Friedrich-Schiller-Universität Jena Winter term 2017/18
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Problems in Advanced Quantum Mechanics

Problem Sheet 10

Problem 23: Path integral for charged particle in elm. field 5 points

The Lagrangian of a charged particle in an external electromagnetic field is

$$L = \frac{m}{2}\dot{\boldsymbol{x}}^2 + L_{\text{int}}, \qquad L_{\text{int}} = \frac{e}{c}\,\dot{\boldsymbol{x}}\cdot\boldsymbol{A}(t,\boldsymbol{x}) - e\varphi(t,\boldsymbol{x}).$$

where the potentials are related to the electromagnetic fields via

$$\boldsymbol{E} = -\nabla\varphi - \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} \quad , \quad \boldsymbol{B} = \nabla\times\boldsymbol{A}.$$

The corresponding Hamilton-Function reads

$$H = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}(t,\boldsymbol{x})\Big)^2 + e\varphi(t,\boldsymbol{x}),$$

The wave function at time t is related to the wave function at time $t-\epsilon$ via

$$\psi(t,\boldsymbol{x}) = \int \mathrm{d}^3y\, K(t,\boldsymbol{x};t-\epsilon,\boldsymbol{y})\psi(t-\epsilon,\boldsymbol{y})$$

We assume that the path integral representation for the evolution kernel K holds true,

$$K(t,\boldsymbol{x};t_0,\boldsymbol{y}) \propto \int_{\boldsymbol{x}(t_0)=\boldsymbol{y}}^{\boldsymbol{x}(t)=\boldsymbol{x}} \mathcal{D}\boldsymbol{x}\, \mathrm{e}^{\mathrm{i}S/\hbar}$$

For small ϵ we may approximate the Riemann-integral as

$$S \approx \frac{m}{2}\frac{\boldsymbol{u}^2}{\epsilon} + \epsilon L_{\text{int}}, \quad L_{\text{int}} = \frac{e}{c}\frac{\boldsymbol{u}}{\epsilon}\cdot\boldsymbol{A}\left(t-\frac{\epsilon}{2},\boldsymbol{x}-\frac{\boldsymbol{u}}{2}\right) - e\varphi\left(t-\frac{\epsilon}{2},\boldsymbol{x}-\frac{\boldsymbol{u}}{2}\right)$$

with $\boldsymbol{u} = \boldsymbol{x}-\boldsymbol{y}$. Then the path integral for the propagation during the time interval ϵ is

$$\psi(t,\boldsymbol{x}) = \lim_{\epsilon\to 0} C_\epsilon^3 \int \mathrm{d}^3u \exp\left(\frac{im}{2\hbar\epsilon}\boldsymbol{u}^2\right)\exp\left(\frac{\mathrm{i}\epsilon}{\hbar}L_{\text{int}}\right)\psi(t-\epsilon,\boldsymbol{x}-\boldsymbol{u}),$$

where $C_\epsilon = (m/2\pi i\hbar\epsilon)^{1/2}$. In the lecture is has been stated that for $\epsilon\to 0$ this wave function obeys the time-dependent Schrödinger equation with above Hamiltonian. The essential steps to prove this statement have been sketched in general and worked for two typical terms appearing in an expansion in ϵ. In this exercise you should now fill the gaps in the arguments and prove the statement in detail.

Submission date: Thursday, 11. January 2018, before the lecture begins.