Prof. Andreas Wipf, MSc. Marc Steinhauser November 20, 2019

Problem sheet „Advanced Quantum Mechanics“

winter term 2019/20

Sheet 5

Problem 12: Interacting Spins 1+2+1=4 points

Given is a system with 2 spin-$\frac{1}{2}$ particles interacting via the Hamilton operator

$$H = \frac{4J}{\hbar^2}(S_{x,1}S_{x,2} + S_{y,1}S_{y,2}) + \frac{4J'}{\hbar^2}S_{z,1}S_{z,2}\,.$$

1. Write the Hamilton operator as a 4×4 matrix with the basis $|+,+\rangle \equiv \vec{e}_1$, $|+,-\rangle \equiv \vec{e}_2$, $|-,+\rangle \equiv \vec{e}_3$, $|-,-\rangle \equiv \vec{e}_4$.
   Remark: In this basis the angular momentum operators $S_i = \frac{\hbar}{2}\sigma_i$ contain the Pauli matrices σ_i in the Hilbert spaces of the two particles.
2. Calculate the eigenstates and energy eigenvalues by diagonalizing the matrix from (1).
3. Sketch for a fixed positive J the eigenvalues of H as a function of J' for $J' \in [-2J, 2J]$. Name the different cases of the ground state including their energy and degree of degeneration.

Problem 13: Addition of 3 angular momenta 2+2=4 points

Investigate the addition of three angular momenta to a total angular momentum, $\vec{J} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3$. Which and how many multiplets (singulet **1**, dublet **2**, triplet **3**,...) contains the system if all three quantum number j_i

1. have the value $\frac{1}{2}$ (i.e. three spin-$\frac{1}{2}$ particles)?
2. have the value 1 (i.e. three spin-1 particles)?

Determine the degree of degeneration for each occurring quantum number j.
Hint: You can first add two angular momenta and then calculate the sum with the third angular momentum, for example $\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2} = (\mathbf{2}\otimes\mathbf{2})\otimes\mathbf{2}$ (the tensor product is associative).

Problem 14: Tensor Operators 2+3=5 points

Let (V_x, V_y, V_z) and (W_x, W_y, W_z) denote the Cartesian components of two commuting vector operators. From the nine operators $\{V_xW_x, V_xW_y, \ldots\}$ one can construct tensor operators of rank zero (scalar), rank one (vector) and of rank two:

$$\begin{cases} S &=& \boldsymbol{V}\cdot\boldsymbol{W} = T^{(0)} \\ \boldsymbol{U} &=& \boldsymbol{V}\wedge\boldsymbol{W} = T^{(1)} \\ T^{(2)}_{ij} &=& V_iW_j + V_jW_i - \frac{2}{3}\delta_{ij}S\,. \end{cases}$$

1. For all three operators one can find the spherical component $T^{(J)}_M$ with maximal M. Show, that

$$\begin{aligned} T^{(0)}_0 &\propto V_0W_0 - (V_1W_{-1} + V_{-1}W_1) \\ T^{(1)}_1 &\propto V_0W_1 - V_1W_0 \\ T^{(2)}_2 &\propto V_1W_1. \end{aligned}$$

2. Determine the remaining spherical (normal) components of $T^{(1)}$ and $T^{(2)}$.

Submission date: Thursday, 28.11.2019, before the lecture