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Problem sheet „Advanced Quantum Mechanics“

winter term 2019/20

Sheet 9

Problem 20: Lorentz transformation of $F^{\mu\nu}$ — 2+2 = 4 points

The contravariant components of the field strength tensor transform under a change of the inertial systems $I \to I'$ according to

$$F^{\mu\nu}(x) \mapsto F'^{\mu\nu}(x') = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}(x)\,.$$

Consider the Lorentz boost

$$\begin{aligned} x'^0 &= \gamma x^0 - \beta\gamma x^1, \qquad & x'^2 &= x^2\,, \\ x'^1 &= \gamma x^1 - \beta\gamma x^0, \qquad & x'^3 &= x^3\,. \end{aligned}$$

How does the electric field $\boldsymbol{E}$ and magnetic field $\boldsymbol{B}$ (which make up the field strength tensor) transform under this Lorentz-transformation? Use the same notation and conventions as in the lecture.

Problem 21: The scalar field — 2+2 = 4 points

In the lectures we defined the current density 4-vector j^μ for a Klein-Gordon field ϕ in presence of an external electromagnetic field with potential A_μ as follows:

$$j^\mu = \frac{\mathrm{i}\hbar}{2m}\left(\phi^* D^\mu \phi - \phi (D^\mu \phi)^*\right)$$

where the covariant derivative is given by

$$D_\mu \phi = \left(\partial_\mu + \frac{\mathrm{i}e}{\hbar c}A_\mu\right)\phi\,.$$

1. Show that the current density is gauge invariant, i.e. invariant under the transformation

   $$A_\mu \mapsto A_\mu - \partial_\mu \lambda, \quad \phi \mapsto \mathrm{e}^{\mathrm{i}e\lambda/\hbar c}\phi$$

   for any arbitrary gauge function λ.
2. Show that the current is conserved

   $$\partial_\mu j^\mu = 0,$$

   if ϕ solves the Klein-Gordon equation

   $$\left(D_\mu D^\mu + \frac{m^2 c^2}{\hbar^2}\right)\phi = 0\,.$$

Problem 22: Lorentz boosts — 1+1+2+1+1 = 6 points

An arbitrary proper orthochrone Lorentz transformation has the form

$$\Lambda(\boldsymbol{\alpha},\boldsymbol{\theta}) = \mathrm{e}^{\omega(\boldsymbol{\alpha},\boldsymbol{\theta})} \qquad \text{with} \qquad \omega(\boldsymbol{\alpha},\boldsymbol{\theta}) = \begin{pmatrix} 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ -\alpha_1 & 0 & -\theta_3 & \theta_2 \\ -\alpha_2 & \theta_3 & 0 & -\theta_1 \\ -\alpha_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix}$$

1. Calculate the Lorentz transformation

$$\Lambda(\alpha\boldsymbol{e},0) = \mathrm{e}^{\omega(\alpha\boldsymbol{e},0)} \qquad \text{with} \qquad \boldsymbol{e} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} .$$

2. Show

$$\Lambda(\mathcal{R}\boldsymbol{\alpha},0) = \Lambda(\mathcal{R})\Lambda(\boldsymbol{\alpha},0)\Lambda^{-1}(\mathcal{R}) , \qquad \text{where} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \mathcal{R} & \\ 0 & & & \end{pmatrix} .$$

Hint: Show first $\omega(\mathcal{R}\boldsymbol{\alpha},0) = \Lambda(\mathcal{R})\omega(\boldsymbol{\alpha},0)\Lambda^{-1}(\mathcal{R})$.

3. The Lorentz boost are

$$\Lambda(\alpha\boldsymbol{e},0) = \mathrm{e}^{\omega(\alpha\boldsymbol{e},0)} = \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha)\boldsymbol{e}^{\mathrm{T}} \\ -\sinh(\alpha)\boldsymbol{e} & \delta_{ij} - (1-\cosh(\alpha))e_i e_j \end{pmatrix} \qquad \text{for} \qquad \alpha \geq 0 .$$

They map the inertial system I to I', $x \mapsto x' = \Lambda x$. What coordinates x has the origin $x' = (x'^0, \boldsymbol{0})$ of I'? Use this result to express α and $\boldsymbol{e}$ as function if the velocity $\boldsymbol{v}$ of I' relatively to I. Write $\Lambda(\alpha\boldsymbol{e},0)$ also as a function of $\boldsymbol{v}$.

4. Consider a prototype meter resting in I'. Its endpoints are given by

$$x' = \begin{pmatrix} ct' \\ 0 \\ 0 \\ 0 \end{pmatrix} , \qquad y' = \begin{pmatrix} ct' \\ 1 \\ 0 \\ 0 \end{pmatrix} .$$

What are the coordinates (x, y) of this prototype meter in I, when I' is moving relatively to I in 1-direction $\boldsymbol{v} = (v, 0, 0)^{\mathrm{T}}$? What length does an observer measure in the inertial system I?

5. A clock rests in the origin of I'. Given$x'^0 = 0$ and $y'^0 = t'$, calculate $\delta t = y^0 - x^0$ in the inertial system I.

Submission date: Thursday, 19.12.2019, before the lecture