AdS/CFT-Correspondence

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Contents

Abstract

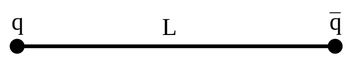
We review the duality between large N Gauge Theories in 4 dimensions and String backgrounds of the form $AdS_5 \times K_5$. This review is essentially taken from Maldacena et.al und de Veccia.

1 The various interactions

Gravity: described by Einstein's theory of general relativity and its supersymmetric extensions, gauge group is Poincare group, non-renormalizable.

Gauge theories: Gauging of internal symmetry group. Strong interaction described by QCD that does not contain gravity. 't Hooft limit for $SU(N)$-gauge theories: $N \to \infty$ and $\lambda = g_{YM}^2 N$ fixed. In leading order described by planar diagrams, gauge invariant variables are determined by a master field that satisfies a classical equation of motion (Witten). Conjecture: large-N gauge theory described by a string theory. Mesons are string excitations. Idea supported by observed Regge behavior, explanation of $U(1)$ problem and other successes.

Old idea of describing Wilson loops in strongly coupled QCD in terms of string partition function dates back to early eighties. It was found [1], that the static potential $V_{q\bar{q}}$ between heavy probe quarks,



defined via

$$e^{-TV_{q\bar{q}}(L)} = \langle W[C] \rangle, \qquad W[C] = \frac{1}{N} \mathrm{tr}\, \mathcal{P} e^{ig \oint A}$$

has the form

$$V_{q\bar{q}}(L) \sim \sigma L - \frac{c}{L} + \mathrm{const} + O(L^{-1}), \tag{1}$$

with a universal (coupling-independent) positive constant c. The universal Lüscher term arises from zero-point fluctuation of the string. O. Alvarez derived an exact expression for the partition function of the NG-action in the large d (=space-time dimension) limit. It yields

$$V_{q\bar{q}}(L) \sim \frac{L}{\alpha'} \sqrt{1 - \frac{2c}{L^2/\alpha'}}. \tag{2}$$

When expanding in $2c\alpha'/L^2$ one finds linear confinement as well as the Lüscher term. This is in accordance with the following exact results:

- $V_{q\bar{q}}(L)$ cannot rise faster than linearly for large L (Seiler 1978)
- $V(L) \geq$const for large L (Yaffee 1982)
- $V(L)$ is monotonically rising and concave: $V' \geq 0$ and $V'' \leq 0$ (Bachas 1986 from reflection positivity).

The strong coupling Hamiltonian of lattice gauge theories,

$$\begin{aligned} H &= \frac{g^2}{2a}E^2(l) + O(1/g^2) \\ [E^a(l), E^b(l)] &= if^{abc}E^c(l), \quad [E^a(l), U(l)] = -T^aU(l), \end{aligned}$$

makes the chromo-electric flux between external charges explicit. The vacuum obeys

$$E^a(l)|0\rangle = 0 \quad \forall \quad \text{links} \quad l.$$

The eigenstates of the Hamiltonian are

$$|l, \alpha\beta\rangle = U^{\alpha\beta}(l)|0\rangle, \qquad H|l, \alpha\beta\rangle = \frac{g^2}{2a}C\,|\,l, \alpha\beta\rangle,$$

where C is the second order Casimir. For the string tension one finds $\sigma = Cg^2/2a^2$. Although the strong coupling expansion converges for $1/g$ sufficiently small (Osterwalder, Seiler) it has a finite radius of convergence. In the continuum limit the coupling becomes small and the strong coupling expansion breaks down. There is some numerical evidence for a Lüscher term in QCD_3 associated with a bosonic string from lattice simulations [2]. The value of c is not well determined numerically.

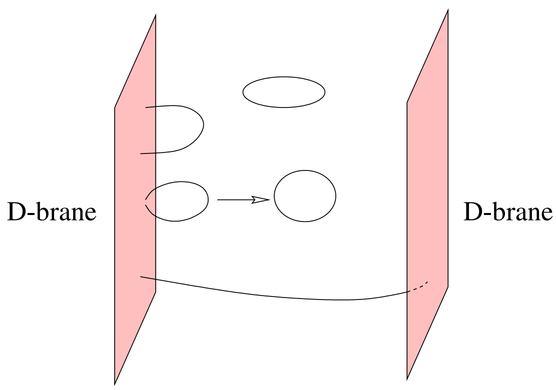
This QCD-strings should not contain gravity. Many attempts have been made to construct string theories directly from QCD. None can be considered sufficiently satisfactory. This is a 30 year old problem.

String theories: Born from attempts of describing strong interaction (dual resonance model). As such inconsistent (anomalies, no massless Pseudo scalar). Later used as consistent way of unifying all interactions, containing quantum gravity. All five consistent string theories in 10 dimensions unify gravity with gauge theories.

TYPE I STRINGS: Open and closed non-oriented strings with anomaly-free gauge group $SO(32)$, Chan-Paton gauge degrees of freedom located at end points of open strings: open strings contain usual gauge theories. Only open strings are not consistent, since non-planar loops corrections generate closed strings. Hence, gravity (zero-slope limit of closed string theory) is necessary consequence of gauge theories (zero slope limit of open string theory).

Heterotic strings: theory of only closed strings, contains supergravity and gauge theories, gravity is fundamental theory and gauge theories are obtained from it through a stringy KK-reduction.

Type IIA and IIB strings: perturbatively contain only closed strings and no gauge degrees of freedom. They contain non-perturbative D-branes. Open strings can end on branes and hence appear in type II strings. Thus, through branes one gets gauge theories. The type II strings contain Ramond-Ramond $(p+1)$-form potentials A_{p+1}. In type IIA (IIB) theory, p is even (odd). The theory contains also magnetically charged $(6-p)$-branes, which are electrically charged under the dual $dA_{7-p} =^* dA_{p+1}$ potential.



-1-brane	instanton	$0+0$ -dimensional
0-brane	point particle magnetic monopole black hole	$1+0$ - dimensional
1-brane	string	$1+1$ - dimensional
⋮	⋮	⋮
3-brane	spacetime $1+3$-dimensional	

N coinciding branes: $U(N)$-gauge theory on brane
All string theories contain gravity and gauge theories.

2 D-branes and Maldacena's conjecture

D-branes yield a connection between gauge theories and gravity. Branes come in various dimensionalities. D-zero-branes have zero spatial dimensions and are like localized, particle like soliton solutions, analogous to the 't Hooft-Polyakov monopole. D-one-branes are D-strings. They are much heavier than ordinary fundamental strings when the string coupling g_s is small. D-branes are defined in string perturbation theory as surfaces where open strings can end. The zero modes of the open strings describe oscillations of the branes. For N coinciding branes the open strings can start and end on different branes, so they carry two indices that run from 1 to N. This implies that the low energy dynamics is describes by $U(N)$ gauge theories. D-p-branes are charged under $p+1$-form gauge fields. The corresponding $p+2$-form field strength are part of the massless closed string modes, which belong to the supergravity multiplet containing the massless fields in flat space string theory before we put in any D-brane. The brane solutions of supergravity are very similar to extremal black holes in ordinary gravity. Like black holes they contain event horizons.

The near horizon geometry of N coincident D-3-branes turns out to be $AdS_5 \times S^5$. The world volume dynamics is governed by a $U(N)$ gauge theory with $\mathcal{N} = 4$ supersymmetry.

Perturbative field theory valid for g_sN small. Low energy gravitational description is valid when $R \ll$string scale. This is equivalent to large g_sN. Low-energy string effective action for metric, dilaton, Ramond-Ramond (RR) $(p+1)$-form potential and other fields in the string frame is

$$S = \frac{1}{(2\pi)^7 l_s^8} \int d^{10}x \sqrt{-g} \Big(e^{-2\phi} (\mathcal{R} + 4(\nabla\phi)^2) - \frac{2}{(8-p)!} F_{p+2}^2 \Big), \tag{3}$$

where $\alpha' = l_s^2$. defines the string length l_s and $F_{p+2} = dA_{p+1}$. One looks for a solution corresponding to a p-dimensional electric source of charge N for A_{p+1}, by requiring the Euclidean symmetry $ISO(p)$ in p-dimensions:

$$ds^2 = e^{\alpha} \sum_1^p dx^i dx^i + ds_{10-p}^2,$$

where ds_{10-p}^2 is a Lorentzian-signature metric in $10-p$ dimensions. We also assume that the metric in $10-p$ dimensions with the R-R-source at the origin,

$$\int_{S^{8-p}} {}^*F_{p+2} = N.$$

For example

$$\text{3-brane: } A_4, F_5 =^* F_5 \to S^5 \quad , \quad \text{1-brane: } A_2, F_3 \to S^7.$$

The resulting metric in the string frame is

$$ds^2 = -\frac{f_+}{\sqrt{f_-}}dt^2 + \sqrt{f_-}\sum_1^p dx^i dx^i + \frac{f_-^{-\frac{1}{2}-\beta}}{f_+}d\rho^2 + \rho^2 f_-^{\frac{1}{2}-\beta} d\Omega_{8-p}^2 \tag{4}$$

with $\beta = (5-p)/(7-p)$ and the dilaton field has the form

$$e^{-2\phi} = g_s^{-2} f_-^{-(p-3)/2}. \tag{5}$$

The functions $f_\pm$ depend only on ρ,

$$f_\pm(\rho) = 1 - \left(\frac{r_\pm}{\rho}\right)^{7-p}. \tag{6}$$

The parameters $r_\pm$ are related to the mass M (per unit volume) and the RR chare N of the solution by

$$\begin{aligned} M &= \frac{1}{(7-p)(2\pi)^7 d_p l_p^8}\big((8-p)r_+^{7-p} - r_-^{7-p}\big) \\ N &= \frac{1}{d_p g_s l_s^{7-p}}(r_+ r_-)^{(7-p)/2}, \end{aligned} \tag{7}$$

where $l_p = g_s^{1/4} l_s$ is the 10-dimensional Planck length and d_p is a numerical factor

$$d_p = 2^{5-p}\pi^{(5-p)/2}\Gamma\big(\frac{7-p}{2}\big).$$

Transforming to the Einstein frame,

$$g_{E\mu\nu} = \sqrt{g_s e^{-\phi}} g_{\mu\nu},$$

one finds the standard Einstein-Hilbert action plus terms for the dilaton fields plus .... The Einstein frame metric has a horizon at $\rho = r_+$. For $p \leq 6$ there is also a curvature singularity at $\rho = r_-$. The singularity is shielded by the horizon if $r_+ > r_-$.

The singularity structure in the critical case $r_+ = r_-$ depends very much on p. For $p = 3$ (and only for this case) is the dilaton constant and the $\rho = r_+$ surface is regular even in the critical case. The absence of a naked singularity, $r_+ > r_-$ transforms into the inequality

$$M \geq \frac{N}{(2\pi)^p g_s l_s^{p+1}}. \tag{8}$$

This is the celebrated BPS-bound with respect to 10-dimensional supergravity. The area of the black hole goes to zero in the extremal case $r_+ = r_-$ in which case (8) becomes an equality. In the extremal limit the metric simplifies to

$$ds^2 = \sqrt{f}\Big(- dt^2 + \sum_1^p dx^i dx^i\Big) + f^{-\frac{3}{2}-\beta} d\rho^2 + r^2 f^{\frac{1}{2}-\beta} d\Omega_{8-p}^2. \tag{9}$$

To describe the geometry of the extremal solution outside the horizons, it is useful to define the new coordinate

$$r^{7-p} \equiv \rho^{7-p} - r_+^{7-p}, \quad \text{3-brane: } r^4 = \rho^4 - r_+^4,\ \beta = \frac{1}{2}, \tag{10}$$

and introduce the isotropic coordinates

$$y^i = r\theta^i$$

Then, the classical solution for the 3-brane takes the simple form

$$\begin{aligned} ds^2 &= H^{-1/2}(r)\eta_{\alpha\beta}dx^\alpha dx^\beta + H^{1/2}(r)\delta_{ij}dy^i dy^j \\ e^{-(\phi-\phi_0)} &= H(r)^{(p-3)/4}, \qquad A_{01\ldots p} = H^{-1}(r), \end{aligned} \tag{11}$$

where

$$H(r) = 1 + \frac{K_p N}{r^{7-p}}, \qquad K_p = \frac{(2\pi\sqrt{\alpha'})^{7-p}}{(7-p)\Omega_{8-p}}\, g_s. \tag{12}$$

In these formulae r measures the distance from the brane, $r^2 = y_i y^i$ and $\Omega_p = 2\pi^{(q+1)/2}/\Gamma((q+1)/2)$.

Scales:

- classical supergravity requires $r_+ \gg l_s$
- no string loop corrections requires $e^\phi \ll 1$. For 3 branes $g_s = e^\phi = l_p^4/l_s^4$ is

constant and we need $l_p \ll l_s$. When $g_s > 1$ we might need to do an S-duality, $g_s \to 1/g_s$.

To summarize, for 3-branes the supergravity approximation is valid when

$$l_p < l_s \ll r_+ \Longleftrightarrow 1 \ll g_s N < N.$$

For $p \neq 3$ the supergravity description is valid only in a limited region of spacetime.

