may rewrite the terms containig second derivatives of the metric as follows

$$-\tfrac{1}{2}(2g_{mn,pq} - g_{mq,pn} - g_{mp,nq})\,\bar\psi^m\psi^n\bar\psi^p\psi^q = -\tfrac{1}{2}(2g_{mn,pq} + g_{mn,qp})\ldots = -\tfrac{3}{2}g_{mn,pq}\ldots.$$

Similarly, the terms containing the Christoffel symbols simplifiy as follows,

$$\begin{aligned} &(\Gamma_{kmn}\Gamma^k_{\ pq} - \Gamma_{kmq}\Gamma^k_{\ pn})\,\bar\psi^m\psi^n\bar\psi^p\psi^q = \tfrac{3}{2}(\tfrac{2}{3}\Gamma_{kmn}\Gamma^k_{\ pq} - \tfrac{1}{3}\Gamma_{kmq}\Gamma^k_{\ pn} - \tfrac{1}{3}\Gamma_{kmp}\Gamma^k_{\ nq})\,\ldots \\ &= \tfrac{3}{2}(\tfrac{2}{3}\Gamma_{kmn}\Gamma^k_{\ pq} + \tfrac{1}{3}\Gamma_{kmn}\Gamma^k_{\ qp})\ldots = \tfrac{3}{2}\Gamma_{kmn}\Gamma^k_{\ pq}\ldots. \end{aligned}$$

If follows that

$$R_{mpnq}\bar\psi^m\psi^n\bar\psi^p\psi^q = -\tfrac{3}{2}\Big(g_{mn,pq} + \Gamma_{kmn}\Gamma^k_{\ pq}\Big)\bar\psi^m\psi^n\bar\psi^p\psi^q \tag{15.29}$$

so that the density aquires the following rather simple form

$$\tfrac{1}{2}g_{ij}(\Phi)\bar D\Phi^i\,D\Phi^j = \tfrac{1}{2}g_{ij}\bar\psi^i\psi^j - ig_{ij}\bar\alpha\gamma^\mu\psi^j\partial_\mu A^i + \bar\alpha\alpha\mathcal{L}$$

with Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}g_{ij}(\partial_\mu A^i\partial^\mu A^j - i\bar\psi^i\gamma^\mu(D_\mu\psi)^j) + \tfrac{1}{12}R_{mpnq}\,(\bar\psi^m\psi^n)(\bar\psi^p\psi^q). \tag{15.30}$$

Clearly, the objects $\partial_\mu A^i$ and ψ^i must transform as target space vectors. In addition, $\partial_\mu A^i$ is a spacetime vector.
To find the susy transformation of $\mathcal{L}$ we note that $g_{ij}(\Phi)\bar D\Phi^i\Phi^j$ is a real superfield, similarly as Φ in (13.5). From (13.7) we read off, that under the (on-shell) susy transformations

$$\delta A^i = \bar\beta\psi^i \quad\text{and}\quad \delta\psi^i = (\tfrac{1}{2}\Gamma^i_{\ pq}\bar\psi^p\psi^q + i\partial\!\!\!/ A^i)\beta$$

the Lagrangian transforms into a derivative, as it must be,

$$\delta\mathcal{L} = \partial_\mu(\bar\beta V^\mu), \qquad V^\mu = \tfrac{1}{2} g_{ij}\gamma^\mu\gamma^\nu\psi^i\partial_\nu A^j.$$

The Noether current is gotten from

$$\begin{aligned}\sum \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\phi &= g_{ij}(\partial^\mu A^i\delta A^j - \tfrac{i}{2}\bar\psi^i\gamma^\mu\psi^l\Gamma^j_{\ kl}\delta A^k - \tfrac{i}{2}\bar\psi^i\gamma^\mu\delta\psi^j)\\ &= \bar\beta g_{ij}(\tfrac{3}{2}\partial^\mu A^i\psi^j + \tfrac{1}{2}\gamma_*\psi^i\epsilon^{\mu\nu}\partial_\nu A^j),\end{aligned}$$

where we made use of $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - \epsilon^{\mu\nu}\gamma_*$. This then leads to the following Noether current,

$$J^\mu = g_{ij}(\eta^{\mu\nu} + \gamma_*\epsilon^{\mu\nu})\partial_\nu A^i\psi^j, \tag{15.31}$$

which is almost identical to the one of the supersymmetric Wess-Zumino model (13.11).

15.3 Supersymmetric 0(3)-model

If the target manifold is $S^2 = SO(3)/SO(2)$, then the model possesses a global $SO(3)$-invariance and is called $O(3)$ model. The model maybe reformulated in terms of fields propagating on CP^1 and in this formulation it is called CP^1-model.
We use the following conformally flat metric on S^2:

$$g_{ij} = \frac{1}{N^2}\delta_{ij} \quad \text{with} \quad e^{2\sigma} = \frac{1}{N^2} = \frac{1}{(1+u^iu^i)^2}, \quad u^i \in \mathbb{R}^2 \tag{15.32}$$

With $\sigma = -\log(1+\vec{u}\vec{u})$ we may immediately apply the formulae (15.3-15.6) which yield

$$\Gamma^p_{\ ij} = -\frac{2}{N}(u^j\delta_{pi} + u^i\delta_{pj} - u^p\delta_{ij}) \quad \text{and} \quad \partial^2\sigma = -\frac{1}{N^2}$$

which leads to the following supersymmetric Lagrangian

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\frac{1}{(1+\vec{u}\vec{u})^2}\Big(\partial_\mu u^i\partial^\mu u^i - i\bar\psi^i\not{\partial}\psi^i + \frac{2i}{N}\bar\psi^i\gamma^\mu\psi^j(u^j\partial_\mu u^i - u^i\partial_\mu u^j)\Big)\\ &\quad + \tfrac{1}{2}\frac{1}{(1+\vec{u}\vec{u})^4}\ \bar\psi^1\psi^1\bar\psi^2\psi^2.\end{aligned}$$

The Lagrangian further simlifies if we introduce complex coordinates on S^2 and combine the two Majorana spinor to a Dirac spinor,

$$u = u^1 + iu^2 \quad \text{and} \quad \psi = \psi^1 + i\psi^2, \quad \bar\psi = \bar\psi^1 - i\bar\psi^2.$$

Then

$$\begin{aligned}&\partial_\mu u^i\partial^\mu u^i = \partial_\mu\bar{u}\partial^\mu u\\ &\bar\psi^i\not{\partial}\psi^i = \bar\psi\not{\partial}\psi - i\partial_\mu(\bar\psi^1\gamma^\mu\psi^2), \qquad \partial_\mu\bar\psi^i\gamma^\mu\psi^i = \partial_\mu\bar\psi\gamma^\mu\psi - i\partial_\mu(\bar\psi^1\gamma^\mu\psi^2)\\ &\bar\psi^i\gamma^\mu\psi^j(u^j\partial_\mu u^i - u^i\partial_\mu u^j) = \tfrac{1}{2}\bar\psi\gamma^\mu\psi(\bar{u}\partial_\mu u - u\partial_\mu\bar{u})\end{aligned}$$

and the Lagrangian density now takes the simple form

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\frac{1}{(1+\bar{u}u)^2}\Big(\partial_\mu\bar{u}\partial^\mu u - \tfrac{i}{2}\bar{\psi}^i\partial\!\!\!/\psi^i + \tfrac{i}{2}\partial_\mu\bar{\psi}\gamma^\mu\psi + \frac{i}{N}\bar{\psi}\gamma^\mu\psi(\bar{u}\partial_\mu u - u\partial_\mu\bar{u})\Big)\\ &\quad+\tfrac{1}{4}\frac{1}{(1+\bar{u}u)^4}\,(\bar{\psi}\psi)^2.\end{aligned}$$

The supersymmetry transformation of u and ψ are

$$\delta u = \bar{\beta}\psi \quad \text{and} \quad \delta\psi = \frac{2}{N}\bar{u}(\bar{\beta}\psi)\psi + i\partial\!\!\!/ u\beta,$$

where β is a Majorana spinor (contrary to ψ). Actually, this model admits an extended $N=2$ supersymmetry.

Kapitel 16

SYM in $d = 2$ dimensions

In the chapter we consider various two-dimensional supersymmetric gauge theories, characterized by the number of supercharges.

16.1 Dimensional Reduction from 10 dimensions

The $\mathcal{N} = (8,8)$ supersymmetric Yang-Mills theory in 2 dimension is obtained from $\mathcal{N} = 1$ supersymmetric Yang-Mills theory in 10 dimensions by a Kaluza-Klein reduction on a 8-dimensional torus. After compactification eight components of the vector potential become scalar fields. A Weyl-Majorana Dirac spinor in 10 dimensions has 16 real or 8 complex components which branch into eight Majorana spinors or equivalently into four Dirac spinors in 2 dimensions. Thus we expect that the on-shell $\mathcal{N} = (8,8)$ model has one gauge potential, 8 real scalars in the adjoint representation and 8 Majorana spinors in the adjoint representation.

16.1.1 First choice of Γ matrices

We start with the Dirac matrices γ^μ in 2-dimensional Minkowski space and give an explicit realization for the matrices $\Gamma^m, m = 0, \dots, 9$ in 10 dimensions: We make the ansatz

$$\Gamma_\mu = \Delta \otimes \gamma_\mu, \quad \Gamma_{1+a} = \Delta_a \otimes \gamma_*, \quad \mu = 0, 1, \quad a = 1, \dots, 8$$

with 16×16 matrices Δ and Δ_a. To see what are the condition on the Δ-factors such that

$$\{\Gamma_m, \Gamma_n\} = 2\eta_{mn}, \qquad \eta = \mathrm{diag}(1, -1, \dots, -1) \tag{16.1}$$

holds, one uses that for $[A, B] = 0$ on has

$$\{A \otimes C, B \otimes D\} = AB \otimes \{C, D\} \quad \text{and} \quad \{C \otimes A, D \otimes B\} = \{C, D\} \otimes AB. \tag{16.2}$$

and for $\{A, B\} = 0$ one has

$$\{A \otimes C, B \otimes D\} = AB \otimes [C, D] \quad \text{and} \quad \{C \otimes A, D \otimes B\} = [C, D] \otimes AB. \tag{16.3}$$

Now it is easy to see that we must demand

$$\Delta^2 = \mathbb{1}_{16}, \quad [\Delta, \Delta_a] = 0, \qquad \{\Delta_a, \Delta_b\} = -2\delta_{ab}\mathbb{1}_{16}.$$

for (16.1) to hold true. Since Γ^0 is hermitean and the $\Gamma^{m>0}$ antihermitean and since $\gamma_5 = \gamma^0\gamma^1$ is hermitean it also follows from

$$(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger.$$

that

$$\Delta^\dagger = \Delta, \quad \Delta_a^\dagger = -\Delta_a.$$

Other useful identities which we shall sometimes need are

$$(A \otimes B)^T = A^T \otimes B^T \quad \text{and for vectors} \quad (x \otimes y, u \otimes v) = (x, u)(y, v). \tag{16.4}$$

Since Δ commutes with all matrices and squares to $\mathbb{1}_{16}$ we may choose it to be the identity,

$$\Delta = \mathbb{1}_{16}.$$

Note that the hermitean $i\Delta_a$ generate the Euclidean Clifford algebra in 8 dimensions and that the $[\Delta_a, \Delta_b]$ generate the group $spin(8)$. Earlier we have shown that in 8 Euclidean dimensions there is a Majorana-Weyl representation. Hence we may choose Δ_a to be real and antisymmetric. With our earlier convention the hermitean $\Gamma_{11} = \Gamma^0 \cdots \Gamma^9$ takes the form

$$\Gamma_{11} = \Gamma_* \otimes \gamma^0\gamma^1 = \Gamma_* \otimes \gamma_*, \qquad \Gamma_* = \Gamma_*^\dagger = \Delta_1 \cdots \Delta_8, \quad \gamma_*^\dagger = \gamma_*. \tag{16.5}$$

