Prof. Andreas Wipf, MSc. Marc Steinhauser November 18, 2019

Problem sheet „Advanced Quantum Mechanics“

winter term 2019/20

Sheet 5

Problem 10: Wave function of a spinning particle and rotations 9 points

Let $U \in$ be an element of the quantum mechanical rotation group SU(2), i.e. $U^\dagger = U^{-1}$ and $\det U = 1$.

1. For $\boldsymbol{n} \in \mathbb{R}^3$ and the Pauli matrices $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, show that $R(U)$ in

$$U\boldsymbol{n}\cdot\boldsymbol{\sigma}\,U^{-1} = (R(U)\boldsymbol{n})\cdot\boldsymbol{\sigma}$$

describes a proper rotation in $\mathbb{R}^3$, i.e. $R^T R = \mathbb{1}$ and $\det R = 1$. The Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Prove that the map $U \to R(U)$ from SU(2) to SO(3) is a „representation“, which means, that $R(\mathbb{1}_2) = \mathbb{1}_3$ and that $R(U_1U_2) = R(U_1)R(U_2)$.

3. Show, that for the quantum mechanical rotation

$$U(\boldsymbol{e},\theta) = \mathrm{e}^{-\mathrm{i}\theta\boldsymbol{e}\cdot\boldsymbol{\sigma}/2}$$

the rotation in space has the form

$$R(\boldsymbol{e},\theta) = \mathrm{e}^{\boldsymbol{e}\cdot\boldsymbol{\Omega}\,\theta}, \qquad \Omega_e = \boldsymbol{e}\cdot\boldsymbol{\Omega} = \begin{pmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{pmatrix}$$

and hence is a rotation with angle θ and axis defined by the unit vector $\boldsymbol{e}$.
Hint: This proof can be performed for infinitesimal rotations, i.e. show

$$\left(\frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\Big|_{\theta=0} R(\boldsymbol{e},\theta)\boldsymbol{n}\right)\cdot\boldsymbol{\sigma} = \frac{\mathrm{d}^n}{\mathrm{d}\theta^n}\Big|_{\theta=0} U(\boldsymbol{e},\theta)(\boldsymbol{n}\cdot\boldsymbol{\sigma})\,U(\boldsymbol{e},\theta)^{-1}.$$

4. The wave function of a particle with spin $\frac{1}{2}$ transforms under rotations as

$$\psi(\boldsymbol{x}) \longmapsto U\psi\big(R(U)^{-1}\boldsymbol{x}\big) \equiv \big(\Gamma(U)\psi\big)(\boldsymbol{x}).$$

Determine the rotation around the 3-axis, given by $U(\boldsymbol{e}_3,\theta)$, where θ varies smoothly from 0 to 2π. What happens with $\theta = 2\pi$?

5. Show that for all U the linear map $\Gamma(U)$ is unitary on $\mathcal{H} = L_2(\mathbb{R}^3)\times\mathbb{C}^2$ with scalar product

$$(\psi,\phi) = \sum_{i=1}^{2}\int \mathrm{d}^3x\,\bar{\psi}_i(\boldsymbol{x})\phi_i(\boldsymbol{x}) = \int \mathrm{d}^3x\,\psi^\dagger(\boldsymbol{x})\phi(\boldsymbol{x})$$

6. Show that the linear map $\Gamma(U)$ defines a representation, i.e. $\Gamma(\mathbb{1}_2) = \mathbb{1}_{\mathcal{H}}$ and $\Gamma(R_1R_2) = \Gamma(R_1)\Gamma(R_2)$.

7. Consider a quantum-mechanical rotation $U(\boldsymbol{e}_3, \theta)$ about the $\boldsymbol{e}_3$-axis, Which operator A (infinitesimal generator) generates these rotations, i.e. satisfies

$$(A\psi)(\boldsymbol{x}) = \frac{\mathrm{d}}{\mathrm{d}\theta}\Big|_{\theta=0} U(\boldsymbol{e}_3, \theta)\psi\big(R^{-1}(\boldsymbol{e}_3, \theta)\boldsymbol{x}\big) ,$$

where (of course) $R(\boldsymbol{e}_3, \theta)$ is the spatial rotation belonging to $U(\boldsymbol{e}_3, \theta)$.
Hint: Don't forget the chain rule.

8. Argue, that the (anti-hermitean) Operator A generates the unitary rotation about the $\boldsymbol{e}_3$-axis,

$$(\mathrm{e}^{\theta A}\psi)(\boldsymbol{x}) = U(\boldsymbol{e}_3, \theta)\psi\big(R^{-1}(\boldsymbol{e}_3, \theta)\boldsymbol{x}\big) .$$

Hint: Use the infinitesimal rotations

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(\Gamma\big(U(\boldsymbol{e}_3, \theta)\psi\Big)(\boldsymbol{x}) = -\frac{\mathrm{i}}{\hbar} J_z \Big(\Gamma\big(U(\boldsymbol{e}_3, \theta)\psi\Big)(\boldsymbol{x}) .$$

9. What do you think is the infinitesimal generator for a rotation $U(\boldsymbol{e}, \theta)$ about a arbitrary axis $\boldsymbol{e}$? What do you think, is the generator for several particles with spin $\frac{1}{2}$?

Literature:

- http://www.tpi.uni-jena.de/qfphysics/homepage/wipf/lectures/qm1/qm18.pdf
  Wipf - Quantum Mechanics 1 Script, Chapter 8
- http://www.tpi.uni-jena.de/qfphysics/homepage/wipf/lectures/qm1/qm110.pdf
  Wipf - Quantum Mechanics 1 Script, Chapter 10.3
- http://www.tpi.uni-jena.de/qfphysics/homepage/wipf/lectures/gruppen/gruppenhead.pdf
  Wipf - Symmetries in Physics Script, Chapter 5.2 & 9.2
- M. Bartelmann, B. Feuerbacher, T. Krüger, D. Lüst, A. Rebhan, A. Wipf - Theoretische Physik, Chapter 27.5
- K. Gottfried, T.-M. Yan - Quantum mechanics fundamentals, Chapter 2.5(d)
- J. J. Sakurai, J. Napolitano - Modern Quantum Mechanics Chapter 3.1 - 3.3

Problem 11: Coupling of three angular momenta 2+2 = 4 points

Consider the eigenvectors of the total angular momentum

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 + \boldsymbol{J}_3,$$

where the individual angular momenta are all 1. An eigenvalue of $\boldsymbol{J}^2$ is written as $\hbar^2 j(j+1)$.

1. What are the possible values of j? How many linearly independent eigenstates belong to each possible value of j?
   Hint: The tensor product and the direct sum are associative and the tensor product distributes over the direct sum, $\mathfrak{h}_1 \otimes (\mathfrak{h}_2 \oplus \mathfrak{h}_3) = (\mathfrak{h}_1 \otimes \mathfrak{h}_2) \oplus (\mathfrak{h}_1 \otimes \mathfrak{h}_3)$
2. Construct the state with $j = 0$ explicitly. If $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ are vectors in $\mathbb{R}^3$, then there is one multilinear scalar, namely $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c})$. Find a connection between this fact and your result for the state with $j = 0$.
   Hint: You may use the relation between cartesian and spherical components of a vector given in the lecture.

Submission date: Thursday, 21.11.2019, before the lecture