It is believed that the extremal p-brane in supergravity and the D-p-brane in string theory are two different descriptions of the same object. The D-brane uses the string worldsheet and, therefore, is a good description in string perturbation theory. For N coinciding D-branes, the effective loop expansion parameter for open strings is $g_s N$ rather than g_s. Thus, the D-brane description is good when $g_s N \ll 1$. This is complementary to the regime $1 \ll g_s N < N$ when the supergravity description is appropriate. The low energy effective theory of open strings on the D-p-brane is the $U(N)$ gauge theory in $p+1$ dimensions with 16 supercharges. The theory has $9-p$ adjoint scalar fields in the adjoint. The scalar potential is

$$V \sim \sum_{I,J} \mathrm{tr}\, [\Phi_I, \Phi_J]^2.$$

A system of N coincident D-branes is described by the non-Abelian Born-Infeld action,

$$S_{BI} = -\tau_p^{(0)} \int d^{p+1}\xi e^{-\phi}\, \mathrm{STr} \sqrt{-\det[G_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}]}. \qquad (13)$$

The brane tension is given by

$$\tau_p = \frac{\tau_p^{(0)}}{\tau_p} = \frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha' g_s}, \qquad g_s \equiv e^{\phi_\infty}.$$

Here $G_{\alpha\beta}$ is the pullback of the metric $G_{\mu\nu}$ and $F_{\alpha\beta}$ is a gauge field on the brane. STr is the symmetrized trace over the group indices. Expanding the logarithm of the determinant in α' yields

$$\begin{aligned} \log\det(G + 2\pi\alpha' F) &= \mathrm{tr}\,\log\big(G[1 + 2\pi\alpha' G^{-1}F]\big) \\ &= \log\det G + \mathrm{tr}\,\log\big(1 + 2\pi\alpha' G^{-1} F\big) \\ &= \log\det G - \frac{(2\pi\alpha')^2}{2}\mathrm{tr}\,(G^{-1}FG^{-1}F), \end{aligned}$$

so that

$$\sqrt{-\det[G_{\alpha\beta}+2\pi\alpha' F_{\alpha\beta}]} \sim \left\{-\det G\big(1-2\pi^2\alpha'^2\,\mathrm{tr}\,(G^{-1}F)^2\big)\right\}^{1/2}$$
$$\sim \sqrt{-\det G}\big(1-(\pi\alpha')^2\mathrm{tr}\,(G^{-1}F)^2\big).$$

Thus we find the second order term[1]

$$S_{BI}\sim \frac{1}{4g_{YM}^2}\int F^a_{\mu\nu}F^{a\mu\nu}, \qquad g_{YM}^2 = 2g_s(2\pi)^{p-2}(\alpha')^{(p-3)/2}.$$

One one hand the D-branes are classical solutions of the supergravity action, while on the other hand they are described by a gauge theory whose action reduces at low energy to the usual YM-action.

For $r\to\infty$ curvature vanishes and spacetime becomes flat so that supergravity is a good approximation of the D-brane. In the NEAR HORIZON LIMIT $r\to 0$ for $p=3$

$$\phi = \text{const},\quad H = 1+4\pi g_s N\frac{\alpha'^2}{r^4}\sim 4\pi g_s N\frac{\alpha'^2}{r^4},\quad g_{YM}^2 = 4\pi g_s$$

More precisely, the near horizon limit is

$$r\to 0\quad \alpha'\to 0\quad \text{and}\quad U=\frac{r}{\alpha'}=\text{fixed}\quad [\alpha']=L^2\Rightarrow [U]=L^{-1}.$$

It follows, that

$$\frac{ds^2}{\alpha'}\sim \frac{U^2}{\sqrt{4\pi N g_s}}\eta_{\alpha\beta}dx^\alpha dx^\beta + \frac{\sqrt{4\pi N g_s}}{U^2}dU^2+\sqrt{4\pi N g_s}d\Omega_5^2 \tag{14}$$

This is the metric of the manifold $AdS_5\times S^5$, where the two radii are equal and given by:

$$R^2_{AdS_5} = R^2_{S^5} = b^2 = \alpha'\sqrt{4\pi N g_s}. \tag{15}$$

Measured in units of the string length,

$$\frac{b^2}{\alpha'} = \sqrt{4\pi N g_s} = \sqrt{N}g_{YM}.$$

The classical solution (14) is a good approximation when the radii are big in string units,

$$\frac{b^2}{\alpha'}\gg 1 \Longrightarrow N g_{YM}^2 \gg 1.$$

[1] how does $\exp(-\phi)$ becomes g_s?

MALDACENA CONJECTURE: strongly interacting ($\lambda \gg 1$) $\mathcal{N} = 4$ SYM with gauge group $U(N)$ is equivalent to ten-dimensional classical supergravity compactified on $AdS_5 \times S^5$. SYM possesses the maximal number of 4 spinor supercharges. Besides the gluons the theory contains four fermions and six scalar fields in the adjoint representation of the gauge group. The Lagrangian of such a theory is completely specified by supersymmetry. There is a global $SU(4) R$-symmetry that rotates the six scalar fields and the four fermions.

Locally there is only one space with $SO(4,2)$ isometries: five dimensional Anti-de-Sitter space, or AdS_5. It a maximally symmetric solution of Einstein's equation with a negative cosmological constant. AdS has a boundary at spatial infinity. The boundary is at an infinite spatial distance, but a light ray can go to the boundary and come back in finite time. Massive particles can never get to the boundary. The curvature radius is proportional to N. Thus, for $N \to \infty$ the curvature tends to zero. The space S^5 has $SO(6) \sim SU(4)$ as isometry group. This becomes the R-symmetry of the SYM-theory on the boundary.

To extend the conjecture to any value of λ one has to find a substitute for supergravity, which is an inconsistent quantum theory. Extend conjecture: $\mathcal{N} = 4$ SUPER YANG-MILLS IS EQUIVALENT TO TYPE IIB STRING THEORY COMPACTIFIED ON THE SPECIAL BACKGROUND $AdS_5 \times S^5$. The parameters of Yang-Mills theory (g_{YM}, N) are related to those of string theory (g_s, α', R_{S^5}) as

$$g_{YM}^2 \equiv \frac{\lambda}{N} = 4\pi g_s \quad \text{and} \quad \sqrt{\lambda} = \frac{R_{S^5}^2}{\alpha'} = \frac{R_{AdS_5}^2}{\alpha'}, \tag{16}$$

where

$$\lambda = g_{YM}^2 N = 4\pi g_s N.$$

The following picture emerges:

- Classical supergravity is a good approximation for $\lambda \gg 1$
- In the 't Hooft limit in which λ is kept fixed and $N \to \infty$ the string coupling $g_s \to 0$ and string loop corrections are negligible. Hence classical string theory is a good approximation.
- Yang-Mills perturbation theory is a good approximation when $\lambda \ll 1$

Strongest evidence: $\mathcal{N} = 4$ SYM and type IIB string compactified on $AdS_5 \times S^5$ have same symmetries. Both are invariant under

- 32 supersymmetries

- the conformal group $SO(4,2) \sim$ isometries of AdS_5
- R-symmetry group $SU(4) \sim SO(6)$ isometries of S^5
- Montonen-Olive duality based on $SL(2,\mathbb{Z})$.

Theory live on different spaces: $\mathcal{N}=4$ super Yang-Mills on boundary of $AdS_4 \sim M_4$ and IIB string theory on $AdS_5 \times S^5$. Technically the main difference between the 'old' string calculations in QCD and the modern ones is the fact, that the spacetime background is no longer flat but rather $AdS_5 \times S^5$. New problem: $\mathcal{N}=4$ super Yang Mills is conformally invariant and is in the *Coulomb phase*, not in the confining phase.

3 The correspondence at work

If theories are equivalent, then it must be possible to specify for each field $O(x)$ of the boundary Minkowski theory the corresponding field $\Phi(y)$ of the bulk string theory such that the corresponding correlators in the two theories agree. One considers the generating functional for boundary theory,

$$Z(\Phi_0) = \left\langle e^{\int d^4x \Phi_0(x) O(x)} \right\rangle, \tag{17}$$

with source Φ_0. According to the recipe in (Gubser, Klebanov, Polyakov; Witten) one needs to find the bulk field $\Phi(y)$ such that

$$\Phi_0(x) = \Phi(y)\big|_{y \in \partial(AdS_5)=M_4},$$

that is the boundary value of the bulk field is the source in the boundary theory. Then the generating functional is just obtained by performing in the bulk theory the functional integral over Φ whose boundary value is Φ_0:

$$Z(\Phi_0) = \int_{\Phi \to \Phi_0} \mathcal{D}\Phi \, e^{-S[\phi]}. \tag{18}$$

3.1 $F(x)F(x')$ for $\lambda \gg 1$

In the conformally invariant super-YM theory the composite field F^2 has dimension 4 and hence

$$\langle F^2(\vec{x}) F^2(\vec{y}) \rangle \sim \frac{N^2}{|\vec{x}-\vec{y}|^8}.$$

Consider the dilaton kinetic term in IIB supergravity in $D = 10$ compactified on $AdS_4 \times S^5$. The volume of S^5 is $\pi^3 \times b^3$, and hence

$$S = \frac{\pi^3 b^3}{4\kappa_{10}^2} \int d^5 x \sqrt{g} g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi,$$

where $g_{\mu\nu} = b^2 \delta_{\mu\nu}/x_0^2$ is the metric on AdS_5 in the Poincare-coordinates and $\mu, \nu = 0 \dots 4$. In the $\lambda \gg 1$ limit classical supergravity is a good approximation, and we just need to solve the dilaton equation of motions,

$$\partial_\mu \left[\sqrt{g} g^{\mu\nu} \partial_\nu \Phi \right] = 0.$$

The solution that is equal to Φ_0 on the boundary ($x_0 \to 0$) is given by

$$\Phi(x_0, \vec{x}) = \int d^4 \vec{x} K(x^0, \vec{x}; \vec{z}) \Phi_0(\vec{z}), \qquad K(x_0, \vec{x}; \vec{z}) \sim \frac{x_0^2}{[x_0^2 + (\vec{x} - \vec{z})^2]^4}.$$

Inserting this solution into the classical action we find that the contribution to the classical action is entirely due to the boundary term

$$S = \frac{\pi^3 b^8}{4\kappa_{10}^2} \int d^4 \vec{x} x_0^3 \Phi \partial_0 \Phi \Big|_\epsilon^\infty \sim -\frac{\pi^3 b^8}{4\kappa_{10}^2} \int d^4 \vec{x} \int d^4 \vec{y} \frac{\Phi_0(\vec{x}) \Phi_0(\vec{y})}{(\vec{x} - \vec{y})^8}.$$

In the classical approximation we get

$$Z(\Phi_0) = \exp \left\{ \frac{\pi^3 b^8}{4\kappa_{10}^2} \int d^4 \vec{x} \int d^4 \vec{y} \frac{\Phi_0(\vec{x}) \Phi_0(\vec{y})}{(\vec{x} - \vec{y})^8} \right\}.$$

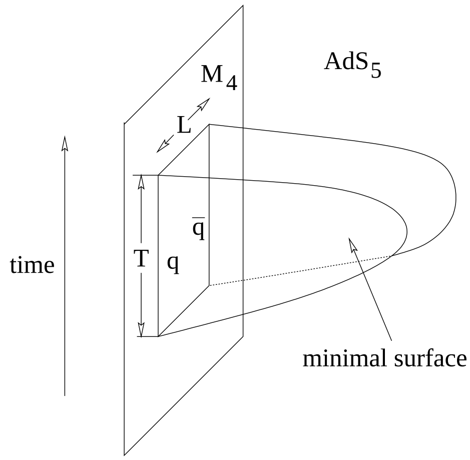
Since $b^2 \sim \alpha' \sqrt{N}$ and $2\kappa_{10}^2 = (2\pi)^7 g_s^2 (\alpha')^4$ one ends up with

$$\langle F^2(x) F^2(y) \rangle = \frac{\delta^2 Z(\Phi_0)}{\delta \Phi_0(\vec{x}) \delta \Phi_0(\vec{y})} \sim \frac{N^2}{(\vec{x} - \vec{y})^8}. \tag{19}$$

3.2 Wilson loops for λ fixed

The form of the loop considered is that of a very long strip along a spatial direction $-L/2 < x \equiv x^1 < L/2$ and a temporal length T with $L \ll T \to \infty$.

We assume time translation invariance and consider classical static string solutions. These are spanned in the (x^1, u)-plane by the function $u(x^1)$ and we set $x^2, \dots, x^p = 0$. We use the static gauge in which we set the world sheet time to be identical to that of the target space, $\tau = x^0 \equiv t$. For the classical solution we take $\sigma = 2\pi x^1$.



We restrict ourselves to diagonal background metrics with components that depend only on the coordinate u. The metric $G_{\mu\nu}$ for the coordinates

$$\{t \equiv x^0, x \equiv x^1, x^2, \dots x^p, u, \zeta^I\}, \quad I = p+2, \dots, 9$$

is given by

$$G_{\mu\nu} = \alpha' \mathrm{diag}\big\{G_{tt}(u), G_{xx}(u), G_{x^ix^i}(u), G_u(u), G_{\zeta^I\zeta^I}(u)\big\}. \tag{20}$$

The Nambu-Goto action with a general background metric takes the form

$$S = \frac{1}{2\pi\alpha'}\int d\sigma d\tau \sqrt{\det h_{\alpha\beta}}, \qquad h_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X). \tag{21}$$

With our assumptions the non-vanishing components of the induced metric takes the form

$$\begin{aligned} h_{\tau\tau} &= (\partial_\tau t)^2 G_{tt} = G_{tt} \\ h_{\sigma\sigma} &= (\partial_\sigma x)^2 G_{xx} + (\partial_\sigma u) G_{uu} = \frac{1}{(2\pi)^2}\big(G_{xx} + (\partial_x u)^2 G_{uu}\big) \end{aligned}$$

Defining the two functions

$$f^2(u) = G_{tt}(u)G_{xx}(u) \quad \text{and} \quad g^2(u) = G_{tt}(u)G_{uu}(u)$$

we have

$$\det h_{\alpha\beta} = \frac{1}{(2\pi)^2}\big(G_{tt}G_{xx} + G_{tt}G_{uu}(\partial_x u)^2\big)$$

$$= \frac{1}{(2\pi)^2}\big(f^2(u)+g^2(u)(\partial_x u)^2\big),$$

and the NG-action becomes

$$S = \frac{T}{4\pi^2\alpha'}\int dx\sqrt{f^2(u)+g^2(u)(\partial_x u)^2}. \tag{22}$$

For finding the stationary strings of this action we use the methods of classical mechanics. Then $u(x)$ corresponds to the trajectory of the particle. The conjugate 'momentum is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial u'} = \frac{g^2 u'}{\sqrt{f^2+g^2u'^2}}.$$

The energy is constant

$$H = \pi u' - \mathcal{L} = \frac{g^2u'^2}{\sqrt{f^2+g^2u'^2}} - \sqrt{f^2+g^2u'^2} = -|f(u_0)|,$$

where $u_0' = 0$, i.e. u_0 is the minimal value of the string amplitude. Solving for the derivative of u yields

$$u'^2 = \frac{f^2}{g^2}\frac{f^2-f_0^2}{f_0^2} \quad \text{or} \quad u' = \pm\frac{f}{g}\frac{\sqrt{f^2-f_0^2}}{f_0}, \qquad f_0 = f(u_0). \tag{23}$$

