Prof. Andreas Wipf, MSc. Marc Steinhauser January 22, 2020

Problem sheet „Advanced Quantum Mechanics“

winter term 2019/20

Sheet 12

Problem 28: Weyl spinors — 2 points

The two-component spinor $\phi(p)$ fulfills the Weyl equation

$$\sigma_0 p_0\, \phi(p) = \boldsymbol{\sigma} \cdot \boldsymbol{p}\, \phi(p)\,.$$

Show that only for

$$p_0 = \pm|\boldsymbol{p}| = \frac{E}{c}$$

non-vanishing solutions exist.
Hint: Act with the helicity operator $\hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma}$ or $\boldsymbol{p} \cdot \boldsymbol{\sigma}$ on the equation.

Problem 29: Relativistic electron in a constant magnetic field — 4 points

We consider the time-independent Dirac-equation in Hamiltonian form

$$E\psi(\boldsymbol{x}) = H\psi(\boldsymbol{x})$$

in a constant (in direction and magnitude) magnetic field with static 4-potential $A^\mu(\boldsymbol{x}) = (0, 0, Bx^1, 0)$. Argue, that the solution have the form $\psi = \exp\big(\mathrm{i}(p_2x^2 + p_3x^3)\big)u(x^1)$ and that the corresponding energies are

$$E^2 = m^2 + p_3^2 + (2n+1)|eB| \pm eB, \quad n \in \{0, 1, 2, \dots\}\,.$$

Hint: if you need an explicit representation for the γ^μ, then you should use the Dirac representation.

Problem 30: Lorentz-Liealgebra and angular momenta — 1+2+1+1 = 5 points

In the lecture the generators of rotations in space Ω_i and of Lorentz boosts Λ_i have been introduced. They fulfill the commutation relations

$$[\Lambda_i, \Lambda_j] = -\epsilon_{ijk}\Omega_k, \quad [\Omega_i, \Omega_j] = \epsilon_{ijk}\Omega_k, \quad [\Lambda_i, \Omega_j] = \epsilon_{ijk}\Lambda_k, \quad i, j, k \in \{1, 2, 3\}\,.$$

In the following we define the generators $\Lambda_{\mu\nu} = -\Lambda_{\nu\mu}$ as follows:

$$\Lambda_{0i} = -\mathrm{i}\Lambda_i \quad \text{and} \quad \Lambda_{ij} = -\mathrm{i}\epsilon_{ijk}\Omega_k\,.$$

1. Check, that they fulfill the commutation relations

   $$[\Lambda_{\mu\nu}, \Lambda_{\rho\sigma}] = \mathrm{i}\big(g_{\mu\rho}\Lambda_{\nu\sigma} + g_{\nu\sigma}\Lambda_{\mu\rho} - g_{\mu\sigma}\Lambda_{\nu\rho} - g_{\nu\rho}\Lambda_{\mu\sigma}\big)\,.$$

   Generators with these commutation relations generate the Lorentz-Liealgebra (this Lie-algebra is the most important Lie-algebra in relativistic quantum mechanics).

2. Proof that the operators (generators)

   $$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \quad \text{and} \quad \Sigma_{\mu\nu} = \frac{1}{4\mathrm{i}}[\gamma_\mu, \gamma_\nu]$$

   and hence $J_{\mu\nu} = \hbar(M_{\mu\nu} + \Sigma_{\mu\nu})$ satisfy the same commutation relations as the $\Lambda_{\mu\nu}$ (up to a factor $\hbar$).
   Hint: Use the antisymmetry in $(\mu \leftrightarrow \nu)$ to shorten your calculation.

3. Which commutation relations fulfill the 3 generators

$$J_i = \epsilon_{ijk} J_{jk} \,?$$

4. The vector operator $\boldsymbol{J}$ can be written as $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ with

$$S_i = \epsilon_{ijk} \Sigma_{jk} \,.$$

   What interpretation has $\boldsymbol{S}$?

Submission date: Thursday, 30.01.2020, before the lecture