If we would choose a Majorana representation with imaginary γ^μ, for example $\{\gamma^0, \gamma^1\} = \{\sigma_2, \mathrm{i}\sigma_1\}$ then $\gamma_* = \gamma^0\gamma^1$ is real. For the given representation $\gamma_* = \sigma_3$.
The eight anti-hermitean matrices Δ_a are 16×16 matrices. We take the following imaginary and symmetric matrices:

$$\Delta_a = \begin{pmatrix} 0 & \tilde{\Delta}_a \\ \tilde{\Delta}_a^T & 0 \end{pmatrix} \Longrightarrow \tilde{\Delta}_a \tilde{\Delta}_b^T + \tilde{\Delta}_b \tilde{\Delta}_a^T = -2\delta_{ab}\mathbb{1}_{16}$$

where the $\tilde{\Delta}_a$ are imaginary 8×8 matrices. We may choose

$$\tilde{\Delta}_1 = \mathrm{i}\sigma_0 \otimes \mathbb{1}_4, \;\; \tilde{\Delta}_2 = \sigma_2 \otimes \mathbb{1}_4, \;\; \tilde{\Delta}_{2+i} = \sigma_1 \otimes \alpha_i, \;\; \tilde{\Delta}_{5+i} = \sigma_3 \otimes \beta_i,$$

with antisymmetric and imaginary matrices

$$\begin{aligned} \alpha_1 &= \sigma_2 \otimes \sigma_1, \qquad & \alpha_2 &= \sigma_0 \otimes \sigma_2, \quad & \alpha_3 &= \sigma_2 \otimes \sigma_3 \\ \beta_1 &= \sigma_1 \otimes \sigma_2, \qquad & \beta_2 &= \sigma_2 \otimes \sigma_0, \quad & \beta_3 &= \sigma_3 \otimes \sigma_2. \end{aligned}$$

With this choice we have

$$\Gamma_* = \mathrm{i}\sigma_0 \otimes \sigma_2 \otimes \mathbb{1}_4\,.$$

Since the Δ_a are imaginary it follows that the Γ_m are imaginary as well such that the charge conjugation matrix is $\mathcal{C}_{10} = -\Gamma^0 = \mathbb{1}_8 \otimes (-\gamma^0)$. Hence we are lead to take the following symmetric charge conjugation matrix

$$\mathcal{C}_{10} = \mathbb{1}_8 \otimes \mathcal{C}_2 \tag{16.6}$$

Indeed, since Γ_0 is anti-symmetric and the $\Gamma_1, \Gamma_2, \dots$ are symmetric, it is easily seen that

$$\mathcal{C}_{10}\Gamma_m\mathcal{C}_{10}^{-1} = \Gamma^0\Gamma_m\Gamma^0 = -\Gamma_m^T. \tag{16.7}$$

Let us first see how Majorana spinors look like. Since

$$\Psi_c = \mathcal{C}_{10}\Gamma^{0T}\Psi^* = (-\Gamma^0)\Gamma^{0T}\Psi^* = \Psi^*$$

the Majorana condition implies that Ψ is real. For $\Psi = \xi \otimes \chi$ this means

$$\Psi_c = \Psi \Longleftrightarrow \ \xi \in \mathbb{R}^{16}, \ \chi_c = \mathcal{C}_2\gamma^{0T}\chi^* = \chi. \tag{16.8}$$

We conclude that a Majorana spinor in 10 space-time dimensions has the expansion

$$\Psi = \sum_{r=1}^{16} E_r \otimes \chi_r, \tag{16.9}$$

where the χ_r are Majorana spinors in 2 dimensions and the E_r form a (real) base in $\mathbb{R}^{16}$. A spinor has positive chirality if

$$\Psi = \Gamma_{11}\Psi = (\Gamma_* \otimes \gamma_5)\Psi. \tag{16.10}$$

16.1.2 A second choice for the Γ-matrices

We begin with a representation of the Euclidean Clifford algebra

$$\{\Delta^I, \Delta^J\} = 2\delta^{IJ}\mathbb{1}_{16}, \quad I, J \in \{1, \dots, 8\} \tag{16.11}$$

in 8 dimensions in terms of real and symmetric matrices Δ^I. A convenient choice is

$$\begin{array}{lcl}
\Delta^1 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2 & , & \Delta^2 = \sigma_2 \otimes \sigma_0 \otimes \sigma_1 \otimes \sigma_2 \\
\Delta^3 = \sigma_2 \otimes \sigma_0 \otimes \sigma_3 \otimes \sigma_2 & , & \Delta^4 = \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_0 \\
\Delta^5 = \sigma_2 \otimes \sigma_3 \otimes \sigma_2 \otimes \sigma_0 & , & \Delta^6 = \sigma_2 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_1 \\
\Delta^7 = \sigma_2 \otimes \sigma_2 \otimes \sigma_0 \otimes \sigma_3 & , & \Delta^8 = \sigma_1 \otimes \sigma_0 \otimes \sigma_0 \otimes \sigma_0 \,,
\end{array} \tag{16.12}$$

and with this choice we have

$$\Delta_* = \Delta^1\Delta^2 \cdots \Delta^8 = \begin{pmatrix} \mathbb{1}_8 & 0 \\ 0 & -\mathbb{1}_8 \end{pmatrix}. \tag{16.13}$$

Now we can construct the 32×32 matrices

$$\Gamma^0 = \gamma^0 \otimes \mathbb{1}_{16}, \quad \Gamma^1 = \gamma^1 \otimes \Delta_*, \quad \Gamma^{I+1} = \gamma^1 \otimes \Delta^I\,. \tag{16.14}$$

The matrix Γ_* takes the simple form

$$\Gamma_* = \Gamma^0\Gamma^1\cdots\Gamma^9 = \gamma^0\gamma^1\otimes\Delta_*^2 = \gamma_*\otimes \mathbb{1}_{16}\,. \tag{16.15}$$

the matrices Γ^m are imaginary. The charge conjugation matrices in 10 and 2 dimensions are related via

$$\mathcal{C}_{10} = \mathcal{C}_2\otimes\mathbb{1}_{16}\quad\text{with}\quad \gamma^{\mu T} = \pm\,\mathcal{C}^{-1}\gamma^\mu\mathcal{C}\,. \tag{16.16}$$

For the particular choice

$$\gamma^0 = \sigma_2,\quad \gamma^1 = \mathrm{i}\sigma_1\quad\text{with}\quad \gamma_* = \sigma_3,\quad \mathcal{C}_2 = -\sigma_2 \tag{16.17}$$

the Γ^m and γ^μ matrices are all imaginary and the the second equation in (16.16) with a minu sign applies. Thus with this choice a Majorana spinor obeying

$$\Psi = \Psi_c = \mathcal{C}\bar{\Psi}^T \tag{16.18}$$

is real.

16.1.3 Reduction of Yang-Mills term

In 10 spacetime dimensions a gauge field and gauge coupling constant have the dimensions

$$[A_m] = L^{-4},\quad [g_{10}] = L^3 \Longrightarrow [g_{10}A_m] = L^{-1}\,. \tag{16.19}$$

We may absorb the coupling constant in the gauge potential, $A_m\to g_{10}A_m$ such that the 10-dimensional coupling constant appears only in front of the the Yang-Mills action,

$$S_{\rm YM} = -\frac{1}{4g_{10}^2}\int \mathrm{d}^{10}x\,\mathrm{Tr}\,F_{mn}F^{mn}. \tag{16.20}$$

Now we do a Kaluza-Kleine reduction of action on $\mathbb{R}^2\times T^8$. As internal space we choose the 8-dimensional torus T^8 with volume V_8. We write

$$A_m = (A_0,\dots,A_9) = (A_0,A_1,\Phi_1,\dots,\Phi_8)\,, \tag{16.21}$$

and assume all fields are *independent of the internal coordinates* $x^2,\dots,x^9$ and only depend on the coordinates x^0 and x^1. We find the following decomposition of the field strength,

$$F_{\mu,1+a} = \partial_\mu\Phi_a - i[A_\mu,\Phi_a]\quad,\quad F_{1+a,1+b} = -i[\Phi_a,\Phi_b]\,. \tag{16.22}$$

Inserting this into the 10-dimensional Yang-Mills action we find

$$S_{\rm YM}\to\frac{1}{4g^2}\int d^2x\ \mathrm{Tr}\left(-F_{\mu\nu}F^{\mu\nu} + 2\sum_a D^\mu\Phi_a D_\mu\Phi_a + \sum_{ab}[\Phi_a,\Phi_b]^2\right), \tag{16.23}$$

where we took into account, that $\Phi_a = -\Phi^a$ and where the dimensionful coupling constant g_{10} and the *dimensionless coupling constant* g are related as

$$g^2 = g_{10}^2/V_8 \Longrightarrow [g] = L^{-1} \tag{16.24}$$

Now we may rescale the fields with g to obtain after the Kaluza-Klein reduction the following Lagrangian in 4-dimensional Minkowski spacetime:

$$\mathcal{L}_{\rm YM} \to \mathcal{L}_{\rm YMH} = {\rm Tr}\left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\sum_a D^\mu\Phi_a D_\mu\Phi_a + \tfrac{1}{4}g^2\sum_{ab}[\Phi_a,\Phi_b]^2\right). \tag{16.25}$$

The covariant derivative is $D_\mu = \partial_\mu - {\rm i}g\ {\rm ad}A_\mu$. Not unexpectedly we have gotten the action for a four-dimensional Yang-Mills-Higgs theory with 8 Higgs fields in the adjoint representation. All fields in (16.25) are dimensionless. The fields in (16.23) have dimension $1/L$.

16.1.4 Reduction of Dirac term

In 10 and 2 space-time dimensions a spinor field has the dimension

$$[\Psi] = L^{-9/2} \quad \text{and} \quad [\psi] = L^{-1/2},$$

respectively. We use Majorana representations with imaginary γ-matrices in 10 and 2 dimensions and parameterize a Weyl spinor $\Gamma_*\Psi = \Psi$ in 10 dimensions as

$$\Psi = \frac{1}{\sqrt{V_8}}\sum_{a=1}^{8}\psi_a\otimes e_a + \frac{\rm i}{\sqrt{V_8}}\sum_{b=9}^{16}\gamma^0\psi_b\otimes e_b\,, \tag{16.26}$$

where the $e_1,\dots,e_{16}$ form the conventional cartesian base in $\mathbb{R}^{16}$, e.g. $e_1 = (1,0,\dots,0)^T$. The spinor is right-handed if

$$\Gamma_*\Psi = \Psi \Longleftrightarrow \gamma_*\psi_a = \psi_a,\ \ \gamma_*\psi_b = -\psi_b\,, \tag{16.27}$$

and it is a Majorana in case

$$\Psi^* = \Psi \Longleftrightarrow \psi_a^* = \psi_a,\ \ \psi_b = \psi_b^*\,. \tag{16.28}$$

The corresponding expansion of the Dirac-conjugate spinor reads

$$\bar\Psi = \frac{1}{\sqrt{V_8}}\sum_a\bar\psi_a\otimes e_a^T - \frac{\rm i}{\sqrt{V_8}}\sum_b\bar\psi_b\gamma^0\otimes e_b^T\,. \tag{16.29}$$

The Dirac term decomposes according to

$${\rm Tr}\,\bar\Psi\,\Gamma^m D_m\Psi = \frac{1}{V_8}\sum_{p=1}^{16}{\rm Tr}\,\bar\psi_p\not{D}\,\psi_p - {\rm i}\sum_{a=1}^{8}\bar\Psi\,\Gamma^{a+1}[\Phi_a,\Psi] \qquad \not{D} = \gamma^\mu D_\mu\,. \tag{16.30}$$

To reduce the last term we insert the decompositions (16.26) and (16.29).