The quark and antiquark are set at the coordinates $x = \pm L/2$. The relation between L and u_0 is given by

$$L = \int_{-L/2}^{L/2} dx = 2\int_{u_0}^{u_s} du\,\Big|\frac{dx}{du}\Big| = 2\int_{u_0}^{u_s}\frac{g(u)}{f(u)}\frac{f(u_0)}{\sqrt{f^2(u)-f^2(u_0)}}.$$

We have taken the boundary plane at some large but finite value $u = u_s$, which later is taken to infinity. The find the action we write $dx = du/|u'|$ in (22) and insert the expression (23) with the result

$$S = \frac{T}{2\pi^2\alpha'}\int_{u_0}^{u_s}\frac{g(u)}{f(u)}\frac{f^2(u)}{\sqrt{f^2(u)-f^2(u_0)}}. \tag{24}$$

The Maldacena conjecture of the string/gauge duality is that the natural candidate for the expectation value of the Wilson loop is proportional to the partition function of the corresponding string action,

$$\langle W \rangle \sim \int \mathcal{D}X^\mu \exp\big(-S\big), \tag{25}$$

where the integral is over all surfaces whose boundary is the given loop. The static potential is

$$V_{q\bar{q}} = -\lim_{T\to\infty} \frac{1}{T}\langle W \rangle - 2m_q.$$

In the classical limit, where configurations with least action dominate, we would get

$$V_{q\bar{q}} = \lim_{T\to\infty} \frac{S}{T} = \frac{1}{2\pi^2\alpha'}\int_{u_0}^{u_s} \frac{g(u)}{f(u)}\frac{f^2(u)}{\sqrt{f^2(u)-f^2(u_0)}} - 2m_q.$$

Unfortunately this expression diverges linearly as $u_s \to \infty$. The mass of one single quark translates in the string language into a straight line from $u = 0$ to the boundary plane and is given by $m_q = \int_0^{u_s} g(u)du$. It is natural to determine the static potential as

$$V_{q\bar{q}} = \frac{1}{2\pi^2\alpha'}\int_{u_0}^{u_s} \Big(\frac{g(u)}{f(u)}\frac{f^2(u)}{\sqrt{f^2(u)-f^2(u_0)}} - g(u)\Big). \tag{26}$$

4 Beyond conformal invariance and supersymmetry

One makes use of Wittens idea [3] of putting the Euclidean time direction on a circle with anti-periodic boundary conditions for the fermions. This way the static potential for the $\mathcal{N} = 4$ theory at finite temperature as well as 3d pure YM theory has been determined. Similarly Wilson loops of 4d YM theory, 't Hooft loops and $V_{q\bar{q}}$ in $MQCD$ have been determined. Concerning the Lüscher term: For a superstring in flat spacetime with Ramond-boundary conditions there is an exact cancellation of bosonic and fermionic contributions to the Casimir energy and the term vanishes. For the Neveu-Schwartz boundary conditions, bosonic and fermionic zero-point energies contribute with the same sign, and that is what leads to a tachyonic state for a free string. When the GSO projection is taken into account and the tachyonic state is removed, we again have a massless ground state and a vanishing Lüscher term. Greensite and Oleson [4] calculated the bosonic determinants and conjectured that there maybe a violation of concavity due to the fermionic determinants. Kinar et.al [5] claim, that the fermions are massive and hence do not contribute to the Lüscher term. Hence the bosonic fluctuation determinants dominate and the static potential is concave. Problems in the calculations: gauge fixing of world sheet diffeomorphisms and κ-symmetry. Only for a particular class of $\kappa-$symmetry fixing schemes there are no divergent correction to the static potential. Using scaling arguments Kinar et.al could write down the general L-dependence for a large class of models. Models based on D_p branes with 16 supersymmetries have a Lüscher type behavior. On $AdS_5 \times S^5$ there is only a partial cancellation between bosonic and fermionic determinants. The coefficient and sign of the Lüscher term has not been determined in this case. Unfortunately there is no Green-Schwartz action which corresponds to the non flat background associated with confining gauge theories.

Finite temperature Finite T breaks both supersymmetry and conformal invariance. To construct the gravity solution describing gauge theory at finite temperature one starts with the general three-brane solution and takes the decoupling limit keeping the energy density above extremality finite. The resulting metric can be written as

$$\begin{aligned} ds^2 &= R^2\Big(u^2(-hdt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{du^2}{hu^2} + d\Omega_5^2\Big) \\ h &= 1 - \frac{u_0^4}{u^4}, \quad u_0 = \pi T. \end{aligned} \tag{27}$$

On the supergravity side, the entropy of near-extremal D3-branes is just the usual Bekenstein-Hawking result $S = A/4G_N$ and this should yield the entropy of the gauge theory at large N and large $g_{YM}^2 N$. This regime is not directly accessible in gauge field theory. If one compares with the *free gauge field theory* then the free energies almost agree

$$F_{\text{sugra}} = -\frac{\pi^2}{8} N^2 V T^4 = \frac{3}{4} F_{\text{YM}}.$$

The factor 3/4 is a long-standing puzzle. There is only a qualitative insight.

Background considered One would like to discuss compactifications of string theory or M theory, which are believed to be consistent theories of quantum gravities, on backgrounds involving AdS_5.

- backgrounds which are related to $AdS_5 \times S^5$ by deformations.
- backgrounds $AdS_5 \times X$. Duality relates it to a conformal field theory. For most X (e.g. CP^3 supersymmetry is not preserved. Non-supersymmetric backgrounds require an understanding of the quantum corrections, which are not

well-understood either in M theory or in type IIB compactifications with RR backgrounds.

- D3-branes in type IIB string theory, e.g. spaces with topology $S^3 \times B^2$ whose boundary is $S^3 \times S^1$, relevant for field theories on S^3 at finite temperature.
- Orbifolds on $AdS_5 \times S^5$. For example, $AdS_5 \times S^5/\mathbb{Z}_3$ has a $SU(N)^3$ gauge theory as counterpart.
- Orientifolds on $AdS_5 \times S^5$
- Conifold theories

5 More severe tests

The stronger form of the duality conjecture which is advocated is that THE FIELD THEORY IS EQUIVALENT TO THE STRING THEORY, AND THE ONLY ISSUE IS UNDERSTANDING THE MAPPING FROM ONE TO THE OTHER.

Natural to ask what in field theory corresponds to non-perturbative objects in string theory (e.g. to D-branes). The idea is, that every object we can exhibit in gauge theory has a stringy counterpart, and vice versa.

QCD_3 The starting point for studying QCD_3 is the $\mathcal{N}=4$ superconformal $SU(N)$ gauge theory in four dimensions which is realized as low energy effective theory of N coinciding parallel D-3 branes. Then one compactifies this theory on $\mathbb{R}^3 \times S^1$, with antiperiodic boundary conditions for the fermions around the circle. For small radius R_0 the fermions decouple from the system and susy is broken. Also, the scalars aquire masses at one loop, and these masses become infinite for $R_0 \to \infty$. In the infrared one is left with only gauge fields and the theory should be pure QCD_3. Upon compactification the $AdS_5 \times S^5$ geometry is replaced by the Euclidean black hole geometry

$$ds^2 = \alpha' \sqrt{4\pi g_s N}\Big(u^2(hdt^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{du^2}{hu^2} + d\Omega_5^2\Big), \tag{28}$$

where $h = 1 - u_0^4/u^4$ and τ parametrizes the compactifying circle (with radius R_0 in field theory) and

$$u_0 = \frac{1}{2R_0}.$$

The effective three dimensional coupling

$$g_3^2 N = \frac{g_4^2 N}{2\pi R_0}.$$

should remain finite in the limit $R_0 \to 0$. Hence one has to take the limit $g_4^2 N \to 0$ as $R_0 \to 0$. The sugra description is valid for $g_s N \gg 1$ in which case the typical mass scale of QCD_3, $g_3^2 N$, in much larger than the cutoff scale $1/R_0$. Therefore, with the present techniques, to suppress the unwanted massive Kaluza-Klein states, on can only study large N QCD_3 with a fixed UV cutoff $1/R_0$ in the strong coupling regime (strong as compared with the cutoff scale).

One expects

$$V_{q\bar{q}}(r \to 0) \sim \log(r), \qquad V_{q\bar{q}}(r \to \infty) \sim \sigma r.$$

The confining property has been shown by lattice simulations. If one repeats the $AdS_5 \times S^5$ computation, but now for the AdS-black hole background. The calculation yields confinement and

$$\sigma = \frac{(g_s N)^{1/2}}{4\sqrt{\pi} R_0^2}.$$

Almost nothing has been done as concerning the Lüscher term. The behavior of the interquark potential on the background is shown in the following table, taken from [6]

Model	Nambu-Goto Lagrangian	Energy
$AdS_5\times S^5$	$\sqrt{u^4+(U')^2},\quad u=\frac{U}{R}$	$-\frac{2\sqrt{2}\pi^{3/2}R^2}{\Gamma(\frac{1}{4})^4}\cdot L^{-1}$
non-conformal D_p brane (16 supers.)	$\sqrt{u^{7-p}+(U')^2}$	$-d'\cdot L^{-2/(5-p)}$ $+O(l^{-2/(5-p)-2(6-p)/(5-p)(7-p)})$
$YM_4, T>0$	$\sqrt{u^4(1-(U_T/U)^4)+U'^2}$	$\sim L^{-1}(1-c(LT)^4)$ for $L<<L_c$ full screening $L>L_c$
Dual YM_3	$\sqrt{u^4+U'^2(1-(U_T/U)^4)^{-1}}$	$\frac{U_T^2}{2\pi R^2}\cdot L-2\kappa+O(\log l\; e^{-\alpha L})$
Dual YM_4	$\sqrt{u^3+U'^2(1-(U_T/U)^3)^{-1}}$	$\frac{U_T^{3/2}}{2\pi R^{3/2}}\cdot L-2\kappa+O(\log L\; e^{-\alpha L})$
Rotating D_3	$\sqrt{C}\sqrt{\frac{U^6}{U_0^4}\Delta+U'^2\frac{U^2\Delta}{1-a^4/U^4-U_0^6/U^6}}$	$4/3\frac{U_T^2}{R^2}CL+...$
D_3+D_{-1}	$\sqrt{u^4+q)+U'^2(1+qR^4/U^4)}$	qL+...
$MQCD$ system	$2\sqrt{2\zeta}\sqrt{\cosh(s/R_{11})}\sqrt{1+s'^2}$	$E=2\sqrt{2\zeta}\cdot L-2\kappa$ $+O(\log L\; e^{-1/\sqrt{2}R_{11}L})$
’t Hooft loop	$\frac{1}{g_{YM}^2}\sqrt{u^3(1-(U_T/U)^3)+U'^2}$	full screening of monopole pair

6 QCD_4

The starting point for obtaining QCD_4 is the $(2,0)$ superconformal theory in six dimensions realized on N parallel coinciding $M5$-planes. Then one compactifies on an S^1 of radius R_1. This yields a 5-dimensional theory whose low-energy limit is the maximally supersymmetric $SU(N)$ gauge theory, with gauge coupling constant $g_5^2 = 2\pi R_1$. Then one compactifies further on another S^1 of radius R_0. The dimensionless coupling constant in 4 dimensions is

$$g_4^2 = \frac{g_5^2}{2\pi R_0} = \frac{R_1}{R_0}.$$

To break susy one imposes the anti-periodic boundary condition on the fermions around the second S^1. To get QCD_4 one needs the condition

$$\text{typical mass scale of } QCD \text{ states} \ll \frac{1}{R_0}, \frac{1}{R_1}.$$

Thus one has to go beyond sugra approximations. The large N limit of the six-dimensional theory is M theory on $AdS_7 \times S^4$. Upon compactification on $\mathbb{T}^2$, we get M theory on a black hole background. In the 't Hooft limit $R_1 \ll R_0$. Using the duality

$$\text{M-theory on a circle} \leftrightarrow \text{Typ IIA string theory}$$

the large N limit of QCD then becomes type IIA string theory on the black hole geometry given by the metric

$$ds^2 = \frac{2\pi\lambda}{3u_0} u\Big(4u^2 \sum_1^4 dx_i^2 + \frac{4}{9u_0^2} u^2 \big(1 - \frac{u_0^6}{u^6}\big) d\tau^2 + 4\frac{du^2}{u^2(1-u_0^6/u^6)} + d\Omega_4^2\Big)$$

with a non-constant dilaton field. The horizon is at u_0 and one has

$$r_0 = \frac{1}{3R_0}.$$

The string tension is found to be

$$\sigma = \frac{4\lambda}{27R_0^2}.$$

7 Summary

Properties of the duality:

- Yang-Mills string (made of gluons) = fundamental strings in higher dimensions
- When we can find a low curvature gravity description we can do numerous calculation in the large N limit.
- One can calculate: spectra, correlators of operators and Wilson loops, thermal properties.
- If field theory is conformal the gravity solution will include a AdS_5 factor.
- Large N limit of a gauge theory should have a string theory description. Whether there is a gravity description, depends on how large the curvature is.

Open problems:

- Better understanding what is the class of field theories which have a gravity approximation
- For non-supersymmetric QCD, or other theories which are weakly coupled, we expect to have curvatures at least of the order of the string scale: a proper understanding of string theories on highly curved spaces (RR-backgrounds) seems crucial.

8 Large N Gauge Theories

String theories were born from the attempt of describing the properties of strong interaction through the construction of the dual resonance models. This was later recognized to correspond to the quantization of a relativistic string. It contained all sort of massless particles as gluons, gravitons and others except a massless pseudoscalar particle corresponding to the pion. Because of this and other unphysical features it became clear that string theories could not provide a theory of strong interactions. It was abandoned and replaced by QCD.

Since the middle of the seventies it has been known that $SU(N)$ gauge theories in the 't Hooft limit $N \to \infty$ with $\lambda = g_{YM}^2 N$ fixed simplify in the sense that only planar diagrams survive in leading order. It has been conjectured that in this limit QCD is described by a string theory: the mesons are string excitations that are free when $N \to \infty$. Although many attempts have been made to construct a QCD-string, none can be considered sufficiently satisfactory.

Main problem: in the ultraviolet ($R \leq 0.25\,\mathrm{fm}$) QCD describes quarks and gluons whereas in the infrared it should describe colorless hadron. At large energies the coupling constant is small whereas it is large at low energies. This holds true as long as the β-function which determines the dependency of the QCD coupling constant on the dimensional cutoff (in lattice theories $\mu \sim 1/a$) through the Gell-Mann-Low equation,

$$\frac{d \log g^2}{d \log \mu^2} = \beta(g^2)$$

is negative. At the one-loop level it reads

$$\beta(g^2) = -\Big(\frac{11}{3}N_c - \frac{2}{3}N_f\Big)\frac{g^2}{16\pi^2}$$

and QCD is asymptotically free for $N_c = 3, N_f \leq 16$. In the IR the standard PT inapplicable since for example the renormalization-invariant measurable scale parameter following from the integration of the Gellmann-Low formula

$$\Lambda_{QCD}^2 = \mu^2 \exp\Big[-\int^{g^2(\mu^2)} \frac{dg'^2}{g'^2\beta(g'^2)}\Big]$$

has no expansion in powers of g. Here the the dimensionless gauge coupling has been dimensionally transmuted into the QCD scale Λ_{QCD}. All observable quantities in QCD are proportional to the corresponding power of Λ_{QCD}. For example, using the one-loop result for the β_{QCD} function yields the string tension

$$\sigma \sim \Lambda_{QCD}^2 = \mu^2 \exp\Big[-\frac{16\pi^2}{(\frac{11}{3}N_c - \frac{2}{3}N_f)g^2(\mu^2)}\Big] \tag{29}$$

which has now power series expansion at $g_{QCD} = 0$. Measurements yield $100\,\mathrm{MeV} < \Lambda_{QCD} < 300\,\mathrm{MeV}$ and lattice simulation give $\sigma \sim 0.2\,\mathrm{GeV}^2$

Since its original formulation [7] the large N expansion has been the most concrete possibility for reaching an analytical understanding of the non-perturbative aspects of QCD including its confinement properties. One may hope that for large N the $SU(N)$ gauge theories simplify and there is a perturbation expansion in terms of $1/N$ (for a nice review see [8]). First we need to understand how to scale the coupling g_{YM} in the large-N limit. In a asymptotically free theory like pure QCD it is naturally to scale g_{YM} such that Λ_{YM} remains constant when $N \to \infty$ and comparing with (29) this implies

$$g_{YM}^2 N \equiv \lambda \quad \text{fixed.}$$

The limit $N \to \infty$ with fixed λ is known as 'T HOOFT LIMIT. The same limit would be valid if we include also matter in the adjoint representation. For conformal gauge theories, as $\mathcal{N} = 4$ SYM theory, also the limit $\lambda \to \infty$ is possible.

Let us study the large N limit for a set of fields $\Phi_i = \Phi_i^a T_a$ in the adjoint representation of $SU(N)$ with Lagrangian

$$\mathcal{L} \sim \tfrac{1}{2}\mathrm{tr}\,(d\Phi_i d\Phi_i) + g_{YM} c^{ijk}\mathrm{tr}\,(\Phi_i\Phi_j\Phi_k) + g_{YM}^2 d^{ijkl}\mathrm{tr}\,(\Phi_i\Phi_j\Phi_k\Phi_l),$$

where $\Phi_i = \Phi_i^a T_a$ is in the Lie algebra and we assumed that the interaction is $SU(N)$ invariant. We rescale the fields $\Phi \to \Phi/g_{YM}$ such that

$$\mathcal{L} \sim \frac{N}{\lambda}\,\mathrm{tr}\,\left\{\tfrac{1}{2}d\Phi_i d\Phi_i + c^{ijk}\Phi_i\Phi_j\Phi_k + d^{ijkl}\Phi_i\Phi_j\Phi_k\Phi_l\right\}.$$

One may believe that one can perform the naive saddle point approximation in the functional integral for $N \to \infty$ and find the classical limit. This is wrong, since the number of fields also increases with increasing N. Unfortunately for Yang-Mills theory with action

$$S[A] = \int d^dx \frac{1}{2g^2}\mathrm{tr}\,F_{\mu\nu}^2\,, \tag{30}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{31}$$

and A_μ is the $N \times N$ matrix

$$A_\mu^{ij}(x) = g \sum_a A_\mu^a(x)[t^a]^{ij}, \tag{32}$$

no change of variables is known (as for zero-, one- or some two-dimensional matrix models) such that the number of dynamical fields is limited. So let us have a look at the Feynman diagrams. For the free model

$$\begin{aligned} Z_0(j) &= \int \mathcal{D}\Phi \exp\Big\{ \int \tfrac{N}{2\lambda} \mathrm{tr}\, \Phi_i \triangle \Phi_i + \int \mathrm{tr}\, j_i \Phi_i \Big\} \\ &= \exp\Big\{ -\tfrac{\lambda}{2N} \mathrm{tr}\, j_i \frac{1}{\triangle} j_i \Big\} = \exp\Big\{ \tfrac{1}{2} j^a_{i\,b}(x) G^{bd}_{ac}(x,y) j^c_{d\,i} \Big\} \end{aligned}$$

where the Green function reads

$$G^{bd}_{ac}(x,y) = \frac{\lambda}{N}(\delta^d_a \delta^b_c - \tfrac{1}{N}\delta^b_a \delta^d_c) G_0(x,y),$$

where G_0 is the propagator for spinless particles (the inverse of $-\triangle$ in coordinate space) so that the two-point function reads (we suppress the index i from now on)

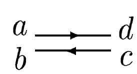
$$\Phi = (\Phi^a{}_b), \qquad \langle \Phi^a{}_b(x) \Phi^c{}_d(y) \rangle = \frac{\lambda}{N}(\delta^a_d \delta^c_b - \tfrac{1}{N}\delta^a_b \delta^c_d) G_0(x,y).$$

With 't Hooft we use a double line notation, in which Φ is regarded as a direct product of a fundamental and anti-fundamental field

$$N \times \bar{N} = \text{ adjoint} + U(1), \qquad \phi \sim \psi \psi^\dagger.$$

In leading order there is no difference between $SU(N)$ and $U(N)$ theories so we neglect this difference here. Using $N \times \bar{N} \sim$ adjoint we may view a Feynman diagram of adjoint fields as a network of double lines. This way one gets a $U(N)$ field with propagator

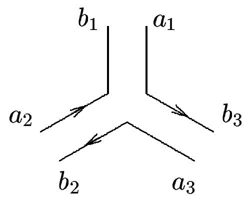
$$\langle \Phi^a{}_b \Phi^c{}_d \rangle \sim \frac{\lambda}{N} \delta^a_d \delta^c_b$$



Each line represents the Kronecker delta-symbol and has an orientation which is indicated by the arrow. This notation is obviously consistent with the space-time structure of the propagator which describes a propagation from x to y.

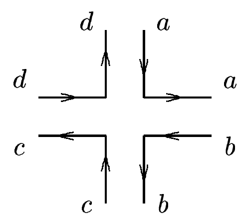
The arrows are due to the fact that the matrix Φ^a_b is Hermitian and its off-diagonal components are complex conjugate. The independent fields are, say, the complex fields Φ^a_b for $a > b$ and the diagonal real fields Φ^a_a. The arrow represents the direction of the propagation of the indices of the complex field Φ^a_b for $a > b$ while the complex-conjugate one, $\Phi^b_a = \Phi^{a*}_b$, propagates in the opposite direction. For the real fields Φ^a_a, the arrows are not essential.

The three-gluon vertex is depicted in the double-line notations as



$$\propto \tfrac{N}{\lambda}\left(\delta^{a_1}_{b_3}\delta^{a_2}_{b_1}\delta^{a_3}_{b_2}\right) \tag{33}$$

where the subscripts 1, 2 or 3 refer to each of the three gluons. The four-gluon vertex has the form



$$\propto \tfrac{N}{\lambda} \tag{34}$$

The diagrams of perturbation theory can now be completely rewritten in the double-line notation [7]. The simplest one, which describes the one-loop correction to the gluon propagator, is shown in Figure 1. It has 4 propagators, 2 vertices and one

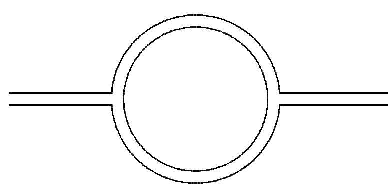


Figure 1: *Double-line representation of a one-loop diagram for the gluon propagator. The sum over the N indices is associated with the closed index line. The contribution of this diagram is λ^2/N.*

loop so that it is multiplied by the factor

$$\frac{\lambda^4}{N^4}\cdot\frac{N^2}{\lambda^2}\cdot N = \frac{\lambda^2}{N}.$$

Note that our counting applied to $\langle \Phi^a_b \Phi^c{}_d\rangle$ with fixed a, b, c and d. This should be compared with the tree diagram, with is goes with a factor λ/N. Hence this one-loop diagram contributes in the same order in N as the tree-diagram.

A typical 4-loop diagram is depicted in (2). It has eight three-gluon vertices, 13

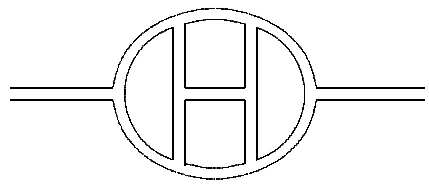


Figure 2: *Double-line representation of a four-loop diagram for the gluon propagator. The sum over the N indices is associated with each of the four closed index lines whose number is equal to the number of loops. The contribution of this diagram is λ^5/N.*

propagators and four closed index lines which coincides with the number of loops. Therefore, the order of this diagram is

$$(\frac{N}{\lambda})^8 \cdot (\frac{\lambda}{N})^{13} \cdot N^4 = \frac{\lambda^5}{N}.$$

It is of the same order in N as the tree-diagram. The diagrams of the type in Figure 2, which can be drawn on a sheet of a paper without crossing any lines, are called the *planar* diagrams. The following figure shows some *vacuum planar diagrams* and their dependency on λ and N. Let us now consider a non-planar

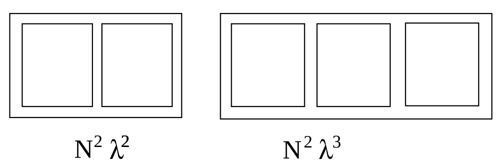


Figure 3: *Planar diagrams*

diagram of the type depicted in Figure 4. This diagram is a three-loop one and has

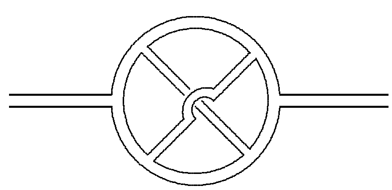


Figure 4: *Double-line representation of a three-loop non-planar diagram for the gluon propagator. The diagram has six three-gluon vertices but only one closed index line (while three loops!). The order of this diagram is λ^4/N^3.*

6 three-vertices and 10 propagators. The crossing of the two lines in the middle does

not correspond to a four-gluon vertex and is merely due to the fact that the diagram cannot be drawn on a sheet of a paper without crossing the lines. The diagram has only one closed index line. Hence its order is

$$(\frac{N}{\lambda})^6 \cdot (\frac{\lambda}{N})^{10} \cdot N = \frac{\lambda^4}{N^3}.$$

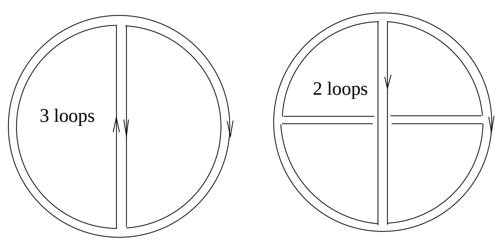
It of order $1/N^2$ relative to the planar diagrams.

Let us analyze the dependency of general vacuum diagrams on λ and N. We may view the double lines as forming the edges in a simplicial decomposition of a surface, if we view each single-loop as edge of a face in the decomposition. The surface will be oriented, since the lines have an orientation. Adding the point at infinity each diagram corresponds to a compact, closed and oriented surface. Let us count the powers of N and λ associated to such a diagram: Each vertex contributes a factor N/λ, each propagator a factor λ/N and each loop a factor N. (for $SU(N)$ each index has N possible values).

For a diagram with V vertices, E propagators (= edges in the simplicial decomposition) and F loops (= faces) has a factor

$$N^{V-E+F}\lambda^{E-V} = \lambda^{E-V}N^{\chi},$$

where $\chi = V - E + F$ is the *Euler character* of the surface corresponding to the diagram. For closed oriented surfaces is $\chi = 2 - 2g$ where g is the genus (number of handles) of the surface. For example, consider the following two vacuum diagrams:



the one on the left goes as λN^2 the one on the right as λ^2. One can see that the dependence on λ varies with the order of the diagram, while the dependence on N is only sensitive to its topological properties. We see that the perturbative expansion may be written as

$$\log Z \sim \sum_{g=0}^{\infty} N^{2-2g} \sum_{i=0}^{\infty} c_{g,i}\lambda^i = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda). \tag{35}$$

The leading contributions come from surfaces with maximal Euler character, that is surfaces with the topology of a sphere. These socalled *planar diagrams* will give a contribution of order N^2. The non-planar diagrams are down by a factor N^{-2} with respect to the planar ones. This large N expansion selects the topology of Feynman diagrams rather than their order and can pick up, at lowest order, important non-perturbative effects. It is often referred to as the *topological expansion* or the *genus expansion.*

Let us finally see how the expansion looks like for correlators of gauge invariant fields, $\langle\prod G_j\rangle$, such that each G_i cannot be written as a product of two gauge-invariant fields. To find the N-dependency we couple the field to sources

$$S \longrightarrow S - N\sum_{i=1}^{n} j^i G_i \sim N$$

and compute the connected correlators by the wellknown formula

$$\langle\prod_{i=1}^{n} G_i\rangle_c = N^{-n}\frac{\delta^n}{\delta j^1\cdots\delta j^n}\log Z(j)\big|_{j=0}.$$

The above counting of powers of N still holds since the additional vertices come also with a factor N. Thus

$$\langle\prod_{i=1}^{n} G_i\rangle_c \sim N^{2-n}. \tag{36}$$

This is not true if one G_i would be the product of two gauge invariant fields since it would lead to two vertices. Because of

$$\langle G_1 G_2\rangle \sim N^0 \quad \text{and} \quad \langle G_1 G_2 G_3\rangle \sim \frac{1}{N}$$

we are again lead to interpret $1/N$ as coupling constant. For more general gauge theories similar results hold. For the gauge groups $SO(N)$ and $USp(N)$ the adjoint is in $N\times N$ and the surfaces are non-oriented.