16.1.5 Reduction of the Dirac term

In 10 and 4 dimensions a spinor field has the dimension

$$[\Psi]=L^{-9/2}\quad\text{and}\quad[\chi]=L^{-3/2},$$

respectively. We start with the general expansions (10.13,10.14) for a 10-dimensional Majorana-Weyl spinor and its adjoint. We rescale the spinors such that the χ_p in

$$\Psi=\frac{1}{\sqrt{V_6}}\sum_{p=1}^4(F_p\otimes P_+\chi_p+F_p^*\otimes P_-\chi_p)\quad,\quad\bar\Psi=\frac{1}{\sqrt{V_6}}\sum_{p=1}^4(F_p^\dagger\otimes\bar\chi_pP_-+F_p^T\otimes\bar\chi_pP_+)$$

have the dimension of a spinorfield in 4-dimensional Minkowski spacetime. The spinor should be independent of the internal coordinates. Again we absorb the 10-dimensional gauge coupling constant in the gauge potential. We find

$$\begin{aligned}D_\mu\Psi&=\frac{1}{\sqrt{V_6}}\sum_p(F_p\otimes D_\mu P_+\chi_p+F_p^*\otimes D_\mu P_-\chi_p),\qquad\mu=0,1,2,3\\ D_{3+a}\Psi&=-\frac{i}{\sqrt{V_6}}\sum_p(F_p\otimes[\Phi_a,P_+\chi_p]+F_p^*\otimes[\Phi_a,P_-\chi_p]),\quad a=1,\dots,6.\end{aligned}$$

Now we may rewrite the Dirac term in 10 dimensions as follows:

$$\begin{aligned}\int d^{10}x\,\mathrm{Tr}\,\bar\Psi\Gamma^mD_m\Psi&=\int d^4x\,\mathrm{Tr}\,\bar\chi_p\not{D}\chi_p\\&-\;i\int d^4x\,\mathrm{Tr}\Big\{(\Delta_+^a)_{pq}\bar\chi_pP_+[\Phi_a,\chi_q]-(\Delta_-^a\Gamma_*)_{pq}\bar\chi_pP_-[\Phi_a,\chi_q]\Big\},\end{aligned}$$

where one should sum over the indices on the right. We have introduced

$$(\Delta_+^a)_{pq}=(F_p^*,\Delta^aF_q)\quad\text{and}\quad(\Delta_-^a)_{pq}=(F_p,\Delta^aF_q^*)\quad\text{with}\quad\Delta_+^a=(\Delta_-^a)^*.\tag{16.31}$$

Putting the various terms together we end up with the following $N=4$ supersymmetric gauge theory in 4-dimensional Minkowski spacetime:

$$\begin{aligned}\mathcal{L}&=\mathrm{Tr}\Big(-\tfrac14F_{\mu\nu}F^{\mu\nu}+\tfrac12D^\mu\Phi_aD_\mu\Phi_a+\tfrac14g^2[\Phi_a,\Phi_b]^2\Big)\\&\quad+\tfrac12\mathrm{Tr}\Big(i\bar\chi_p\not{D}\chi_p+g(\Delta_+^a)_{pq}\bar\chi_pP_+[\Phi_a,\chi_q]-g(\Delta_-^a)_{pq}\bar\chi_pP_-[\Phi_a,\chi_q]\Big),\end{aligned}\tag{16.32}$$

where $a=1,\dots,6$ and $p=1,\dots,4$.

Kapitel 17

Susy sigma models in 4 dimensions

In 4 dimensions a real superfield is not irreducible and therefore we used a chiral superfield to construct the Wess-Zumino model. The Wess-Zumino model is just a linear σ-model and we expect that nonlinear σ-models also make use of chiral superfields. This is indeed the case as we shall demonstrate now.

17.1 Superfield formulation

We start with the simplest sigma models in 4 dimensions, namely the ones for which the target space is Kähler manifold with one complex dimension.
Let us discress and explain the notion of a Kähler manifold. One conveniently starts with an *almost complex manifold*: A $2n$-dimensional real manifolds M is called almost complex manifold if we can assign to each point $x \in M$ a linear and bijective map

$$J : TM_x \longrightarrow TM_x$$

of the tangential space at x with the property

$$J^2 = -\mathbb{1}.$$

Almost complex manifolds are always orientable. A hermitean metric on an almost complex manifold M is a Riemannian metric with the additional property that

$$g_x(Jv, Jw) = g_x(v, w) \qquad \forall v, w \in TM_x$$

and for all $x \in M$. Next one may define a 2-form ϕ by

$$\phi_x(v, w) = g_x(v, Jw) \qquad \forall v, w \in TM_x, \quad x \in M.$$

It is called the *fundamental form* of the hermitean metric. If this 2-form is closed,

$$d\phi = 0,$$

the g is called a *Kähler metric* and M almost Kähler manifold.

Now let Φ be a chiral superfield, the body of which propagates on a Kähler manifold,

$$\begin{aligned}\Phi &= A(y)+\sqrt{2}\theta\psi(y)+\theta^2F(y)\\ &= A+i\theta\sigma^\mu\bar\theta\partial_\mu A-\tfrac14\theta^2\bar\theta^2\Box A+\sqrt{2}\theta\psi+i\sqrt{2}\theta\sigma^\mu\bar\theta\theta\partial_\mu\psi+\theta^2F=A+\delta\Phi\end{aligned}$$

such that

$$\begin{aligned}\Phi^\dagger &= A^*(z)+\sqrt{2}\bar\theta\bar\psi(z)+\bar\theta^2F^*(z)\\ &= A^*-i\theta\sigma^\mu\bar\theta\partial_\mu A^*-\tfrac14\theta^2\bar\theta^2\Box A^*+\sqrt{2}\bar\theta\bar\psi-i\sqrt{2}\theta\sigma^\mu\bar\theta\bar\theta\partial_\mu\bar\psi+\bar\theta^2F^*\equiv A^*+\delta\Phi^\dagger.\end{aligned}$$

The argument of the component fields in the last two lines of the previous formulae is x. Now we expand the superfield

$$K(\Phi,\Phi^\dagger)=K(A+\delta\Phi,A^*+\delta\Phi^\dagger)$$

in powers of $\delta\Phi$ and $\delta\Phi^\dagger$. In the course of calculation one needs the Fierz identity

$$(\theta\sigma^\mu\bar\theta)(\theta\psi)(\bar\theta\bar\psi)=\tfrac14\theta^2\,\bar\theta^2\,(\psi\sigma^\mu\bar\psi).$$

After some algebra one finds

$$\begin{aligned}K(\Phi,\Phi^\dagger) &= K+\sqrt{2}(K_A\theta\psi+K_{\bar A}\bar\theta\bar\psi)\\ &+\theta^2(K_AF-\tfrac12K_{AA}\psi^2)+\bar\theta^2(K_{\bar A}F^*-\tfrac12K_{\bar A\bar A}\bar\psi^2)+i\theta\sigma^\mu\bar\theta(K_A\partial_\mu A-K_{\bar A}\partial_\mu A^*+K_{A\bar A}\psi\sigma_\mu\bar\psi)\\ &+i\sqrt{2}\,\theta\sigma^\mu\bar\theta\Big(\partial_\mu\{K_A\theta\psi-K_{\bar A}\,\bar\theta\bar\psi\}+2K_{A\bar A}\{\partial_\mu A\,\bar\theta\bar\psi-\partial_\mu A^*\theta\psi\}\Big)\\ &+\sqrt{2}\,\theta^2\,\bar\theta\bar\psi(K_{A\bar A}F-\tfrac12K_{AA\bar A}\psi^2)+\sqrt{2}\,\bar\theta^2\,\theta\psi(K_{A\bar A}F^*-\tfrac12K_{\bar A\bar AA}\bar\psi^2)\\ &+\tfrac14\theta^2\bar\theta^2\Big(-\Box K+4K_{A\bar A}\{\partial_\mu A\partial^\mu A^*-\tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi+\tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi+FF^*\}\\ &\qquad+2K_{AA\bar A}\{i\psi\sigma^\mu\bar\psi\partial_\mu A-\psi^2F^*\}-2K_{\bar A\bar AA}\{i\psi\sigma^\mu\bar\psi\partial_\mu A^*+\bar\psi^2F\}+K_{AA\bar A\bar A}\psi^2\bar\psi^2\Big),\end{aligned}$$

where we have used

$$\begin{aligned}\partial_\mu(K_A\theta\psi) &= K_A\theta\partial_\mu\psi+K_{AA}\partial_\mu A\theta\psi+K_{A\bar A}\partial_\mu A^*\theta\psi\\ \Box K &= K_{AA}\partial_\mu A\partial^\mu A+K_{\bar A\bar A}\partial_\mu A^*\partial^\mu A^*+K_A\Box A+K_{\bar A}\Box A^*+2K_{A\bar A}\partial_\mu A\partial^\mu A^*\end{aligned}$$

to simplify the terms trilinear and quartic in θ. Now we may read off the D-term

$$\begin{aligned}\tfrac12K(\Phi,\Phi^\dagger)|_D &= -\tfrac18\Box K+\tfrac12K_{A\bar A}(\partial_\mu A^*\partial^\mu A+\tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar\psi-\tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar\psi+F^*F)\\ &+\tfrac14K_{AA\bar A}(i\psi\sigma^\mu\bar\psi\partial_\mu A-\psi^2F^*)-\tfrac14K_{A\bar A\bar A}(i\psi\sigma^\mu\bar\psi\partial_\mu A^*+\bar\psi^2F)\\ &+\tfrac18K_{AA\bar A\bar A}\psi^2\bar\psi^2,\end{aligned}$$

The corresponding action should be invariant under

$$\delta A=\sqrt{2}\zeta\psi,\quad \delta\psi=\sqrt{2}\zeta F+i\sqrt{2}(\sigma^\mu\bar\zeta)\partial_\mu A,\quad \delta F=i\sqrt{2}\bar\zeta\tilde\sigma^\mu\partial_\mu\psi.$$

The algebraic equations of motion for the auxiliary field can easily be solved,

$$F=\tfrac12\partial_A\log(K_{A\bar A})\psi^2\quad\text{or}\quad F^*=\tfrac12\partial_{\bar A}\log(K_{A\bar A})\bar\psi^2.$$

17.1.1 Example: The O(3)-Model

Let us now have a closer look at the $O(3)$ sigma model with target manifold S^2. S^2 is a Kähler manifold with Kähler potential

$$K = \log(1 + \bar{u}u),$$

where the complex scalar field is denoted by u instead of A. We need

$$K_u = \frac{\bar{u}}{1 + \bar{u}u}, \quad K_{u\bar{u}} = \frac{1}{(1 + \bar{u}u)^2}, \quad K_{uu\bar{u}} = -2\frac{\bar{u}}{(1 + \bar{u}u)^3}, \quad K_{uu\bar{u}\bar{u}} = 2\frac{2\bar{u}u - 1}{(\bar{u}u + 1)^4}.$$

Clearly, $K_{\bar{u}}$ is just the complex conjugate of K_u. Now we can easily solve for the auxiliary field,

$$F = -\frac{\bar{u}}{1 + \bar{u}u}\psi^2$$

and insert this solution into the supersymmetry transformations and the Lagrangian density. On shell the susy transformations simplify to

$$\delta u = \sqrt{2}\zeta\psi, \quad \delta\psi = -\sqrt{2}\,\frac{\bar{u}\psi^2}{1 + \bar{u}u}\zeta + i\sqrt{2}(\sigma^\mu\bar{\zeta})\partial_\mu u,$$

and the Lagrangian takes the form

$$\begin{aligned} \mathcal{L} \;=\; & +\tfrac{1}{2}\frac{1}{(1 + \bar{u}u)^2}\Big(\partial_\mu u\partial^\mu\bar{u} + \tfrac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\psi} - \tfrac{i}{2}\psi\sigma^\mu\partial_\mu\bar{\psi}\Big) \\ & +\tfrac{i}{2}\frac{\psi\sigma^\mu\bar{\psi}}{(1 + \bar{u}u)^3}(u\partial_\mu\bar{u} - \bar{u}\partial_\mu u) - \tfrac{1}{4}\frac{1}{(1 + u\bar{u})^4}\psi^2\bar{\psi}^2. \end{aligned}$$

Similarly as in 2 dimensions we find a 4-Fermi interaction and a coupling of the current $\sim \psi\sigma^\mu\bar{\psi}$ to the bosonic field u.