The formula (35) is the same as one finds in a perturbative theory with closed oriented strings (type II), if we identify $1/N$ with g_s. In string theory the analog of the fields G_j in (36) would be the vertex operators inserted on the string world sheet.

Virtual quark loops can be easily incorporated in the $1/N$-expansion One distinguishes between the 't Hooft limit when the number of quark flavors N_f is fixed as $N\to\infty$ and the Veneziano limit [9] when

$$N\to\infty,\ N_f\to\infty \quad \text{with} \quad \frac{N_f}{N},\ g_{YM}^2 N,\ g_{YM}^2 N_f \quad \text{fixed.}$$

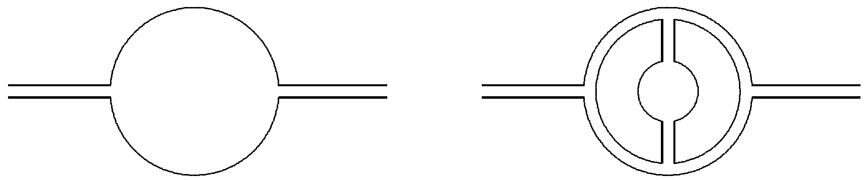


Figure 5: Diagrams for the gluon propagator with a quark loop which is represented by the single lines. The diagram a) involves one quark loop and has no closed index lines so that its order is λ^2/N^2. The diagram b) involves three loops one of which is a quark loop and two closed index lines. Its order is λ^4/N^2.

The Veneziano model provides a better explanantion of certain aspects of low energy phenomenology. The 't Hooft limit is, however, much simpler and has been studied in much more detail.

Virtual quark loops are suppressed in the 't Hooft limit as $1/N$ and lead in the Veneziano limit to the same topological expansion as dual-resonance models of strong interaction.

If one includes fermion in the fundamental representation, then diagrams with quark loops are down by a factor $1/N$ with respect to those without quark loops. Hence the diagrams that dominate the large N limit are the planar ones with the minimum of quark loops. Let us see, how the suppression of fermion loops comes about. It is easy to incorporate quarks in the topological expansion. A quark field belongs to the fundamental representation of the gauge group SU(N) and its propagator is represented by a single line

$$\left\langle \psi_i \bar{\psi}_j \right\rangle \propto \delta_{ij} \;=\; i \longrightarrow j \quad . \tag{37}$$

The arrow indicates, as usual, the direction of propagation of a (complex) field ψ. We shall omit these arrows for simplicity.

The diagram for the gluon propagator which involves one quark loop is depicted in Figure 5a. It has two $A\psi^2$ vertices and no closed index lines so that its order is 1, Analogously, the relative order of a more complicated tree-loop diagram in Figure 5b, which involves one quark loop and two closed index lines, is

$$\text{Figure 5b} \qquad N \cdot \frac{\lambda^6}{N^6} \cdot \frac{N^4}{\lambda^4} = \lambda^2 \, .$$

More generally, if we replace a gluon propagator by the diagram 5a, then

$$\frac{\lambda}{N} \longrightarrow \frac{\lambda^2}{N^2}$$

and the new graph containing one more quark loop is down by a factor λ/N. Therefore diagrams with L quark loops are suppressed at large N by

$$L \text{ quark loops } \sim \lambda^{E-V+L} N^{\chi-L} . \tag{38}$$

8.1 Large-N factorization

The vacuum expectation values of several colorless operators, which are singlets with respect to the gauge group, factorize in the large-N limit of QCD (or other matrix models). The simplest gauge-invariant operator in a pure SU(N) gauge theory is the square of the non-Abelian field strength:

$$O\left(x\right)=\frac{1}{N}\,\mathrm{tr}\,F_{\mu\nu}^{2}\left(x\right). \tag{39}$$

The natural normalization has been chosen, such that

$$\left\langle \frac{1}{N}\,\mathrm{tr}\,F_{\mu\nu}^{2}\left(x\right)\right\rangle \sim 1 \tag{40}$$

exists for $N \to \infty$. The contribution of all planar graphs to the average on the LHS of (40) is of order 1 in accord with the general formula.

In order to verify the factorization in the large-N limit, let us consider the diagrams for the average of the product of two colorless operators $O\left(x_1\right)$ and $O\left(x_2\right)$ given by (39). It involves a factorized part when gluons are emitted and absorbed by the same operators. The contribution of the factorized part is of order 1 as above.

Let us assume, that

$$\langle O(x_1)O(x_2)\rangle - \langle O(x_1)\rangle \langle O(x_2)\rangle \neq 0.$$

For this connected correlator not to vanish at least one gluon line is emitted and absorbed by different operators $O\left(x_1\right)$ and $O\left(x_2\right)$. If one remove a connecting gluon propagator one removes one vertex and one propagator from each subdiagram and also the connecting propagator. Independently to which two lines in the subdiagrams the connecting propagator has been attached, one alway gains a factor N/λ when removing this propagator.

Alternatively, the connected correlator of the two operators is associated with the general formula

$$\text{generic graph} \sim \left(\frac{1}{N}\right)^{\chi+L+2B} . \tag{41}$$

It does not depend on its order in the coupling constant and is completely expressed via the Euler number χ, the number of virtual quark loops L and external boundaries B.

This example illustrates the general property that only (planar) diagrams with gluon lines emitted and absorbed by the same operators survive as $N \to \infty$. Since correlations between the colorless operators $O(x_1)$ and $O(x_2)$ are of order $1/N^2$, the *factorization* property holds as $N \to \infty$:

$$\begin{aligned} &\left\langle \frac{1}{N} \operatorname{tr} F^2(x_1) \frac{1}{N} \operatorname{tr} F^2(x_2) \right\rangle \\ &= \left\langle \frac{1}{N} \operatorname{tr} F^2(x_1) \right\rangle \left\langle \frac{1}{N} \operatorname{tr} F^2(x_2) \right\rangle + \mathcal{O}\left(\frac{1}{N^2}\right) \end{aligned} \tag{42}$$

For a general set of gauge-invariant operators $O_1, \ldots, O_n$, the factorization property can be represented by

$$\langle O_1 \cdots O_n \rangle = \langle O_1 \rangle \cdots \langle O_n \rangle + \mathcal{O}\left(\frac{1}{N^2}\right) .$$

The factorization in large-N QCD was first discovered by A.A. Migdal in the late seventies. An important observation that the factorization implies a semiclassical nature of the large-N limit of QCD was done by Witten [10]. The factorization property also holds for gauge-invariant operators constructed from quarks.

The large-N factorization has been shown to all orders of perturbation theory. It can be also verified at all orders of the strong coupling expansion in the SU(N) lattice gauge theory.

Phenomenology: The phenomenology of the $N = \infty$ model is remarkably similar to that of the real world. The dominant Feynman graphs contributing to the connected part of a n-point function of fermionic current, e.g. $\bar{\psi}\psi$ or $\bar{\psi}\gamma_5\psi$, are all $O(N^{1-n})$ and have the following properties

- They are planar
- There are no internal fermion loops

- All current insertion are on a single fermion loop which forms the boundary of the graph

Similary the graphs contributing to Greens functions of gauge-invariant operators constructed out of gauge fields alone are $O(N^{2-n})$ and

- are planar
- contain no fermion loops.

As we have seen, each fermion loop costs a factor of $1/N$, while each non-planar crossing is suppressed by $1/N^2$. Assuming that the $N = \infty$ theory confines so that propagating states are color singlets, it is possible to study properties of hadrons. This is done by applying the above rules and analyzing the intermediate states that contribute to the various n-point functions [11]. I simply quote the relevant results:

(a) Mesons:

- Mesons are stable: their decay amplitude are $O(1/N)$
- Mesons are non-interacting: scattering amplitudes are $O(1/N)$
- Meson masses are finite; i.e. they are $O(1)$
- The number of mesons is infinite
- Zweigs rule holds

(b) Glueballs:

- Glueballs are stable
- Glueballs are non-interacting: a vertex involving l Glueballs is suppressed by $O(1/N^{l-1})$
- There are infinitely many glueballs
- Glueballs do not mix with mesons: a vertex involving k mesons and l glueballs is of $O(1/N^{l+k/2-1})$.

(c) Baryons:

Baryons pose a problem at $N = \infty$, since a baryon in a $SU(N)$ theory must be made out of N quarks while a meson is always made out of a quark-antiquark pair, irrespective of N. This make baryons behave differently from mesons

- Baryon masses are $O(N)$
- The splitting of various excited baryonis states is $O(1)$
- Baryons interact strongly amongst themselves: the typical baryon-baryon vertex is $O(N)$
- Baryons interact with mesons with $O(1)$ couplings

Baryons behave similar to solitons in weakly coupled theories. For example, a monopole mass is $O(1/g^2)$, but the energies of excitations around the monopole background are $O(1)$. The $M\bar{M}$ scattering amplitude is $O(1/g^2)$, while monopole-electron scattering amplitudes are $O(1)$. This lead Witten to suggest that baryons are in some sense solitons of large-N QCD, with N playing the role of $1/g_{YM}^2$ [12]. A further bonus of the large N expansion is, that the $U(1)$-anomly

$$\partial_\mu j_5^\mu = \frac{g_{YM}^2 N_f}{16\pi^2} \mathrm{tr}\,({}^*F_{\mu\nu}F^{\mu\nu}) \sim \frac{\lambda}{N}$$

vanishes for $N \to \infty$. Then the η' becomes a Goldstone boson of the chiral symmetry breaking similarly as π, K and η. For finite N the η' is a pseudo-Goldstone boson, with a (mass)^2 proportional to the symmetry breaking term which is of order $1/N$.

Factorization once more: Let J_i denote fermionic curent operators and G_i denote gauge invariant operators made out of gluon fields only. Then, according to the previous rules

$$\begin{aligned} &\langle J_1 \cdots J_n \rangle_c = O(N^{1-n}), \quad \langle G_1 \cdots G_m \rangle_c = O(N^{2-n} \\ &\langle J_1 \cdots J_n \cdot G_1 \cdots G_m \rangle_c = O(N^{1-n-m}) \end{aligned}$$

From these equations it immediately follows:

$$\frac{\langle J_1 \cdots J_n \rangle_c}{\langle J_1 \rangle \cdot \langle J_n \rangle} = O(1/N^{n-1}) \qquad \frac{\langle G_1 \cdots G_m \rangle_c}{\langle G_1 \rangle \cdot \langle G_n \rangle} = O(1/N^{2m-2})$$

$$\frac{\langle J_1 \cdots J_n G_1 \cdots G_m \rangle_c}{\langle J_1 \rangle \cdot \langle J_n \rangle \langle G_1 \rangle \cdots \langle G_m \rangle} = O(1/N^{n+2m-1})$$

The leading diagram for the scattering of n mesons is of the order N^{-1-n}. The tree diagrams for the scattering of n mesons in string theory is of order g_s^{n-2}. Again this suggests that g_s^2 should be identified with $1/N$. In the limit $N \to \infty$ the mesons have vanishing width precisely as it is the case in tree-level string theory.

From discussion at the Combo:
How can the eigenvalues of the hermitean operator D be purely imaginary?

9 $\mathcal{N} = 4$ super Yang-Mills

Before discussing the supersymmetry extension of conformal conformal field theory let recall some facts about conformal field theories. The representation theory for the superconformal case is rather complicated and I refer to [14] for detail.

9.1 Conformal algebra

Conformal transformations leave the light cone invariant. The consists of translations, Lorentz transformations, scale transformations (dilatations) and special conformal transformation. In $d > 2$ dimensions the form a $\frac{1}{2}(d+1)(d+2)$-dimensional group $SO(2,d)$. In 4 spacetime dimensions the conformal group has dimensions 15. An infinitesimal conformal transformations

$$y^\mu \sim x^\mu + X^\mu(x)$$

is generated by a conformal Killing fields

$$X_{\mu,\nu} + X_{\nu,\mu} = \frac{1}{2}\,\eta_{\mu\nu}\partial_\rho X^\rho.$$

In 4 dimensions there are 15 conformal Killing fields. These are

$$X^\mu = a^\mu, \quad X^\mu = \omega^\mu{}_\nu x^\nu, \quad X^\mu = \lambda x^\mu, \quad X^\mu = 2(c\cdot x)x^\mu - x^2 c^\mu. \tag{43}$$

The corresponding hermitian generators are

$$iX^\mu\partial_\mu = \left\{a^\mu P_\mu, 2\omega^{\mu\nu}M_{\mu\nu}, \lambda D, c^\mu K_\mu\right\}. \tag{44}$$

They fulfill the conformal algebra

$$\begin{aligned}
[P_\mu, P_\nu] &= [K_\mu, K_\nu] = [D,D] = 0 \quad , \quad [P_\mu, D] = iP_\mu \\
[P_\mu, K_\nu] &= 2iM_{\mu\nu} + 2i\eta_{\mu\nu}D \qquad , \quad [M_{\mu\nu}, D] = 0 \\
[P_\rho, M_{\mu\nu}] &= -i\big(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu\big) \quad , \quad [K_\mu, D] = -iK_\mu \\
[M_{\mu\nu}, M_{\rho\sigma}] &= i\big(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\big) \\
[K_\rho, M_{\mu\nu}] &= -i\big(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu\big).
\end{aligned} \tag{45}$$

These are just the $SO(2,4)$ commutation relations

$$[M_{mn}, M_{pq}] = i\big(\eta_{mp}M_{nq} + \eta_{nq}M_{mp} - \eta_{mq}M_{np} - \eta_{np}M_{mq}\big)$$

with the identifications

$$2M_{mn} = -2M_{nm} = \begin{pmatrix} 0 & P_\mu & D \\ -P_\mu & M_{\mu\nu} & P_\mu \\ -D & -P_\mu & 0 \end{pmatrix} + \begin{pmatrix} 0 & K_\mu & D \\ -K_\mu & M_{\mu\nu} & -K_\mu \\ -D & K_\mu & 0 \end{pmatrix}.$$

Note that the generators of the Lorentzgroup commute with the Dilatation operator D.

9.2 Transformation of fields and scaling dimensions

Conformal transformations are very particular coordinate transformations and tensor fields transform under small transformations as

$$\phi_{\mu\nu\dots} \longrightarrow \phi_{\mu\nu\dots} + L_X \phi_{\mu\nu\dots},$$

The infinitesimal transformations of a matter field is given by the Lie derivative L_X. For example, for a vector field V_μ it is

$$L_X V_\mu = X^\rho \partial_\rho V_\mu + X^\rho_{\ ,\mu} V_\rho. \tag{46}$$

Inserting the corresponding Killing fields one finds

$$\begin{array}{rl} \text{transl.} & iL_a V_\mu = a^\rho P_\rho V_\mu \\ \text{Lorentztrf.} & iL_\omega V_\mu = \frac{1}{2}\omega^{\rho\sigma} M_{\rho\sigma} V_\mu + i\omega^\rho{}_\mu V_\rho \\ \text{dilatation} & iL_\lambda V_\mu = \lambda D V_\mu + i\lambda V_\mu \\ \text{spez. Trf.} & iL_c V_\mu = c^\rho K_\rho V_\mu + 2i\big(c_\mu x^\rho - x_\mu c^\rho + c\cdot x\, \delta^\rho_\mu\big) V_\rho, \end{array} \tag{47}$$

and similarly for tensor fields of higher rank. The algebra of infinitesimal transformations of the fields is the same as that of the generators of the symmetry since

$$[L_X, L_Y] = L_{[X,Y]}.$$

The transformations

$$x^\mu \longrightarrow x^\mu + X^\mu \quad \text{and} \quad \phi \longrightarrow \phi + L_X \phi$$

are symmetries of any diffeomorphism invariant theory if one also maps the metric $\eta_{\mu\nu}$ to $(1 - \frac{1}{2}\partial_\rho X^\rho)$. For conformally invariant theories the change of the metric can be un-done by a compensating Weyl transformations such that

$$\delta_X \phi_{\mu\nu\dots} = \big(L_X + \frac{\alpha}{2}\partial_\rho X^\rho\big)\phi_{\mu\nu\dots}. \tag{48}$$

The real constant α is the *Weyl-weight* of $\phi_{\mu\nu\dots}$. It can be determined by the following recipe: If the metric transforms as $g_{\mu\nu} \to e^{\varphi} g_{\mu\nu}$ then the field must transforms as $\phi \to e^{-\alpha\varphi}\phi$ in order for action to stay invariant. In particular, for dilatations

$$\phi_{\mu\nu\dots} \longrightarrow \lambda(x^\rho\partial_\rho + s + 2\alpha)\phi_{\mu\nu\dots}, \tag{49}$$

where s is the number of covariant minus the number of contravariant indices of ϕ. The number $\Delta = s + 2\alpha$ is called *scaling dimension* of ϕ. One finds

$$\begin{array}{rl} \text{scalar field: } \Delta(\phi) = \frac{1}{2}(d-2) & \text{Dirac field: } \Delta(\psi) = \frac{1}{2}(d-1) \\ \text{energy-momentum: } \Delta(T_{\mu\nu}) = d & \text{gauge potential (d=4) : } \Delta(A_\mu) = 1. \end{array}$$

9.3 Hilbert space

We assign to a classical field $\phi(x)$ the operator $\hat{\phi}(x)$ on a Hilbert space $\mathcal{H}$. We assume that the conformal transformations are represented by unitary representation on $\mathcal{H}$. For the Lorentz group

$$U(\Lambda)\hat{\phi}(x)U^{-1}(\Lambda) = S\hat{\phi}\big(\Lambda^{-1}x\big) = e^{\frac{i}{2}(\omega,J)}\hat{\phi}(x), \quad J_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \tag{50}$$

and for the the scale transformation

$$U(\lambda)\hat{\phi}(x)U^{-1}(\lambda) = e^{\lambda\Delta}\hat{\phi}(e^\lambda x) = e^{-i\lambda D}\hat{\phi}, \quad D = i(r\partial_r + \Delta).$$

For the Lorentz transformation

$$U(\Lambda) = \exp\big(\tfrac{i}{2}(\omega,\hat{J})\big) \Longrightarrow [\hat{J}_{\nu\nu},\hat{\phi}] = J_{\mu\nu}\hat{\phi}$$

and for the scale transformations

$$U(\lambda) = \exp\big(i\lambda\hat{D}\big) \Longrightarrow [\hat{D},\hat{\phi}] = -D\hat{\phi}.$$

The same relations hold for the translations and special conformal transformations. Let us see what we can say about the two-point function of a scalar field in a conformally invariant theory. We assume, that the vacuum state is invariant under conformal transformations. Then

$$G(|x-y|) \;=\; \langle 0|\phi_1(x)\phi_2(y)|0\rangle$$

$$
\begin{aligned}
&= e^{\lambda(\Delta_1+\Delta_2)} \langle 0|U^{-1}\phi(e^\lambda x)UU^{-1}\phi_2(e^\lambda y)U|0\rangle \\
&= e^{\lambda(\Delta_1+\Delta_2)} G(e^\lambda |x-y|) \\
&\Longrightarrow G(x,y) = c|x-y|^{-\Delta_1-\Delta_2}.
\end{aligned}
$$

Using the invariance of $|0\rangle$ under special conformal transformation implies then, that c vanishes unless $\Delta_1 = \Delta_2$

9.4 Representations

The irreducible representation of the conformal group are classified by the Lorentz-group representation and the eigenvalues of D which is proportional to the so-called scaling dimension Δ,

$$D|\psi\rangle = -i\Delta|\psi\rangle.$$

The conformally invariant vacuum $|0\rangle$ has scaling dimension zero. The state

$$\hat{\phi}(0)|0\rangle$$

has scaling dimension $-i\Delta$:

$$
\begin{aligned}
e^{i\lambda\hat{D}}\,\hat{\phi}(0)|0\rangle &= e^{i\lambda\hat{D}}\hat{\phi}(0)e^{-i\lambda\hat{D}}\,|0\rangle = e^{\lambda\Delta}\hat{\phi}(0)\,|0\rangle \\
\Longrightarrow \quad & \hat{D}\,\hat{\phi}(0)|0\rangle = -i\Delta\,\hat{\phi}(0)|0\rangle.
\end{aligned}
$$

One can argue, that Δ must be bounded below (for scalar fields $\Delta \geq (d-2)/2$ must hold) and in each representation there must be at least one state with smallest scaling dimensions. Let $|\psi\rangle$ be a state with scale dimension Δ, Then P_μ increases the scaling dimension by one and K_μ decreases it by one:

$$D(P_\mu|\psi\rangle) = -i(\Delta+1)|\psi\rangle \quad \text{and} \quad D(K_\mu|\psi\rangle) = i(\Delta-1)|\psi\rangle.$$

An operator of lowest dimension Δ in a multiplet is called *primary field*. Since D commutes with the Lorentz transformations one can classify the representations of the conformal group corresponding to primary operators by the Lorentz representation and the scaling dimension Δ (these determine the Casimirs of $SO(2,4)$). These representations include the primary field and all the descendent fields which are obtained by acting on it with the generators of the conformal group. All fields have a definite scaling dimension. The operators can in general not be eigenfunctions of P_0 or P^2, since $[D, P_\mu] \neq 0$.

9.5 Correlation functions

Since the conformal group is much larger than the Poincare group, it severely restricts the correlation functions of primary fields, which must be invariant under conformal transformations. It has been shown by Lüscher and Mack that the Euclidean Green's function of a CFT maybe analytically continued to Minkowski spacetime, and that the resulting Hilbert space carries a unitary representation of the Lorentzian conformal group.

Under conformal transformations

$$\delta\phi_{\mu\nu\dots} = \Big(L_X + \frac{\alpha}{2}\,\partial_\rho X^\rho\Big)\phi_{\mu\nu\dots}.$$

In the absence of anomalies this implies the conformal *Ward identities.* For example, the 2-point functions of two tensor fields

$$< 0|\phi^{(1)}_{\mu\nu\dots}(x_1)\phi^{(2)}_{\alpha\beta\dots}(x_2)|0 >\equiv G^{(2)}_{\mu\nu\dots,\alpha\beta\dots}(x_1,x_2)$$

fulfills the Ward identity

$$\sum_i \Big[L_{X(x_i)} - \frac{2\alpha_i}{d}\partial_\alpha X^\alpha(x_i)\Big]G^{(2)}(x_1,x_2) = 0.$$

It is the infinitesimal form of the following conformal transformation of the two-point functions:

$$G^{(2)}_{\mu\nu,\alpha\beta}(x_1,x_2) = e^{-\alpha_1\phi(y_1)}e^{-\alpha_2\phi(y_2)}\frac{\partial x_1^\sigma}{\partial y_1^\mu}\frac{\partial x_1^\rho}{\partial y_1^\nu}\frac{\partial x_2^\gamma}{\partial y_2^\alpha}\frac{\partial x_2^\delta}{\partial y_2^\beta}G^{(2)}_{\sigma\rho,\gamma\delta}(y_1,y_2),$$

where α_1 and α_2 are the Weyl-weights of $\phi^{(1)}$ and $\phi^{(2)}$. This differential equation determines the 2-point function uniquely. For two scalar fields $\phi^{(1)}$ and $\phi^{(2)}$, the conformal Ward-identity simplifies to

$$\sum_i \Big[X^\mu(x_i)\frac{\partial}{\partial x_i^\mu} + \frac{\Delta_i}{d}\partial X(x_i)\Big]G^{(2)} = 0.$$

Translations- and Lorentz invariant imply that $G = G(\xi)$, where $\xi = |x_2 - x_1|$. The dilatation Ward identity then simplifies to

$$\Big(\xi\partial_\xi + (\Delta_1 + \Delta_2)\Big)G^{(2)} = 0$$

and has the solution

$$G^{(2)} = C_{12}|x_2 - x_1 - i\epsilon|^{-\Delta_1-\Delta_2}.$$

With a linear transformation of the $\phi^{(a)}$ one can transform the symmetric matrix C_{ab} to δ_{ab}. If we use the Ward-identities belonging to the special conformal transformations then it follows that

$$(\Delta_1 - \Delta_2)\, G^{(2)} = 0.$$

Hence, the conformal Ward identities finally yield

$$< 0|\phi^{(1)}(x_1)\phi^{(2)}(x_2)|0 >= \begin{cases} 0 & \text{if } \Delta_1 \neq \Delta_2; \\ |x_2 - x_1 - i\epsilon|^{-2\Delta} & \text{if } \Delta_1 = \Delta_2 \equiv \Delta. \end{cases}$$

If one does a similar calculation for the 3-point function, then one obtains

$$< 0|\phi^{(1)}(x_1)\phi^{(2)}(x_2)\phi^{(3)})(x_3)|0 >= C_{123}\, \xi_{12}^{\Delta_3-\Delta_2-\Delta_1}\, \xi_{23}^{\Delta_1-\Delta_2-\Delta_3}\, \xi_{31}^{\Delta_2-\Delta_3-\Delta_1}.$$

As expected for a conformal field theory, the correlation function show a power like behavior and show that the theory has no mass-gap. The higher correlation functions are restricted but not determined by conformal invariance.

9.6 $\mathcal{N} = 4$ conformal field theory.

An interesting generalization of the conformal algebra is the super conformal algebra. Superconformal algebras exist only in $d \leq 6$ dimensions. The fundamental representations are a generalization of the remarkable representations of the AdS_4 group $SO(3,2)$ discovered by Dirac [13] some time ago, which were later named singleton (indicating that the representations of Dirac corresponding to fields on the boundary of AdS_4 are singular when the Poincare limit is taken). The singleton representation require a single set of oscillators transforming in the fundamental representation of the maximal compact subgroup of the covering group $Sp(4,\mathbb{R})$ of $SO(2,3)$. The fundamental representations in for the AdS_5-group or conformal group in four dimensions require two sets of oscillators transforming in the fundamental representations of $SU(2,2)$.

The superalgebra of type IIB supergravitation on $AdS_5 \times S^5$ is the supergroup $SU(4|2,2)$ with the bosonic subgroup $SU(4) \times SU(2,2) \times U(1)_Z$, where $SU(4)$ is the double cover of $SO(6)$, the isometry group of the five sphere and $SU(2,2) \sim SO(2,4)$ is the cover of the conformal group in 4 dimensions or the anti-de Sitter group in 5 dimensions. The Abelian $U(1)_Z$ generator Z commutes with all the other generators and acts like a central charge. Therefore, $SU(4|2,2)$ is not simple. One may factor out this Abelian ideal and obtains a simple Lie superalgebra, denoted by $PSU(4|2,2)$, whose bosonic subalgebra is $SU(4) \times SU(2,2)$. The representations of $PSU(4|2,2)$ correspond to representations of $SU(4|2,2)$ with $Z = 0$. $SU(4|2,2)$ can be interpreted as $\mathcal{N}=8$ extended AdS superalgebra symmetry of IIB supergravity in $d=5$ or as the $\mathcal{N}=4$ extended conformal superalgebra in $d=4$.