17.1.2 Chiral superfield revisited

Here I want to rewrite a chiral superfield in terms of Majorana spinors and real fields for an easier comparison with the lower dimensional cases. Let α denote the Majorana parameter with corresponding Weyl component θ and recall that

$$\bar{\alpha}\gamma^\mu\gamma_5\alpha = 2\theta\sigma^\mu\bar{\theta}, \qquad \bar{\alpha}\alpha = \theta^2 + \bar{\theta}^2, \quad \bar{\alpha}\gamma_5\alpha = \bar{\theta}^2 - \theta^2$$

from which immediately follows that a chiral superfield has the expansion

$$\begin{aligned} \Phi \;=\; & \Big(A + \frac{1}{\sqrt{2}}(\bar{\alpha}\psi - \bar{\alpha}\gamma_5\psi) + \tfrac{1}{2}(\bar{\alpha}\alpha - \bar{\alpha}\gamma_5\alpha)F\Big)(x^\mu + \tfrac{i}{2}\bar{\alpha}\gamma^\mu\gamma_5\alpha) \\ =\; & A + \tfrac{i}{2}\bar{\alpha}\gamma^\mu\gamma_5\alpha\partial_\mu A - \tfrac{1}{8}(\bar{\alpha}\gamma_\mu\gamma_5\alpha)(\bar{\alpha}\gamma_\nu\gamma_5\alpha)\partial_\mu\partial_\nu A \\ & + \frac{1}{\sqrt{2}}(\bar{\alpha}\psi - \bar{\alpha}\gamma_5\psi) + \frac{i}{2\sqrt{2}}(\bar{\alpha}\gamma_\mu\gamma_5\alpha)(\bar{\alpha}\partial_\mu\psi - \bar{\alpha}\gamma_5\partial_\mu\psi) + \tfrac{1}{2}(\bar{\alpha}\alpha - \bar{\alpha}\gamma_5\alpha)F. \end{aligned}$$

To continue we use the Fierz identities

$$\begin{aligned}
\tfrac{3}{2}\bar\alpha\gamma_{(\mu}\gamma_5\alpha\,\bar\alpha\gamma_{\nu)}\gamma_5\alpha &= \tfrac{1}{4}((\bar\alpha\alpha)^2-(\bar\alpha\gamma_5\alpha)^2+\bar\alpha\gamma^\sigma\gamma_5\alpha\bar\alpha\gamma_\sigma\gamma_5\alpha)\eta_{\mu\nu}\\
\bar\alpha\gamma^\sigma\gamma_5\alpha\,\bar\alpha\gamma_\sigma\gamma_5\alpha &= 2(\bar\alpha\alpha)^2-2(\bar\alpha\gamma_5\alpha)^2 \Longrightarrow\\
\bar\alpha\gamma_{(\mu}\gamma_5\alpha\,\bar\alpha\gamma_{\nu)}\gamma_5\alpha &= \tfrac{1}{2}((\bar\alpha\alpha)^2-(\bar\alpha\gamma_5\alpha)^2)\eta_{\mu\nu}
\end{aligned}$$

and rewrite the chiral field as follows

$$\begin{aligned}
\Phi &= A+\tfrac{i}{2}\bar\alpha\gamma^\mu\gamma_5\alpha\partial_\mu A-\tfrac{1}{8}(\bar\alpha\alpha)^2\Box A\\
&\quad+\sqrt{2}\bar\alpha P_-\psi+\frac{i}{\sqrt{2}}(\bar\alpha P_-\alpha)(\bar\alpha\not{\partial}P_-\psi)+\bar\alpha P_-\alpha F
\end{aligned}$$

The corresponding representation for a antichiral superfield reads

$$\begin{aligned}
\Phi^\dagger &= A^*-\tfrac{i}{2}\bar\alpha\gamma^\mu\gamma_5\alpha\partial_\mu A^*-\tfrac{1}{8}(\bar\alpha\alpha)^2\Box A^*\\
&\quad+\frac{1}{\sqrt{2}}(\bar\alpha\psi+\bar\alpha\gamma_5\psi)-\frac{i}{2\sqrt{2}}(\bar\alpha\gamma_\mu\gamma_5\alpha)(\bar\alpha\partial_\mu\psi+\bar\alpha\gamma_5\partial_\mu\psi)+\tfrac{1}{2}(\bar\alpha\alpha+\bar\alpha\gamma_5\alpha)F^*
\end{aligned}$$

The sum is a real superfield

$$\begin{aligned}
\Phi+\Phi^\dagger &= A_1-\tfrac{i}{2}\bar\alpha\gamma^\mu\gamma_5\alpha\partial_\mu A^*-\tfrac{1}{16}((\bar\alpha\alpha)^2-(\bar\alpha\gamma_5\alpha)^2)\Box A^*\\
&\quad+\frac{1}{\sqrt{2}}(\bar\alpha\psi+\bar\alpha\gamma_5\psi)-\frac{i}{2\sqrt{2}}(\bar\alpha\gamma_\mu\gamma_5\alpha)(\bar\alpha\partial_\mu\psi+\bar\alpha\gamma_5\partial_\mu\psi)+\tfrac{1}{2}(\bar\alpha\alpha+\bar\alpha\gamma_5\alpha)F^*
\end{aligned}$$

17.1.3 Higher derivatives

Let Φ be a chiral superfield. On finds the following results for the supercovariant derivatives of Φ:

$$\begin{aligned}
D_\alpha\Phi &= 2i(\sigma^\mu\bar\theta)_\alpha\partial_\mu\Phi(y)+\sqrt{2}\psi_\alpha(y)+2\theta_\alpha F(y)\\
D^\alpha\Phi &= -2i(\bar\theta\tilde\sigma^\mu)^\alpha\partial_\mu\Phi(y)+\sqrt{2}\psi^\alpha(y)+2\theta^\alpha F(y).
\end{aligned}$$

Similarly, for the nonvanishing supercovariant derviatives of $\Phi^\dagger$ we obtain

$$\begin{aligned}
\bar D_{\dot\alpha}\Phi &= -2i(\theta\sigma^\mu)_{\dot\alpha}\partial_\mu\Phi^\dagger(z)+\sqrt{2}\bar\psi_{\dot\alpha}(z)+2\bar\theta_{\dot\alpha}\bar F(z)\\
\bar D^{\dot\alpha}\Phi^\dagger &= 2i(\tilde\sigma^\mu\theta)^{\dot\alpha}\partial_\mu\Phi^\dagger(z)+\sqrt{2}\bar\psi^{\dot\alpha}(z)+2\bar\theta^{\dot\alpha}\bar F(z).
\end{aligned}$$

It follows that

$$\begin{aligned}
D^\alpha\Phi D_\alpha\Phi &= 2\psi^2+4\sqrt{2}F\theta\psi+4\theta^2F^2\\
&\quad+4\sqrt{2}i(\psi\sigma^\mu\bar\theta)\partial_\mu\Phi+8i(\theta\sigma^\mu\bar\theta)F\partial_\mu\Phi+4\bar\theta^2\partial_\mu\Phi\partial^\mu\Phi,\\
\bar D_{\dot\alpha}\Phi^\dagger\bar D^{\dot\alpha}\Phi^\dagger &= 2\bar\psi^2+4\sqrt{2}\bar F\bar\theta\bar\psi+4\bar\theta^2\bar F^2\\
&\quad-4\sqrt{2}i(\theta\sigma^\mu\bar\psi)\partial_\mu\Phi^\dagger-8i(\theta\sigma^\mu\bar\theta)\bar F\partial_\mu\Phi^\dagger+4\theta^2\partial_\mu\Phi^\dagger\partial^\mu\Phi^\dagger
\end{aligned}$$

where the arguments of the fields in the first and second formula are y and z, respectively.

17.2 Models with Faddeev-Niemi term

The purely bosonic model has the Lagrangian

$$\mathcal{L} = \frac{m^2}{2}\partial_\mu \boldsymbol{n}\,\partial^\mu \boldsymbol{n} - \frac{1}{4g^2}H_{\mu\nu}H^{\mu\nu}, \quad \text{where} \quad H_{\mu\nu} = \boldsymbol{n}(\partial_\mu \boldsymbol{n} \wedge \partial_\nu \boldsymbol{n}) = \epsilon_{abc}n^a\partial_\mu n^b\partial_\nu n^c,$$

and the target manifold is S^2, i.e. $\boldsymbol{n}\cdot\boldsymbol{n} = 1$. The constant m has the dimension of a mass and g is dimensionaless. To proceed, we define the Cho-connection[1]

$$A_\mu = \epsilon_{abc}\, n^b\partial_\mu n^c \tau_a \quad \text{or} \quad A = \tfrac{1}{2i}[n, dn], \quad n = n^a\tau_a,$$

where we took as basis for the Lie algebra the Pauli-matrices, so that

$$[\tau_a, \tau_b] = 2i\epsilon_{abc}\tau_c, \quad \mathrm{Tr}\,(\tau_a\tau_b) = 2\delta_{ab}, \qquad \mathrm{Tr}\,(nn) = 2\boldsymbol{n}\cdot\boldsymbol{n}.$$

We find

$$\partial_\mu A_\nu - \partial_\nu A_\mu = 2\epsilon_{abc}\partial_\mu n^b\partial_\nu n^c\tau_a \quad \text{and} \quad [A_\mu, A_\nu] = 2in\,\epsilon_{abc}n^a\partial_\mu n^b\partial_\nu n^c,$$

so that

$$\mathrm{Tr}\,n(\partial_\mu A_\nu - \partial_\nu A_\mu) = 4\epsilon_{abc}n_a\partial_\mu n_b\partial_\nu n_c = -i\mathrm{Tr}\,n([A_\mu, A_\nu]) \equiv 4H_{\mu\nu}$$

Next we rewrite the potential and field strength in conformally flat coordinates on the target space S^2. To do that we introduce coordinates on $\mathbb{C}$ related to $\boldsymbol{n}$ by the stereographic projection defined by

$$n^a = \frac{1}{1+\bar{u}u}\begin{pmatrix} 2u^1 \\ 2u^2 \\ 1-\bar{u}u \end{pmatrix} \Longleftrightarrow n = n^a\tau_a = \frac{1}{1+\bar{u}u}\begin{pmatrix} 1-\bar{u}u & 2\bar{u} \\ 2u & \bar{u}u-1 \end{pmatrix}, \qquad u = u^1 + iu^2.$$