The superconformal algebra is generated by $M_{mn}, A_{ij}, B_{ij}, Z$ and Q^i_α. The range of indices is

$$m,n:\ 0,\dots,5;\quad i,j:\ 1,2,3,4;\quad \alpha:\ 0,\dots,6.$$

The M_{mn} generates the conformal $SO(2,4)$ algebra, the $A+iB$ the $SU(4)$ R-symmetry, Z generates a $U(1)$ and Q^i_α are the $4\times 8=32$ supercharges. These supercharges can be grouped into chiral components of Lorentz spinors Q and S, the generators of Poincare and S type supersymmetry. The $SU(4)$ generators commute with the spacetime symmetry generators M_{mn} and the generator Z commutes with both the $SU(4)$ and the $SO(2,4)\sim SU(2,2)$ generators. The part of the conformal superalgebra (in the $SO(2,4)$ notation) which contains the 32 supercharges $Q^i_\alpha, \alpha=1,\dots,8$ reads as follows

$$\begin{aligned}\{Q^i_\alpha, Q^j_\beta\} &= -(\mathcal{C}^{-1})_{\alpha\beta}A_{ij} + (\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}B_{ij}\\ &+ 2(\mathcal{C}^{-1}\gamma_*)_{\alpha\beta}\delta_{ij}Z + \tfrac{1}{2}\delta_{ij}(\mathcal{C}^{-1}\gamma^{mn})_{\alpha\beta}M_{mn}\end{aligned}$$

The generators $A+iB$ generate the $SU(4)$ part of the bosonic superalgebra. $\mathcal{C}$ is the charge conjugation matrix of $SO(2,4)$. The commutators of the bosonic charges with the supercharges read

$$\begin{aligned}[A_{jk}, Q^i_\alpha] &= (a_{jk})_{pi}Q^p_\alpha, \quad [B_{jk}, Q^i_\alpha] = -(s_{jk})_{pi}(\gamma_*)_{\sigma\alpha}Q^p_\sigma\\ [M_{mn}, Q^i_\alpha] &= \tfrac{1}{2}(\gamma_{mn})_{\beta\alpha}Q^i_\beta.\end{aligned}$$

The $U(1)$ charge Z commutes with all other generators and the anti de-Sitter superalgebra becomes non-simple.

The unique irreducible CPT self-conjugate doubleton supermultiplet of $SU(4|2,2)$ is the supermultiplet of $\mathcal{N}=4$ supersymmetric Yang-Mills theory in $d=4$. This theory is uniquely determined by specifying the gauge group, and its field content is a vector multiplet in the adjoint representation. The easiest way to construct $\mathcal{N}=4$ super Yang-Mills theory is by dimensional reduction of $\mathcal{N}=1$ super Yang-Mills in 10 dimensions to 4 dimensions. By construction it is invariant under an internal $SU(4)$ *R-symmetry.* On shell it contains

1 gluon field   4 Majorana spinors   6 real scalars

all in the adjoint representation of the gauge group. Since $SU(4)\sim SO(6)$ the 6 scalars ϕ^I will transform according to the vector representation of $SO(6)$. Each Φ^I

transforms according to the adjoint representation of $SU(N)$. The interaction of the theory contains a scalar potential proportional to

$$V \sim \sum_{I,J} \mathrm{tr}\left([\phi^I, \phi^J]\right)^2.$$

The moduli space of the theory is the space of commuting matrices Φ^I and hence is

$$\text{Moduli space} \sim \mathbb{R}^{6(N-1)}/S^N.$$

In the $SU(4)$ representation they form a (quasi) antisymmetric selfdual tensor

$$\bar{\Phi}_{AB} \sim \Big(\sum_{i=1}^{3} \left(\eta^i_{AB}\phi^i - \bar{\eta}^i_{AB}\phi^{i+3}\right), \quad A, B = 1, 2, 3, 4$$

i.e. a antisymmetric tensor which, up to complex conjugation, is selfual

$$\Phi^{AB} = \frac{1}{2}\epsilon^{ABCD}\bar{\Phi}_{CD} = \Phi^*_{AB}.$$

The Weyl spinors transform according th the 4 representation of $SU(4)$. The Lagrangian $L = L_B + L_F$ has the following form

$$\begin{aligned} L_B &= \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a + (D_\mu\bar{\Phi}_{AB})_a(D^\mu\Phi^{AB})_a - g^2 f^{abc}\Phi^{AB}_b\Phi^{CD}_c f^{ade}\bar{\Phi}^d_{AB}\bar{\Phi}^e_{CD} \\ L_F &= -i\psi^{\alpha A}_a\sigma^\mu_{\alpha\dot{\alpha}}(D_\mu\bar{\psi}^{\dot{\alpha}}_A)_A - g\sqrt{2}f^{abc}\big[\psi^{\alpha A}_a\bar{\Phi}^b_{AB}\psi^{cB}_\alpha + \bar{\psi}^a_{\dot{\alpha}A}\Phi^{AB}_b\bar{\psi}^{\dot{\alpha}}_{cB}\big]. \end{aligned}$$

It is manifestly invariant under R-symmetry,

$$\begin{aligned} \psi^A_\alpha &\longrightarrow U^A{}_B\psi^B_\alpha & \bar{\psi}_{\dot{\alpha}A} &\longrightarrow (U^*)_A{}^B\bar{\psi}_{\dot{\alpha}B} \\ \Phi^{AB} &\longrightarrow U^A{}_B\psi^{CD}(U^T)_D{}^B & \bar{\Phi}_{AB} &\longrightarrow (U^*)_A{}^C\Phi_{CD}(U^\dagger)^D{}_B. \end{aligned}$$

Spectrum: It consists of all gauge invariant quantities that can be formed from the basic fields of the vector multiplet. We focus on local operators which involve fields taken at the same point in spacetime. These are product of traces of products of fields. In the 't Hooft large N limit the correlation functions involving multiple-trace operators are down by powers of N compared to those of single-trace operators involving the fields. Thus we concentrate on single-trace operators. To continue we recall the structure of the commutation relations of the superconformal algebra

$$\begin{aligned} &[D, K] \sim iK, \quad [D, P] \sim -iP, \quad [D, Q] \sim -\tfrac{i}{2}Q, \quad [D, S] \sim \tfrac{i}{2}S \\ &[K, Q] \sim S, \quad [P, S] \sim Q, \quad \{Q, Q\} \sim P, \quad \{S, S\} \sim K \\ &\{Q, S\} \sim M + D + R. \end{aligned}$$

As in the bosonic case one groups the operators into *primary and secondary operators*. To construct representations one starts with some state of lowest scale dimension which annihilated by the lowering operators $\bar{S}^{\dot{\alpha}i}$ and K_μ. Now one acts on this state with the raising operators Q^i_α and P_μ. One distinguishes *chiral primary operators* and non-chiral primaries. The representations of the chiral primaries sit in short representations of $SU(4|2,2)$ and are annihilated by some of the supercharges. It is not hard to see, that if the primary operator has helicity λ then in a long supermultiplett the helicity will have range $\lambda - 4$ to $\lambda + 4$. There is one ultrashort representation of the $\mathcal{N} = 4$ algebra whose range of helicities is from -1 to 1. This is the vector multiplet of the theory itself.

So what are the known chiral primaries of $\mathcal{N} = 4$ SU(N) SYM theory? The lowest component $\mathcal{O}$ of a superconformal-primary multiplet is characterized by the fact that

$$\mathcal{O} \neq Q\mathcal{O}'.$$

Since

$$[Q^A_\alpha, \phi^I] \sim \lambda_{\alpha B} \qquad \{Q^A_\alpha, \lambda_{\beta B}\} \sim \sigma^{\mu\nu}_{\alpha\beta} F_{\mu\nu} + \varepsilon_{\alpha\beta}[\phi^I, \phi^J]$$
$$\{Q^A_\alpha, \bar{\lambda}^B_{\dot{\beta}}\} \sim (\sigma^\mu)_{\alpha\dot{\beta}} D_\mu \phi^I \qquad [Q^A_\alpha, A_\mu] \sim (\sigma_\mu)_{\alpha\dot{\alpha}} \bar{\lambda}^A_{\dot{\beta}} \varepsilon^{\dot{\alpha}\dot{\beta}}$$

one would believe that operators built from the fermions and gauge fields are no good candidates. We expect the lowest chiral primaries to be constructed only from the scalar fields. Indeed, one can prove that the operators

$$\mathcal{O}^{I_1 \ldots I_n} = \mathrm{tr}\left(\Phi^{(I_1} \cdots \Phi^{I_n)}\right),$$

which are traceless with respect to all indices, exactly correspond to the short chiral primary representations. The scale dimension of these operators is n, the same as in the free theory. The symmetry must be imposed, since $\{Q, \lambda\} \sim .. + [\Phi^I, \Phi^J]$. One can further argue, that if $n > N$ then $\mathcal{O}$ can be expressed in terms of operators with $n \leq N$ and descendents. Hence a list of chiral primaries (giving rise to short multiplets) is

$$\mathcal{O}_n \equiv \mathcal{O}^{I_1 \ldots I_n} = \mathrm{tr}\left(\Phi^{(I_1} \cdots \Phi^{I_n)}\right) \quad \text{with} \quad n \leq N, \quad \text{traceless.}$$

This operators transform according to the representations $[0, n, 0]$ of $SU(4)$ which has the dimension

$$\dim([0, n, 0]) = \tfrac{1}{2}(n+1)^2 \cdot (n+2) \cdot (n+3).$$

The representations are obtained by acting with the generators of the superalgebra (Q, P) on $\mathcal{O}_n$. The algebra built on $\mathcal{O}_n$ contains a total of

$$2^8 \times \tfrac{1}{12} n^2 (n^2 - 1)$$

primary states. Out of the $\mathcal{O}_n$ one can construct bosonic primaries by acting on $\mathcal{O}_n$ with a even number of super charges (recall, that $\Delta(Q) = \frac{1}{2}$). This way one gets operators of the form

$$\mathrm{tr}\left(\sigma^{\mu\nu} F_{\mu\nu} \phi^{I_2} \cdots \phi^{I_n}\right) \quad \text{or} \quad \mathrm{tr}\left(\lambda_{\alpha A} \lambda_{\beta B} \phi^{I_3} \cdots \phi^{I_n}\right).$$

9.7 Montonen-Olive duality

There is strong evidence that $\mathcal{N} = 4$ SYM is invariant under $SL(2, \mathbb{Z})$ transformations:

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}, \qquad \tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}, \quad \text{where} \quad ad - bc = 1.$$

For vanishing θ-parameter this transformation relates weak with strong coupling. This duality implies, that the theory can be equivalently formulated as a theory of fundamental gluons having magnetic monopoles as solitons or as a theory of fundamental magnetic monopoles with gluons appearing as solitons.

10 Testing the Maldacena conjecture

What could be the test of the duality conjecture that string theory on $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ superconformal Yang-Mills theory in the 't Hooft limit? Possible checks shoud compare

- symmetries
- correlation functions
- spectrum of operators
- moduli space
- thermal properties
- beyond conformal theories: masses, confinement, ...

10.1 Symmetries

The $\mathcal{N} = 0$ conformal SYM possesses a $SO(2,4)$ spacetime symmetry. These is the group of isometries AdS_5. The AdS group preserves the boundary, which is compactified Minkowski spacetime, $\partial(AdS_5) = CM_4$, and acts on CM_4 like the conformal group. The global R symmetry on the field theory side is $SU(4) \sim SO(6)$ which is the group of isometries of S^5. Both theory very probably have an $SL(2, \mathbb{Z})$ selfduality symmetry. The Yang-Mills coupling is related to the string coupling through

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi} = \frac{i}{g_s} + \frac{\chi}{2\pi}, \tag{51}$$

where χ is the expectation value of the RR-scalar. In both theories the duality transformation is

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1.$$

The $SL(2, \mathbb{Z})$ is a conjectured strong-weak duality symmetry of IIB string theory in flat space. But since the $AdS \times S^5$ background is invariant under $SL(2, \mathbb{Z})$, this symmetry should survive on this symmetric background. For the gauge theory the symmetry under (51) is just the Olive-Montonen duality.

11 Spectra

Unfortunately the full spectrum of type IIB string theory on $AdS_5 \times S^5$ in not known. The only known states are those who arise from the dimensional reduction of $10d$ IIB supergravity multiplet. These fields have helicity form -2 to 2, so they are in small multipletts of the superconformal algebra. They match to the small multiplets of the field theory. Take the dilaton field of supergravity. During the dimensional reduction is it expanded as

$$\chi(x,y) = \sum \chi_n(x) Y^{(n)}(y), \quad x \in AdS_5,\ y \in S^5.$$

The scalar spherical harmonics $Y^{(n)}$ are symmetric, homogeneous and traceless poloynoms of degree n (in the imbedding coordinates) and hence transform according to the $(0,n,0)$ of $SU(4) \sim SO(6)$. In each representation $(0,n,0)$ we find a field χ_n and the mass of this field is

$$m_n = n(n+4)/R^2.$$

A similar expansion maybe performed for the other field in the supergravity multiplett. This way we match the states of type IIB supergravity compactified on $AdS_5 \times S^5$ with the representations of the superconformal algebra on M_4. One finds the representations described above which are built on the superconformal primaries $\mathcal{O}_n$ which are scalars in the $(0,n,0)$ representation of $SU(4)_R$ for $n = 2, 3, \ldots, \infty$. Later we shall see that the scaling dimension of an operator correponding to field modes of mass m is

$$\Delta = 2 + \sqrt{4 + (mR)^2}.$$

In particular, the modes χ_n correponds to a conformal primary of scaling dimension

$$\Delta = 4 + n,$$

Besides this constraint the operator and field must transform according to the same representaions of $SU(4)_R \sim SO(6)_{S^5}$. For example the field $\chi_0 Y^{(0)}$, which is constant on S^5 must correspond to a scalar operator with scaling dimension 4 and which transforms trivially under $SU(4)_R$. This is just $\mathrm{tr}\, F_{\mu\nu}^2$.

Similarly, the dilaton field $\chi_n Y^{(n)}$ corresponds to a complex scalar field of scale dimension $n+4$ and we find the correspondence

$$\chi_n Y^{(n)} \longleftrightarrow \mathrm{tr}\left(F_{\mu\nu}^2 \phi^{I_1} \cdots \Phi^{I_n}\right), \quad n \geq 2 \tag{52}$$

The operator on the right is in the multiplet built from $\mathcal{O}_{n+2}$. The lowest dimensional scalar fields corresponding to the chiral primaries $\mathcal{O}_n$ are linear combination of the graviton and the 4-form field A_{abcd}.

Te massless graviton field which transforms trivially under the $SO(6)$ corresponds to an operator which is a singlet under $SU(4)_R$ and has scaling dimension $\Delta = 4$. This way we are lead to relate

$$g_{\mu\nu} \longleftrightarrow T_{\mu\nu}.$$

More generally, the $n = 2$ *supergravity representation* includes the field content of $d = 5, \mathcal{N} = 8$ gauged supergravity. On the field theory side it corresponds to the representation containing the superconformal current.