The metric on on $\mathbb{C}$ reads

$$ds^2 = dn^a dn^a = g_{ab}du^a du^b, \qquad g_{ab} = \sqrt{g}\,\delta_{ab}, \quad \sqrt{g} = \frac{4}{(1+\bar{u}u)^2}$$

In terms of the u-coordinates the Cho-connection reads

$$A = \frac{2}{i}\frac{1}{(1+\bar{u}u)^2}\begin{pmatrix} \bar{u}du - ud\bar{u} & d\bar{u} + \bar{u}^2 du \\ -du - u^2 d\bar{u} & ud\bar{u} - \bar{u}du \end{pmatrix},$$

so that

$$dA = \frac{4}{i}\frac{1}{(1+\bar{u}u)^3}\begin{pmatrix} 1-\bar{u}u & 2\bar{u} \\ 2u & \bar{u}u-1 \end{pmatrix} d\bar{u}\wedge du = \frac{4}{i}\frac{1}{(1+\bar{u}u)^2}n\, d\bar{u}\wedge du.$$

[1] actually, this is only part of the Cho-connection

The Field strength introduced by Faddeev and Niemi becomes then reads

$$\begin{aligned}
\mathrm{Tr}\,(ndA) &= \frac{8}{i}\frac{1}{(1+\bar{u}u)^2}d\bar{u}\wedge du = 2H_{\mu\nu}dx^\mu\wedge dx^\nu\\
H_{\mu\nu} &= \frac{2}{i}\frac{1}{(1+\bar{u}u)^2}(\partial_\mu u\partial_\nu\bar{u}-\partial_\nu u\partial_\mu\bar{u}),\quad H=\tfrac{1}{2}H_{\mu\nu}dx^\mu dx^\nu.
\end{aligned}$$

The Faddeev-Niemi term reads

$$H_{\mu\nu}H^{\mu\nu} = -\frac{8}{(1+\bar{u}u)^4}\Big((\partial_\mu\bar{u}\,\partial^\mu\bar{u})(\partial_\nu u\partial^\nu u)-(\partial_\mu\bar{u}\partial^\mu u)^2\Big).$$

For static configurations

$$H_{ij}H_{ij} = \frac{8}{(1+\bar{u}u)^4}\Big((\nabla\bar{u}\cdot\nabla u)^2-(\nabla\bar{u})^2(\nabla u)^2\Big).$$

Now we are ready to rewrite the full Lagrangian density in terms of the u field: With

$$\partial_\mu\boldsymbol{n}\,\partial^\mu\boldsymbol{n} = \frac{4\partial_\mu u\partial^\mu\bar{u}}{(1+u\bar{u})^2}$$

we obtain

$$\mathcal{L} = 2m\frac{\partial_\mu u\,\partial^\mu\bar{u}}{(1+u\bar{u})^2}-\frac{2}{g^2}\frac{1}{(1+\bar{u}u)^4}\Big((\partial_\mu\bar{u}\partial^\mu\bar{u})(\partial_\nu u\partial^\nu u)-(\partial_\mu\bar{u}\partial^\mu u)^2\Big). \tag{17.1}$$

The energy density for static configurations is just

$$\mathcal{E} = 2m\frac{\nabla u\,\nabla\bar{u}}{(1+u\bar{u})^2}+\frac{2}{g^2}\frac{1}{(1+\bar{u}u)^4}\Big((\nabla\bar{u}\cdot\nabla u)^2-(\nabla\bar{u})^2(\nabla u)^2\Big). \tag{17.2}$$

17.2.1 Systematic rewriting the invariants

In the derivative expansion of the $\boldsymbol{n}$-field model the following terms are present in fourth order

$$(\boldsymbol{n}\partial_\mu\partial_\mu\boldsymbol{n})(\boldsymbol{n}\partial_\mu\partial_\nu n^a),\qquad (\boldsymbol{n}\Box\boldsymbol{n})(\boldsymbol{n}\Box\boldsymbol{n})\quad\text{and}\quad(\Box\boldsymbol{n},\Box\boldsymbol{n}).$$

We use the previous parametrization of the n field, such that

$$\partial_\mu\boldsymbol{n} = \frac{2}{N}\begin{pmatrix}\partial_\mu u^1\\ \partial_\mu u^2\\ 0\end{pmatrix}-\frac{4(u,\partial_\mu u)}{N^2}\begin{pmatrix}u^1\\ u^2\\ 1\end{pmatrix},\qquad N=1+(u,u),$$

where for the moment beeing we switch to the real representation for the u-field, $u=(u^1,u^2)$. A straightforward calculation shows that

$$\boldsymbol{n}\,\partial_\mu\partial_\nu\boldsymbol{n} = -g_{ab}\partial_\mu u^a\partial_\nu u^b$$

holds true. To continue we observe that for a conformally flat metric in 2 dimensional target space

$$\Gamma^a_{bc} = \frac{1}{2e}(\delta^a_b \partial_c e + \delta^a_c \partial_b e - \delta^{ad}\delta_{bc}\,\partial_d e), \qquad e = \sqrt{g}$$

so that

$$\nabla_\mu \partial^\mu u^a = \partial\partial u^a + \Gamma^a_{bc}\partial_\mu u^b \partial^\mu u^c = \partial\partial u^a + \partial_\mu \log(e)\,\partial^\mu u^a - \tfrac{1}{2}\delta^{ad}\,\partial_d \log(e)\,(\partial u, \partial u).$$

For the 2-sphere $e = 4/N^2$ and

$$\partial_d \log e = -\frac{4}{N} u^d \quad \text{and} \quad \partial_\mu \log e = -\frac{4}{N}(u, \partial_\mu u),$$

so that

$$\begin{aligned} g_{ab}\nabla^2 u^a \nabla^2 u^b &= \frac{4}{N^2}(\Box u, \Box u) + \frac{16}{N^3}\Big((\partial u)^2(\partial u)^2 - 2(\Box u, \partial_\mu u)(u, \partial^\mu u) + (\Box u, u)(\partial u)^2\Big) \\ &+ \frac{16}{N^4}\Big(4(u, \partial_\mu u)(u, \partial_\nu u)(\partial^\mu u, \partial^\nu u) - (\partial u)^2(\partial u)^2 - 4(u, \partial u)^2(\partial u)^2\Big) \end{aligned} \tag{17.3}$$

We want to compare with the invariant constructed out of

$$\Box \boldsymbol{n} = \Big(\frac{16}{N^3}(u, \partial u)^2 - \frac{4}{N}\{(\partial u)^2 + (u, \Box u)\} + \frac{2}{N}\Box - \frac{8}{N^2}(u, \partial_\mu u)\,\partial^\mu\Big)\begin{pmatrix} u^1 \\ u^2 \\ 1 \end{pmatrix}.$$

After a rather lengthy calculation one finds

$$\Big(\Box \boldsymbol{n} - (\boldsymbol{n}, \Box \boldsymbol{n})\boldsymbol{n}, \Box \boldsymbol{n} - (\boldsymbol{n}, \Box \boldsymbol{n})\boldsymbol{n}\Big) = (\Box \boldsymbol{n}, \Box \boldsymbol{n}) - (\boldsymbol{n}, \Box \boldsymbol{n})^2 = g_{ab}\nabla^2 u^a \nabla^2 u^b. \tag{17.4}$$

Let us summarize our findings: Up to fourth derivatives there are four invariant actions operators, and their form in the u-variables are

$$\begin{aligned} S_1 &= \int (\partial \boldsymbol{n}, \partial \boldsymbol{n}) = \int g_{ab}\partial_\mu u^a \partial^\mu u^b \\ S_2 &= \int (\Box \boldsymbol{n}, \Box \boldsymbol{n}) = \int g_{ab}\nabla^2 u^a\, \nabla^2 u^b + \int \Big(g_{ab}\partial_\mu u^a \partial^\mu u^b\Big)^2 \\ S_3 &= \int (\boldsymbol{n}\Box \boldsymbol{n}, \boldsymbol{n}\Box \boldsymbol{n}) = \int \Big(g_{ab}\partial_\mu u^a \partial^\mu u^b\Big)^2 \\ S_4 &= \int (\boldsymbol{n}\partial_\mu \partial_\nu \boldsymbol{n}, \boldsymbol{n}\partial^\mu \partial^\nu \boldsymbol{n}) = \int g_{ab}\partial_\mu u^a \partial_\nu u^b\, g_{cd}\partial^\mu u^c \partial^\nu u^d. \end{aligned}$$

The celebrated Faddeev-Niemi term is just the difference $S_3 - S_4$:

$$S_{FN} = \int H_{\mu\nu}H^{\mu\nu} = S_3 - S_4 = \int \Big((g_{ab}\partial_\mu u^a \partial^\mu u^b)^2 - g_{ab}\partial_\mu u^a \partial_\nu u^b\, g_{cd}\partial^\mu u^c \partial^\nu u^d\Big).$$

Not suprisingly these expression find a simpler form in term of the complex field $u = u^1 + iu^2$, since the target space is Kähler. First we rewrite the covariant Laplacian. With

$$(\nabla^2 u)^1 + i(\nabla^2 u)^2 = \partial^2 u - \frac{4}{N}(u, \partial^\mu u)\partial_\mu u + \frac{2}{N}(\partial u, \partial u)u$$

and

$$(u, \partial^\mu u) = \tfrac{1}{2}\partial^\mu(u\bar{u}) \quad \text{and} \quad (\partial u, \partial u) = \partial_\mu u \partial^\mu \bar{u}$$

we obtain

$$\begin{aligned}(\nabla^2 u)^1 + i(\nabla^2 u)^2 &= \partial^2 u - \frac{2}{N}\partial^\mu(u\bar{u})\partial_\mu u + \frac{2}{N}(\partial u \partial^\mu \bar{u})u = \partial^2 u - \frac{2}{N}(\partial u)^2\,\bar{u} \\ (\nabla^2 u)^1 - i(\nabla^2 u)^2 &= \partial^2 \bar{u} - \frac{2}{N}\partial^\mu(u\bar{u})\partial_\mu \bar{u} + \frac{2}{N}(\partial u \partial^\mu \bar{u})\bar{u} = \partial^2 \bar{u} - \frac{2}{N}(\partial \bar{u})^2\, u. \end{aligned} \tag{17.5}$$

Hence that the above invariants can be rewritten as

$$\begin{aligned} S_1 &= \int e\, \partial u \partial \bar{u} & S_2 - S_3 &= \int e^2\, |\partial^2 u - \tfrac{2}{N}(\partial u)^2 \bar{u}|^{\,2} \\ S_3 &= \int e^2\, (\partial u \partial \bar{u})^2 & S_4 - \tfrac{1}{2}S_3 &= \tfrac{1}{2}\int e^2\, (\partial u \partial u)(\partial \bar{u}\partial \bar{u}). \end{aligned} \tag{17.6}$$

The Faddeev-Niemi term has the simple form

$$S_{FN} = \tfrac{1}{2}\int e^2\, \Big((\partial u \partial \bar{u})(\partial u \partial \bar{u}) - (\partial u \partial u)(\partial \bar{u}\partial \bar{u})\Big). \tag{17.7}$$