The massless $SO(6)$ gauge fields correpond to the global $SU(4)_R$ currents,

$$A^I_\mu \longleftrightarrow J^I_\mu.$$

If one goes through the list of field modes, one finds that one has the same spectrum of chiral primary operators for $n = 2, \ldots, N$. There seem to be no matching for $n > N$. However we can only trust the supergravity results for masses m below the string scale. With

$$\frac{R^4}{l_s^4} \sim g_s N \gg 1,$$

this means

$$m_n^2 \sim \frac{n^2}{R^2} \ll \frac{1}{l_s^2} \sim \frac{\sqrt{g_s N}}{R^2}, \quad \text{or for} \quad n^2 \ll \sqrt{g_n N}.$$

If $\mathcal{N} = 4$ SYM in the *t'Hooft limit* is dual to the IIB string on $AdS_5 \times S^5$, then we would expect that *string spectrum* matches the supergravity spectrum up to a mass scale

$$m^2 \leq \frac{N^2}{R^2} \sim \frac{N^{3/2}}{l_p^2 \sqrt{4\pi}} \gg m_p^2, \quad \text{since} \quad R^2 = \alpha' \sqrt{4\pi N g_s} = l_p^2 \sqrt{4\pi N},$$

which is much bigger than the string- and Planck scale, and that there are *no chiral fields above this scale.* This matches with model calculation of string propagation on $AdS_3 \times X$ which we considered some time ago [15]. There one also needs to cut the spectrum at some conformal weight to get a consistent string propagation on $SU(1,1) \times X \sim AdS_3 \times X$.

11.1 Fields in supergravity $\leftrightarrow$ operators in $\mathcal{N}=4$ SYM

As motivation let us remind that changing the coupling constant

$$g_{YM}=4\pi g_s, \qquad g_s=e^{\langle\phi\rangle}=l_p^4/l_s^4$$

is to be interpreted as deformation of the corresponding marginal operator F^2. The expectation value of the dilaton field is determined by the boundary conditions on ϕ. Changing g_{YM} amount to changing the boundary field ϕ_0. More generally, consider adding the term

$$\int d^4x\phi_0(x)\mathcal{O}(x)$$

to $\mathcal{L}$. It is natural to assume that that this will change the boundary condition of the dilation at the boundary of AdS to

$$\begin{aligned} \phi(x,z)|_{z=0} &= \phi_0(x), \quad \text{with } AdS \text{ metric} \\ ds^2 &= \big(\frac{R}{z}\big)^2\big(dz^2+dx_1^2+\ldots+dx_{p+1}^2\big). \end{aligned}$$

In [16] it was proposed, that

$$\Big\langle \exp\int d^4x\,\phi_0(x)\mathcal{O}(x)\Big\rangle_{CFT}=\mathcal{Z}_{string}\big[\phi(x,z)|_{z=0}=\phi_0(x)\big]. \tag{53}$$

The left hand side is the generating functional of correlation functions in the field theory. The right hand side is the full partition function of string theory with the boundary condition that the field ϕ has the value ϕ_0 on the boundary of AdS. There is a little problem whose solution yields the above mentioned $m\leftrightarrow\Delta$ relation. Consider a bulk field with mass parameter m. We study its behaviour near the boundary $z=0$ of AdS. We use the Poincare coordinates on AdS_{d+1}:

$$\begin{aligned} ds^2 &= \frac{R^2}{z^2}\big(dz^2+dx_1^2+\ldots dx_d^2\big) \quad \text{so that} \\ \triangle &= \frac{1}{\sqrt{g}}\partial_m\big(\sqrt{g}g^{mn}\partial_n\big)=\frac{1}{R^2}\big(z^{d+1}\partial_z(z^{1-d}\partial_z)+z^2\triangle_x\big) \end{aligned}$$

The wave equation $(-\triangle+m^2)\phi=0$ for a function $\phi=f(z)g(x)$ yields

$$z^2f''g+(1-d)zf'g+z^2f\triangle_x g-(mR)^2fg=0.$$

For $f=z^\Delta$ and $z\to0$ this yieds the algebraic relation $\alpha(\alpha-d)-(mR)^2=0$ with the two solutions

$$z^{\Delta} \quad \text{and} \quad z^{d-\Delta}, \qquad \Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + (mR)^2}. \tag{54}$$

To get a finite value for the boundary field belonging to a massive field the correct boundary condition on the bulk field on the right hand side of (53) should be changed to

$$\phi(x, \epsilon) = z^{d-\Delta}\phi_0(x). \tag{55}$$

Since ϕ is dimensionless (?) the boundary field has length dimension

$$[\phi_0] = [\text{length}]^{\Delta - d}.$$

This in turn requires that the operator $\mathcal{O}$ in (53) had scaling dimension Δ. In particular, a field of mass m in AdS_5 couples to an operator $\mathcal{O}$ of scaling dimension

$$\Delta(\mathcal{O}) = 2 + \sqrt{4 + (mR)^2}.$$

A massless field must couple to an operator of scaling dimensions $\Delta = 4$ (more generally $\Delta = d$). These results square with the correpondences listed above. For example, the mass of χ_n is $n(n+4)/R^2$ and hence it must couple to an operator with scaling dimensions

$$\Delta_n = n + 4.$$

This agrees with (52). The graviton is massless and must couple to an operator of scaling dimension 4 which is the scaling dimension of the energy-momentum operator.

11.2 $N, g_{YM}^2 N \to \infty$: Matching of correlation function

We assume

$$N \quad \text{large} \quad , \quad g_{YM}^2 N \quad \text{large}$$

which amounts to neglect all stringy α' corrections that cure the divergences of supergravity and also all string loop corrections. In this regime *classical supergravity* is a good approximation and the basic relation (53) reads

$$W_{YM}[\phi_0] = -\log\left\langle \exp\int d^4x\phi_0(x)\mathcal{O}(x)\right\rangle_{CFT} \sim S_{SUGRA}[\phi_{cl}]\big|_{\phi(z=\epsilon)=\phi_0}, \tag{56}$$

where ϕ_{cl} is the solution of the classical supergravity equations with the boundary condition indicated. We performed the classical limit for supergravity on the right hand side and assumed that there is only one saddle point. Hence the generator of the connected Greenfunctions in the gauge theory, in the large $N, g_{YM}^2 N$ limit, is the on-shell supergravity action. We have not made explicit the rescaling of ϕ when $\epsilon \to 0$. This can be accounted for by a wave function renormalization on $\mathcal{O}$ or ϕ, such that the final answer is independent of the cutoff ϵ.

Two-point functions: We consider only the part of the sugra action which is quadratic in the relevant field perturbation which is needed. For massive scalar fields this action has the generic form

$$S = \eta \int dz d^4x \sqrt{g} \left[g^{mn} \partial_m \phi \partial_n \phi + \tfrac{1}{2} m^2 \phi^2 \right] \Rightarrow (-\triangle + m^2)\phi = 0.$$

Again we choose Poincare coordinates

$$ds^2 = \frac{R^2}{z^2}\big(dz^2 + dx^2\big), \qquad R^2 \triangle = \big(z^5 \partial_z \frac{1}{z^3} \partial_z + z^2 \triangle_x - m^2 R^2\big) \qquad z \geq \epsilon.$$

The general solution is a superposition of modes with fixed 4-momentum, $\phi = Z(u)\exp(ipx)$. The resulting equation for $Z(u) = Z(pz)$ reads

$$u^5 \big(\frac{1}{u^3} Z'\big)' - u^2 Z - (mR)^2 Z = 0, \quad Z = Z(u). \tag{57}$$

The two solutions are

$$Z(u) = u^2 I_{\Delta - 2}(u) \quad \text{and} \quad Z(u) = u^2 K_{\Delta-2}(u),$$

where

$$\Delta = 2 + \sqrt{4 + (mR)^2}.$$

Since the Bessel functions have the following asymptotic behavior,

$$I_\nu(u) \sim \big(\frac{u}{2}\big)^\nu, \qquad K_\nu(u) \sim \big(\frac{u}{2}\big)^{-\nu} \qquad \text{for} \quad u \to 0$$
$$I_\nu(u) \sim \frac{1}{\sqrt{2\pi u}}\, e^u, \qquad K_\nu(u) \sim \sqrt{\frac{\pi}{2z}}\, e^{-u} \qquad \text{for} \quad u \to \infty,$$

we must discard the solution containing I_ν since it explodes in the bulk the corresponding solution has infinite action. Imposing the boundary condition $\phi(x, \epsilon) = \exp(ipx)$, we find

$$\phi_{cl}(z, x) = \frac{(pz)^2 K_{\Delta-2}(pz)}{(p\epsilon)^2 K_{\Delta-2}(p\epsilon)}\, e^{ipx}.$$

To calculate the two point function we note, that for the particular boundary field

$$\phi_0 = \lambda_1 e^{ipx} + \lambda^2 e^{iqx}$$

the perturbation of the action reads

$$\int d^4x \phi_0(x)\mathcal{O}(x) = \lambda_1 \hat{\mathcal{O}}(p) + \lambda_2 \hat{\mathcal{O}}(q),$$

so that the second variation of the Schwinger functional becomes

$$\frac{\partial^2 W[\phi_0]}{\partial\lambda_1\partial\lambda_2}\Big|_{\lambda_1=\lambda_2} = -\big\langle \hat{\mathcal{O}}(p)\hat{\mathcal{O}}(q)\big\rangle_c.$$

To calculate the action of ϕ on AdS_5 we observe, that

$$\begin{aligned} S[\phi_{cl}] &= \frac{\eta}{2}\int dzd^4x\,\sqrt{g}\;\nabla^\mu(\phi_{cl}\partial_\mu\phi_{cl}) \\ &+ \frac{\eta}{2}\int dzd^4x\,\sqrt{g}\;\big(-\phi_{cl}\triangle\phi_{cl} + m^2\phi_{cl}^2\big). \end{aligned}$$

The last bulk terms vanishes since ϕ fulfils the equation of motion so that

$$S[\phi_{cl}] = \frac{\eta}{2}\int\limits_{z=\epsilon} d^4x\sqrt{h}\; n^\mu \phi_{cl}\partial_\mu\phi_{cl} = -\frac{\eta}{2}\frac{R^3}{\epsilon^3}\int d^4x \phi_{cl}(\epsilon, x)\frac{\partial\phi_{cl}}{\partial z}(\epsilon, x).$$

Now we must calculate this surface integral for

$$\phi_{cl} = \frac{z^2}{\epsilon^2}\Big(\lambda_1 \frac{K_{\Delta-2}(pz)}{K_{\Delta-2}(p\epsilon)}e^{ipx} + \lambda_2\frac{K_{\Delta-2}(qz)}{K_{\Delta-2}(q\epsilon)}e^{iqx}\Big).$$

We find for those terms which contribute to the x integral (for example e^{2ipx} does not contribute)

$$\begin{aligned} \phi_{cl}(\epsilon)\phi'_{cl}(\epsilon) &= \left(\frac{4}{\epsilon} + \frac{K'_{\Delta-2}(p\epsilon)}{K_{\Delta-2}(p\epsilon)} + \frac{K'_{\Delta-2}(q\epsilon)}{K_{\Delta-2}(q\epsilon)}\right)\lambda_1\lambda_2\; e^{i(p+q)x} \\ &= \left(\frac{4}{\epsilon} + \frac{\Delta-2}{\epsilon p} + \frac{\Delta-2}{\epsilon q} - \frac{K_{\Delta-1}(p\epsilon)}{K_{\Delta-2}(p\epsilon)} - \frac{K_{\Delta-1}(q\epsilon)}{K_{\Delta-2}(q\epsilon)}\right)\lambda_1\lambda_2\; e^{i(p+q)x} \end{aligned}$$

We use the expansion of K_ν for small arguments,

$$K_\nu(z) = -\frac{\pi}{2\sin(\nu\pi)}\sum_{k=0}^{\infty}\frac{1}{k!}\Big[\frac{(z/2)^{2k+\nu}}{\Gamma(\nu+k+1)} - \frac{(z/2)^{2k-\nu}}{\Gamma(k+1-\nu)}\Big]$$

from which can derive, that

$$\frac{K_{\nu+1}}{K_\nu} \sim \frac{2\nu}{z} + \frac{z}{2\nu-2} - \frac{z^3}{8(\nu-1)^2(\nu-2)}$$

Now the classical action becomes

$$\begin{aligned} S &= -\frac{\eta R^3}{2\epsilon^3}\lambda_1\lambda_2(2\pi)^4\delta(p+q)\Big(\frac{4}{\epsilon} - \frac{\Delta-2}{\epsilon p} - \frac{\epsilon p}{2\Delta-6} \\ &\qquad + \frac{\epsilon^3 p^3}{8(\Delta-3)^2(\Delta-4)} + (p \leftrightarrow q)\Big) \end{aligned}$$

After some algebra one ends up with

$$\langle \tilde{\mathcal{O}}(p)\tilde{\mathcal{O}}(q)\rangle = -\eta\epsilon^{2\Delta-8}(2\Delta-4)\frac{\Gamma(3-\Delta)}{\Gamma(\Delta-1)}\,\delta^4(p+q)\,\big(\frac{p}{2}\big)^{2\Delta-4},$$

or after a Fourier transformatin with

$$\langle \mathcal{O}(x)\mathcal{O}(y)\rangle = -\eta\epsilon^{2\Delta-8}\frac{2\Delta-4}{\Delta}\,\frac{\Gamma(3-\Delta)}{\Gamma(\Delta-1)}\,\frac{1}{|x-y|^{2\Delta}}.$$

Correlation functions of non-scalar operators have also been studies. It should be enough to study the correlators corresponding to the chiral primary fields. The others can by applying supersymmetry. For a calculation of 3 and 4 point function I refer to the literature. All in accordance with the conjectured duality.

11.3 Wilson loop

As in large N QCD we expect the Wilson loop to be related to the string running from the quark to the antiquark. Result of correspondence

$$V_{q\bar{q}}(L) = -\frac{4\pi^2(2g_{YM}^2N)^{1/2}}{\Gamma(1/4)^4L}.$$

It goes as $1/L$ as expected by conformal invariance. In YM perturbation theory V would proportional to g_{YM}^2N and not to the square root of it as it is here. This indicates a screening of the charges at strong coupling.

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