17.2.2 Equations of motion and fluctuations

We slightly generalize the model and assume that spacetime is curved with metric $h_{\mu\nu}$ in which case

$$\mathcal{L} = \frac{m^2}{2}\sqrt{-h}\, h^{\mu\nu} g_{ab}\partial_\mu u^a \partial_\nu u^b - \frac{1}{4e^2}\sqrt{-h}\, h^{\mu\alpha}h^{\nu\beta}H_{\mu\nu}H_{\alpha\beta} = \mathcal{L}_\sigma + \mathcal{L}_{FN}$$

where we have introduced the antisymmetric tensor

$$H_{\mu\nu} = \eta_{ab}\, \partial_\mu u^a \partial_\nu u^b.$$

Actually, the have now changed the sign of $H_{\mu\nu}$! We made use of the skewsymmetric pseudotensor

$$\eta_{ab} = \sqrt{g}\, \epsilon_{ab}, \quad \epsilon_{12} = 1$$

which is related to the closed volume form η on the target manifold via

$$\eta = \tfrac{1}{2}\eta_{ab} du^a \wedge du^b.$$

We are going to expand the action

$$S[u+tv] = S[u] + t(S',v) + \tfrac{1}{2}t^2(v,S''v) + O(t^3)$$

to determine the first and second derivative of S. The first derivative yields the equations of motion and the second derivative is necessary for the stability analysis and the semiclassical quantisation. First we vary the sigma-model term S_σ:

$$\begin{aligned}
S_\sigma[u+v] &= \frac{m^2}{2}\int \sqrt{-h}h^{\mu\nu}g_{ab}(u+v)\partial_\mu(u^a+v^a)\partial_\nu(u^b+v^b) \\
&= \frac{m^2}{2}\int \sqrt{-h}(g_{ab}+g_{ab,c}v^c+\tfrac{1}{2}g_{ab,cd}v^cv^d)\{\partial_\mu u^a\partial^\mu u^b + 2\partial_\mu v^a\partial^\mu u^b + \partial_\mu v^a\partial^\mu v^b\} \\
&= S_\sigma[u] + \frac{m^2}{2}\int \sqrt{-h}\,h^{\mu\nu}\{2g_{ab}\,\partial_\mu v^a\partial_\nu u^b + g_{ab,c}v^c\partial_\mu u^a\partial_\nu u^b\} \\
&\quad + \frac{m^2}{2}\int \sqrt{-h}\,h^{\mu\nu}\{g_{ab}\partial_\mu v^a\partial_\nu v^b + 2g_{ab,c}v^c\partial_\mu v^a\partial_\nu u^b + \tfrac{1}{2}g_{ab,cd}\,v^cv^d\partial_\mu u^a\partial_\nu u^b\}
\end{aligned}$$

Next we replace the fluctuation v^a by the tangent ξ^a to the geodesic from u to $u+v$ (we set $t=1$):

$$v^a = \xi^a - \tfrac{1}{2}\Gamma^a_{\;pq}\xi^p\xi^q + O(\xi^3)$$

Inserting $v(\xi)$ the first order terms partly transforms into the second order terms. With the identities used in section (15.1.1) we arrive at the following expansion

$$\begin{aligned}
S_\sigma[u+v] &= S_\sigma[u] + \frac{m^2}{2}\int \sqrt{-h}\,h^{\mu\nu}\{g_{ab}(\partial_\mu\xi^a\partial_\nu u^b + \partial_\mu u^a\partial_\nu\xi^b) + g_{ab,c}\xi^c\partial_\mu u^a\partial_\nu u^b\} \\
&\quad + \frac{m^2}{2}\int \sqrt{-h}\,h^{\mu\nu}(g_{ab}\nabla_\mu\xi^a\nabla_\nu\xi^b - R_{acbd}\partial_\mu u^a\partial_\nu u^b\xi^c\xi^d).
\end{aligned}$$

We read off the first derivative of S_σ as follows

$$(S'_\sigma,\xi) = m^2\int \sqrt{-h}g_{ab}\nabla_\mu\xi^a\partial^\mu u^b, \quad \text{where} \quad \nabla_\mu\xi^a = \partial_\mu\xi^a + \Gamma^a_{\;bc}\partial_\mu u^b\xi^c.$$

Here $\Gamma^a_{\;bc}$ are the Christoffel symbols of the target manifold. We need the following formula: If A^a_μ is a vector with respect to spacetime and target space diffeomorphisms, e.g. $A^a_\mu = \partial_\mu u^a$, then the following formula holds up to surface terms,

$$\int \sqrt{-h}\,g_{ab}\,\nabla_\mu\xi^a A^{b\nu} = -\int \sqrt{-h}\,g_{ab}\,\xi^a\nabla_\mu A^{b\nu}, \quad \nabla_\mu A^{b\nu} = \partial_\mu A^{b\mu} + \gamma^\nu_{\mu\alpha}A^{b\alpha} + \Gamma^b_{\;cd}\partial_\mu u^c A^{d\nu}.$$

Here $\gamma^\nu_{\mu\alpha}$ are the Christoffel symbols corresponding to the spacetime metric. I leave the proof for an exercise in partial integration. Hence we may partially integrate in (S'_σ,ξ) which results in

$$(S'_\sigma,\xi) = -m^2\int \sqrt{-h}\,g_{ab}\xi^a\nabla_\mu\partial^\mu u^b = -m^2\int \sqrt{-h}\,g_{ab}\xi^a(\Box u^b + \Gamma^b_{\;cd}h^{\mu\nu}\partial_\mu u^c\partial_\nu u^d).$$

Without Faddeev-Niemi term the field equations read

$$\Box u^a + \Gamma^a_{\ bc}\, \partial_\mu u^b \partial^\mu u^c = 0. \tag{17.8}$$

Hence u would be a harmonic map from spacetime to the 2-dimensional target manifold. Let us see that this equation is covariant: under spacetime diffeomorphism $x \to y$ the equation is covariant since the $\partial_\mu u^a$ are vector fields,

$$\frac{\partial u^a}{\partial x^\mu} = (\frac{\partial y^\nu}{\partial x^\mu})\frac{\partial u^a}{\partial y^\nu}$$

Under field transformations

$$u^a \longrightarrow u^a(w^1, w^2).$$

the objects $\partial_\mu u^a$ are vectors as well,

$$\partial_\mu u^a = (\frac{\partial u^a}{\partial w^b})\partial_\mu w^b$$

and the metric transforms as usual

$$g_{ab}(u)\partial_\mu u^a \partial_\nu u = \frac{\partial u^a}{\partial w^c}\frac{\partial u^b}{\partial w^d}\, g_{ab}(u)\, \partial_\mu w^c \partial_\nu w^d = \tilde{g}_{cd}(w)\, \partial_\mu w^c \partial_\nu w^d.$$

Now it is a easy exercise to prove, that

$$D_\mu \partial_\nu u^a, \qquad D_\mu v^a_\nu = \partial_\mu v^a_\nu - \gamma^\alpha_{\ \mu\nu} v^a_\alpha + \Gamma^a_{bc}\partial_\mu u^b v^c_\nu,$$

where γ are the Christoffel sympols of spacetime and Γ are those of the target space, transforms as a 2 tensor under spacetime diffeomorphism and as vector under target space diffeomorphism, provided the target space connection transforms properly,

$$\tilde{\Gamma}^a_{bc} = \tau^a_{\ p}(t^p_{\ b} t^q_{\ c} \Gamma^p_{\ pq} + \frac{\partial^2 u^p}{\partial w^b \partial w^c}), \qquad \tau^a_{\ p} = \frac{\partial w^a}{\partial u^p}, \quad t^p_{\ b} = \frac{\partial u^p}{\partial w^b}.$$

It follows that the wave operator compatibel with both set of diffeomorphism is

$$D^2 u^a = h^{\mu\nu} D_\mu \partial_\nu u^a = \Box u^a + \Gamma^a_{\ bc}\partial_\mu u^b \partial^\mu u^c.$$

Remark: The equations (17.8) should be compared with equation

$$\ddot{x}^a + \Gamma^a_{\ bc}\dot{x}^b \dot{x}^c = 0$$

for the geodesic motion of a particle on a curved target space.
Now we turn to the second derivative of the sigma-model term

$$\begin{aligned}(\xi, S''\xi) &= m^2 \int \sqrt{-h}\, h^{\mu\nu}(g_{ab}\nabla_\mu \xi^a \nabla_\nu \xi^b - R_{cadb}\partial_\mu u^c \partial_\nu u^d \xi^a \xi^b) \\ &= -m^2 \int \sqrt{-h}\, \xi^a (g_{ab}\nabla^\mu \nabla_\mu \xi^b + R_{cadb}\partial_\mu u^c \partial^\mu u^d \xi^b).\end{aligned}$$

We read off the differential equations for fluctuation about a background field u^a:

$$\nabla^\mu\nabla_\mu\xi^a - R^a_{\ cdb}\,\partial_\mu u^c\,\partial^\mu u^d\,\xi^b = \lambda\xi^a. \tag{17.9}$$

Now we turn to the variation of the Faddeev-Niemi term: We use the elegant method of Mukhi, see section (15.1.3), to find the expansion of $\mathcal{L}_{FN}$. First we recall that the antisymmetric pseudo tensor η_{ab} is covariantly conserved. In 2 dimensions the proof is elementary:

$$\nabla_c\eta_{ab} = \partial_c(\sqrt{g})\epsilon_{ab} - \Gamma^d_{\ ca}\eta_{db} - \Gamma^d_{\ cb}\eta_{ad} = \sqrt{g}(\Gamma^d_{\ cd}\,\epsilon_{ab} - \Gamma^d_{\ ca}\epsilon_{db} - \Gamma^d_{\ cb}\epsilon_{ad}).$$

We only need to check that this vanishes for $a = 1, b = 2$:

$$\nabla_c\eta_{12} = \sqrt{g}(\Gamma^d_{\ cd} - \Gamma^1_{\ c1} - \Gamma^2_{\ c2}) = 0.$$

Again we consider a geodesic $u(s)$ which interpolates between $u = u(0)$ and $u(1) = u + v$ and calculate the change of $H^2(s) = H_{\mu\nu}(s)H^{\mu\nu}(s)$ along this curve:

$$\begin{aligned} H^2(s) &= H^2(u) + s\nabla_s H^2(s)|_{s=0} + \tfrac{1}{2}s^2(\nabla_s)^2 H^2(s)|_{s=0} + O(s^3) \\ &= H^2(u) + 2sH^{\mu\nu}(u)\nabla_s H_{\mu\nu}|_{s=0} + s^2(\nabla_s H_{\mu\nu}\nabla_s H^{\mu\nu} + H^{\mu\nu}(\nabla_s)^2 H_{\mu\nu}). \end{aligned}$$

Clearly we need the variation of $H_{\mu\nu}$ along the geodesic: The first derivative is

$$\nabla_s H_{\mu\nu}(s) = \eta_{ab}\nabla_s(\partial_\mu u^a\nabla_\nu u^b) = \eta_{ab}(\nabla_\mu\dot{u}^a\partial_\nu u^b + \partial_\mu u^a\nabla_\nu\dot{u}^b)$$

and its value at $s = 0$ becomes

$$\nabla_s H_{\mu\nu}(0) = \eta_{ab}(\nabla_\mu\xi^a\partial_\nu u^b + \partial_\mu u^a\nabla_\nu\xi^b). \tag{17.10}$$

The second derivative is found to be

$$\begin{aligned} (\nabla_s)^2 H_{\mu\nu}(s) &= \eta_{ab}\nabla_s(\nabla_\mu\dot{u}^a\partial_\nu u^b + \partial_\mu u^a\nabla_\nu\dot{u}^b) \\ &= \eta_{ab}([\nabla_s,\nabla_\mu]\dot{u}^a\partial_\nu u^b + \partial_\mu u^a[\nabla_s,\nabla_\nu]\dot{u}^b + \nabla_\mu\dot{u}^a\nabla_\nu\dot{u}^b + \nabla_\mu\dot{u}^a\nabla_\nu\dot{u}^b) \end{aligned}$$

so that its value at the origin reads

$$(\nabla_s)^2 H_{\mu\nu}(0) = \eta_{ab}(R^a_{\ cdp}\partial_\mu u^p\partial_\nu u^b\xi^c\xi^d + R^b_{\ cdp}\partial_\nu u^p\partial_\mu u^a\xi^c\xi^d + 2\nabla_\mu\xi^a\nabla_\nu\xi^b) \tag{17.11}$$

Now we are ready to calculate the first variation of the Faddeev-Niemi term

$$(S'_{FN},\xi) = -\frac{1}{e^2}\int\sqrt{-h}H^{\mu\nu}\eta_{ab}\nabla_\mu\xi^a\nabla_\nu u^b = \frac{1}{e^2}\int\sqrt{-h}\nabla_\mu(\eta_{ab}H^{\mu\nu}\nabla_\nu u^b)\xi^a.$$

Together with the previous result we obtain the formula for the first order variations of the complete action:

$$(S',\xi) = \int\sqrt{-h}g_{bc}\Big(-m^2(\Box u^c + \Gamma^c_{\ pq}\partial_\mu u^p\partial_\nu u^q) + \frac{1}{e^2}\,\nabla_\nu(H^{\mu\nu}\partial_\mu u^a)\eta_a^{\ c}\Big)\xi^b$$

and this gives rise to the following equations of motion

$$m^2(\Box u^c + \Gamma^c_{\ pq}\partial_\mu u^p \partial_\nu u^q) = \frac{1}{e^2}\nabla_\nu(H^{\mu\nu}\partial_\mu u^a)\eta_a^{\ c} \tag{17.12}$$

Now we come to the most difficult part, the calculation of the second order terms induced by the FN-action. We find

$$\begin{aligned} \nabla_s H_{\mu\nu}\nabla_s H^{\mu\nu} + (\nabla_s)^2 H_{\mu\nu}H^{\mu\nu} &= 2\eta_{ab}\eta_{pq}(\partial_\nu u^b \partial^\nu u^q\, \nabla^\mu\xi^p \nabla_\mu \xi^a + \partial_\mu u^a \partial^\nu u^q\, \nabla^\mu \xi^p \nabla_\nu \xi^b) \\ &+ 2\eta_{ab}H^{\mu\nu}(R^a_{\ cdp}\partial_\mu u^p \partial_\nu u^b \xi^c \xi^d + \nabla_\mu \xi^a \nabla_\nu \xi^b) \end{aligned}$$

Adding the second variation of the sigma-model term results in

$$\begin{aligned} (\xi S'', \xi) &= -m^2 \int \sqrt{-h}\,\xi^a(g_{ab}\nabla^\mu\nabla_\mu \xi^b + R_{cadb}\partial_\mu u^c \partial^\mu u^d \xi^b) \\ &- \frac{1}{e^2} \int \sqrt{-h}\Big(\eta_{ab}\eta_{pq}(\partial_\nu u^b \partial^\nu u^q\, \nabla^\mu \xi^p \nabla_\mu \xi^a + \partial_\mu u^a \partial^\nu u^q\, \nabla^\mu \xi^p \nabla_\nu \xi^b) \\ &\qquad + \eta_{ab}\, H^{\mu\nu}(R^a_{\ cdp}\partial_\mu u^p \partial_\nu u^b \xi^c \xi^d + \nabla_\mu \xi^a \nabla_\nu \xi^b)\Big) \end{aligned} \tag{17.13}$$

or after partial integration

$$\begin{aligned} (\xi S'', \xi) &= -m^2 \int \sqrt{-h}\,\xi^a(g_{ab}\nabla^\mu\nabla_\mu \xi^b + R_{cadb}\partial_\mu u^c \partial^\mu u^d \xi^b) \\ &+ \frac{1}{e^2} \int \sqrt{-h}\xi^a \eta_{ab}\eta_{cd}\nabla^\mu\Big(\partial_\mu u^c \partial^\nu u^b \nabla_\nu \xi^d - \partial_\nu u^b \partial^\nu u^c \nabla_\mu \xi^d\Big) \\ &+ \frac{1}{e^2} \int \sqrt{-h}\xi^a\Big(\eta_{ab}\nabla_\mu(H^{\mu\nu}\partial_\nu \xi^b) - \eta_{bc}H^{\mu\nu}R^b_{\ adp}\partial_\mu u^p \partial_\nu u^c \xi^d\Big), \end{aligned} \tag{17.14}$$

where we used that the covariant derivative of the pseudotensor η_{ab} vanishes. This then yields the following equations for the fluctuations

$$\begin{aligned} &-m^2\nabla^2\xi^a + m^2 R^a_{\ bcd}\partial_\mu u^b \partial^\mu u^c \xi^d \\ &+ \frac{1}{e^2}\eta^a_{\ b}\eta_{cd}\nabla^\mu\Big(\partial_\mu u^c \partial^\nu u^b \nabla_\nu \xi^d - \partial_\nu u^b \partial^\nu u^c \nabla_\mu \xi^d\Big) \\ &+ \frac{1}{e^2}\eta^a_{\ b}\nabla_\mu(H^{\mu\nu}\partial_\nu\xi^b) - \frac{1}{e^2}\eta_c^{\ b}H^{\mu\nu}R^a_{\ bdp}\partial_\mu u^p \partial_\nu u^c \xi^d = \lambda^2 \xi^a, \end{aligned} \tag{17.15}$$

17.2.3 Hopfinvariant

In this subsection we are going to express the Hopfinvariant in terms of the u-field. For that we need a one-form, denoted by a, the differential of which is the abelian field strength H,

$$da = H.$$

Actually, we may easily write down such an abelian potential,

$$a = d\theta + \frac{1}{i}\frac{\bar{u}du - ud\bar{u}}{1+\bar{u}u}.$$

Indeed, we find

$$da = \frac{2}{i}\frac{1}{(1+\bar{u}u)^2}d\bar{u}\wedge du = H.$$

For the Hopf-density we find

$$h \sim a\wedge H = \frac{2}{i}\frac{1}{(1+\bar{u}u)^2}\, d\theta\wedge d\bar{u}\wedge du$$

At this point I do not know what to insert for θ. To understand this apparent ambiguity we switch to yet another parametrization. We use that the Hopf invariant invariant has a simple geometric interpretation in terms of $g\in SU(2)$ defined (up to a $U(1)$-factor) by

$$n = g\tau_3 g^{-1},\quad g = \frac{1}{(1+\bar{u}u)^{1/2}}\begin{pmatrix}1 & -\bar{u}\\ u & 1\end{pmatrix}\begin{pmatrix}e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix} = \frac{1}{(1+\bar{u}u)^{1/2}}\begin{pmatrix}e^{i\phi} & -\bar{u}e^{-i\phi}\\ ue^{i\phi} & e^{-i\phi}\end{pmatrix}.$$

The Hopfinvariant is just the winding number of $g\in SU(2)$ as function of the spatial variables. Hence we assume that g is time-independent and tends to a constant element for large distances from the origin. From

$$g^{-1}dg = i\tau_3 d\phi + \tfrac{1}{2}\frac{1}{1+\bar{u}u}\begin{pmatrix}\bar{u}du - ud\bar{u} & -2e^{-2i\phi}d\bar{u}\\ 2e^{2i\phi}du & ud\bar{u}-\bar{u}du\end{pmatrix}$$

we may calculate the density giving rise to the Hopf invariant. The powers of the potential $g^{-1}dg$ are:

$$\begin{aligned}(g^{-1}dg)^2 &= \frac{2i}{1+\bar{u}u}\begin{pmatrix}0 & e^{-2i\phi}d\bar{u}\\ e^{2i\phi}du & 0\end{pmatrix}\wedge d\phi + \frac{1}{(1+\bar{u}u)^2}\begin{pmatrix}-1 & e^{-2i\phi}\bar{u}\\ e^{2i\phi}u & 1\end{pmatrix}d\bar{u}\wedge du\\ (g^{-1}dg)^3 &= -\frac{3i}{(1+\bar{u}u)^2}\mathbb{1}\, d\bar{u}\wedge du\wedge d\phi\end{aligned}$$

such that the winding number density is proportional to

$$\mathrm{Tr}\,(g^{-1}dg)^3 = -6i\frac{1}{(1+\bar{u}u)^2}d\bar{u}\wedge du\wedge d\phi.$$

To check the normalization we use the standard Hopf map

$$g = i\sin\theta\begin{pmatrix}e^{-i\phi_{34}} & e^{-i\phi_{12}}\cot\theta\\ e^{i\phi_{12}}\cot\theta & -e^{i\phi_{34}}\end{pmatrix} = \begin{pmatrix}x_3+ix_2 & x_1+ix_0\\ -x_1+ix_0 & x_3-ix_2\end{pmatrix}.$$

where we introduced the coordinates

$$\begin{aligned}\hat{x}^\mu &= (\cos\theta\cos\phi_{12}, \cos\theta\sin\phi_{12}, \sin\theta\cos\phi_{34}, \sin\theta\sin\phi_{34})\\ & \quad 0\le\theta<\pi/2,\quad 0\le\phi_{12},\phi_{34}<2\pi\end{aligned}$$

on the 3-sphere. Comparing with the parametrization of g used above we find the identifications

$$e^{i\phi} = ie^{-i\phi_{34}}, \quad \sin^{-2}\theta = 1 + \bar{u}u, \quad u = \cot\theta\, e^{i(\phi_{12}+\phi_{34})},$$

such that the winding number density is proportional to

$$\mathrm{Tr}\,(g^{-1}dg)^3 = 6\sin 2\theta d\theta \wedge d\phi_{12} \wedge d\phi_{34}.$$

On the other hand, since

$$\frac{1}{4\pi^2}\int_{S^3} \sin 2\theta d\theta \wedge d\phi_{12} \wedge d\phi_{34} = 1$$

the correct normalization for the Hopf-density is

$$q = \frac{1}{24\pi^2}\mathrm{Tr}\,(g^{-1}dg)^3 = \frac{1}{4i\pi^2}\frac{1}{(1+\bar{u}u)^2}d\bar{u} \wedge du \wedge d\phi.$$

There are problems when one tries to find the Hopfinvariant in terms of u. It is nonolocal, therefore we turn to the

17.3 Alternative formulations

In this section we shall investigate alternative formulations of the models just considered. We begin with the

17.3.1 CP_1 formulation

We introduce

$$n_a = z^\dagger \tau_a z, \quad z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \in \mathbb{C}^2, \quad z^\dagger z = 1.$$

More explicitly

$$n_1 - in_2 = 2\bar{z}_1 z_0, \quad n_3 = \bar{z}_0 z_0 - \bar{z}_1 z_1, \quad u = \frac{z_1}{z_0}, \quad 1 + \bar{u}u = 1/|z_0|^2.$$

First we rewrite the σ-model term in terms of the CP_1 field. Since

$$\partial_\mu u = \frac{1}{z_0^2}(z_0\partial_\mu z_1 - z_1\partial_\mu z_0)$$

we find

$$\begin{aligned}\frac{\partial_\mu u \partial^\mu \bar{u}}{(1+\bar{u}u)^2} &= |z_0|^2\partial_\mu z_1 \partial^\mu \bar{z}_1 + |z_1|^2 \partial_\mu z_0 \partial^\mu \bar{z}_0 - z_1 \partial_\mu \bar{z}_1 \bar{z}_0 \partial^\mu z_0 - z_0 \partial_\mu \bar{z}_0 \bar{z}_1 \partial^\mu z_1 \\ &= \partial_\mu z^\dagger \partial_\mu z + (z^\dagger \partial_\mu z)(z^\dagger \partial^\mu z),\end{aligned}$$

where we used that z has length one. Introducing the $U(1)$ gauge potential and covariant derivative as

$$A_\mu = -iz^\dagger\partial_\mu z, \quad D_\mu z = (\partial_\mu - iA_\mu)z,$$

this can be rewritten as

$$\frac{\partial_\mu u \partial^\mu \bar{u}}{(1+\bar{u}u)^2} = (D_\mu z)^\dagger (D^\mu z).$$

The proof is simple,

$$(D_\mu z)^\dagger (D^\mu z) = \partial_\mu z^\dagger \partial^\mu z + A_\mu A^\mu z^\dagger z - iA^\mu \partial_\mu z^\dagger z + iA_\mu z^\dagger \partial^\mu z = \partial_\mu z^\dagger \partial^\mu z - A_\mu A^\mu.$$

Next we wish to express the Faddeev-Niemi term in terms of the CP_1 field. We calculate

$$\begin{aligned}
&\frac{\partial_\mu \bar{u}\partial_\nu u - \partial_\nu \bar{u}\partial_\mu u}{(1+\bar{u}u)^2} = (\bar{z}_0\partial_\mu \bar{z}_1 - \bar{z}_1\partial_\mu \bar{z}_0)(z_0\partial_\nu z_1 - z_1\partial_\nu z_0) - (\mu\leftrightarrow\nu)\\
&\quad = |z_0|^2\partial_\mu \bar{z}_1\partial_\nu z_1 + |z_1|^2\partial_\mu \bar{z}_0\partial_\nu z_0 - z_1\partial_\mu \bar{z}_1\bar{z}_0\partial_\nu z_0 - z_0\partial_\mu \bar{z}_0\bar{z}_1\partial_\nu z_1 - (\mu\leftrightarrow\nu)\\
&\quad = \{\partial_\mu \bar{z}_1\partial_\nu z_1 + \partial_\mu \bar{z}_0\partial_\nu z_0 + (z^\dagger\partial_\mu z)(z^\dagger\partial_\nu z)\} - \{\mu\leftrightarrow\nu\}\\
&\quad = \partial_\mu \bar{z}_1\partial_\nu z_1 + \partial_\mu \bar{z}_0\partial_\nu z_0 - \partial_\nu \bar{z}_1\partial_\mu z_1 - \partial_\nu \bar{z}_0\partial_\mu z_0 = i(\partial_\mu A_\nu - \partial_\nu A_\mu)
\end{aligned}$$

Hence we have proven the nice formula

$$\frac{\partial_\mu \bar{u}\partial_\nu u - \partial_\nu \bar{u}\partial_\mu u}{(1+\bar{u}u)^2} = iF_{\mu\nu}.$$

Now we proceed to calculate the Faddeev-Niemi density in terms of u. After some algebra one obtains

$$\begin{aligned}
d\vec{n}\wedge d\vec{n} &= \frac{8}{(1+\bar{u}u)^2}\vec{n}du_1\wedge du_2 \quad \text{or}\\
(d\vec{n}\wedge d\vec{n})_a &= \epsilon_{abc}\partial_\mu n^b\partial_\nu n^b dx^\mu\wedge dx^\nu = \frac{8}{(1+\bar{u}u)^2}\,n_a\,du_1\wedge du_2
\end{aligned}$$

Since

$$id\bar{u}\wedge du = 2du_1\wedge du_2 \Longrightarrow du_1\wedge du_2 = \tfrac{i}{4}(\partial_\mu \bar{u}\partial_\nu u - \partial_\nu \bar{u}\partial_\mu u)dx^\mu\wedge dx^\nu$$

we may rewrite

$$\begin{aligned}
\vec{n}(d\vec{n}\wedge d\vec{n}) &= \epsilon_{abc}n^a\partial_\mu n^b\wedge\partial_\nu n^c dx^\mu\wedge dx^\nu\\
&= \frac{8}{(1+\bar{u}u)^2}\tfrac{i}{4}(\partial_\mu \bar{u}\partial_\nu u - \partial_\nu \bar{u}\partial_\mu u)dx^\mu\wedge dx^\nu = -2F_{\mu\nu}dx^\mu\wedge dx^\nu.
\end{aligned}$$

As a consequence, the antisymmetric tensor $H_{\mu\nu}$ is proportional to the field strength of A_μ,

$$H_{\mu\nu} = \epsilon_{abc}n^a(\partial_\mu n^b\wedge\partial_\nu n^c) = -2F_{\mu\nu}.$$

For the standard Hopf map

$$\vec{n} = \begin{pmatrix} 2(x_1x_3 - x_2x_4) \\ 2(x_1x_4 + x_2x_3) \\ x_3^2 + x_4^2 - x_1^2 - x_2^2 \end{pmatrix} = \begin{pmatrix} \sin 2\theta \cos(\phi_{12} + \phi_{34}) \\ \sin 2\theta \sin(\phi_{12} + \phi_{34}) \\ -\cos 2\theta \end{pmatrix} = \begin{pmatrix} \bar{z}_0 z_1 + \bar{z}_1 z_0 \\ i\bar{z}_1 z_0 - i\bar{z}_0 z_1 \\ \bar{z}_0 z_0 - \bar{z}_1 z_1 \end{pmatrix}$$

we may take

$$z = \begin{pmatrix} \sin\theta\, e^{-i\phi_{12}} \\ \cos\theta\, e^{i\phi_{34}} \end{pmatrix}.$$

For the gauge potential and field strength we obtain

$$A = -i\dot{2}dz = \cos^2\theta\, d\phi_{34} - \sin^2\theta\, d\phi_{12}, \qquad F = -\sin 2\theta\, d\theta \wedge (d\phi_{12} + d\phi_{34}).$$

It follows that

$$A \wedge F = \sin 2\theta\ d\theta \wedge d\phi_{12} \wedge d\phi_{34} \quad \text{and} \quad \frac{1}{4\pi^2}\int_{S^3} A \wedge F = 1.$$

Let us finally summarize what we have gotten so far. The effective Lagrangian density takes the form

$$\mathcal{L} = \frac{m^2}{2}(D_\mu z)^\dagger D^\mu z - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu},$$

where

$$A = -iz^\dagger dz, \quad D_\mu = \partial_\mu - iA_\mu, \quad F = dA.$$

For static fields and in the Weyl gauge we obtain

$$-\mathcal{L} = \frac{m^2}{2}(D_i z)^\dagger D_i z + \frac{1}{4e^2}F_{ij}F^{ij},$$

The properly normalized Hopf invariant reads

$$H = \frac{1}{4\pi^2}\int d^3x A \wedge F = \frac{1}{8\pi^2}\int \epsilon_{ijk}A_i F_{jk}\, d^3x.$$

We may write the energy as a square. For that we notice that

$$z^\dagger D_i z = z^\dagger \partial_i z - iA_i = iA_i - iA_i = 0 = -(D_i z)z^\dagger,$$

such that with

$$\tilde{D}_i z \equiv D_i z - i\epsilon_{ijk}\, F_{jk}$$

we obtain

$$\begin{aligned} |\tilde{D}_i z|^2 &= (D_i z)^\dagger D_i z + \epsilon_{ijk}\epsilon_{ipq}F_{jk}F_{pq} - i\epsilon_{ijk}F_{jk}(D_i z)^\dagger z + i\epsilon_{ipq}F_{pq}z^\dagger D_i z \\ &= (D_i z)^\dagger D_i z + 2F_{ij}F_{ij}. \end{aligned}$$

17.3.2 Yet another formulation

Assume that $z \in \mathbb{C}^2$ with $z^\dagger z = 1$ and A_μ are independent field and consider the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\mathcal{D}_i z)^\dagger \mathcal{D}_i z + 2i\theta\epsilon_{ijk}A_i(\partial_j z^\dagger \partial_k z),$$

where $\mathcal{D}$ is the covariant derivative taken with respect to the apriori independent potential A. The first term becomes

$$(\partial_i z^\dagger + iA_i z^\dagger)(\partial_i z - iA_i z) = \partial_i z^\dagger \partial_i z + A_i A_i + 2iA_i Z_i, \quad \text{where} \quad Z_i = z^\dagger \partial_i z.$$

Later Z_i will be an induced connection. Its fieldstrength is

$$Z_{ij} = \partial_i Z_j - \partial_j Z_i = \partial_i z^\dagger \partial_j z - \partial_j z^\dagger \partial_i z.$$

Hence we obtain

$$\mathcal{L} = \tfrac{1}{2}\partial_i z^\dagger \partial_i z + \tfrac{1}{2}A_i A_i + iA_i Z_i + i\theta\epsilon_{ijk}A_i Z_{jk}.$$

Now we eliminate the field A_i by its algebraic equation of motion, which read

$$A_i + iZ_i + i\theta\epsilon_{ijk}Z_{jk} = 0$$

which yield

$$\mathcal{L} = \tfrac{1}{2}\partial_i z^\dagger \partial_i z + \tfrac{1}{2}(Z_i + \theta\epsilon_{ijk}Z_{jk})^2 = \tfrac{1}{2}(D_i z)^\dagger D_i z + \theta^2 Z_{pq}Z_{pq} - \theta\epsilon_{ijk}Z_i Z_{jk}.$$

Hence we recover the sigma model term, the Faddeev-Nieme term and the Hopf invariant.

17.3.3 Coadjoint orbit variables

We introduce the hermitean and traceless field

$$Q = \tfrac{1}{2}\mathbb{1} - zz^\dagger, \qquad \mathrm{Tr}\,Q = 0, \quad \det Q = -\tfrac{1}{4},$$

such that

$$\partial_\mu Q = -(\partial_\mu z z^\dagger - z\partial_\mu z^\dagger)$$

and

$$\partial_\mu Q \partial^\mu Q = \partial_\mu z z^\dagger \partial^\mu z z^\dagger + \partial_\mu z \partial^\mu z^\dagger + z\partial_\mu z^\dagger \partial^\mu z z^\dagger + z\partial_\mu z^\dagger z \partial^\mu z^\dagger.$$

Using $\mathrm{Tr}\,uv^\dagger = v^\dagger u$ and introducing $Z_\mu = z^\dagger \partial_\mu z$ we obtain

$$\tfrac{1}{2}\mathrm{Tr}\,(\partial_\mu Q \partial^\mu Q) = \partial_\mu z^\dagger \partial^\mu z + Z_\mu Z^\mu.$$

Next we rewrite the Faddeev-Niemi term and the Hopf invariant. With

$$dQdQ = ZdQ + dZ(1 + zz^\dagger)$$

we obtain

$$\mathrm{Tr}\,(dQ)^2 = \mathrm{Tr}\,(dQ)^2 zz^\dagger = 2dZ \quad \text{so that} \quad \mathrm{Tr}\,Q(dQ)^2 = -dZ.$$

Furthermore, using $Z^2 = 0$ we find

$$(dQ)^4 = dZdZ(1+zz^\dagger)(1+zz^\dagger) + ZdZdQ(1+zz^\dagger) + dZZ(1+zz^\dagger)dQ$$

so that

$$\begin{aligned} \mathrm{Tr}\,(dQ)^4 &= 4dZdZ + ZdZ(\mathrm{Tr}\,dQ + \mathrm{Tr}\,dQzz^\dagger) = 4dZdZ \\ \mathrm{Tr}\,\{(dQ)^4 zz^\dagger\} &= 4dZdZ + 4ZdZ\mathrm{Tr}\,(dQzz^\dagger) = 4dZdZ \end{aligned}$$

and hence

$$\mathrm{Tr}\,\{Q(dQ)^4\} = -2dZdZ = -2(dz^\dagger \wedge dz) \wedge (dz^\dagger \wedge dz) = -2d(Z \wedge dZ).$$

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