October 23, 2019

Problem sheet „Advanced Quantum Mechanics"

Wintersemester 2019/20

Sheet 2

Problem 2: Symmetry and time-evolution 2 points

Explain, why an initially completely symmetric or anti-symmetric wave function describing a system of identical particles remains symmetric or anti-symmetric at later times.
Remark: problem from an earlier exam

Problem 3: Non-Interacting Particles 4 points

The 3-dimensional Hilbert space of a quantum mechanical system is spanned by the base $|1\rangle$, $|2\rangle$ and $|3\rangle$. Three particles can occupy these states. How many different physically admitted states exist in the following situations:

- 3 identical fermions
- 3 identical bosons
- 2 identical fermions and 1 boson
- 2 identical bosons and 1 fermion

Problem 4: Thomas-Fermi Atoms: test functions 9 points

We seek the optimal solution for the electron density $n(\boldsymbol{x})$ of a Thomas-Fermi atom within a family of test functions. More precisely, we consider the following family of test functions,

$$n(\boldsymbol{x}) = A\,\frac{\mathrm{e}^{-y}}{y^3}, \qquad y = \sqrt{\frac{r}{\lambda}}, \qquad r = |\vec{x}|\,,$$

where λ is a variational parameter and the constant A is fixed by the normalization $\int \mathrm{d}^3x\, n = N$. For a neutral atom we have $N = Z$.

1. Calculate the energy of the atom (ion) as function of λ.
2. Find the minimizing values of the variational parameter.
3. Calculate the corresponding energy as function of N and Z. What do you obtain for an (neutral) atom.

Hints: Express the result as function of the TF-parameter γ entering the expression for the kinetic energy. The most demanding part is the calculation of the Coulomb interaction between the electrons,

$$V_{ee} = \frac{e^2}{2}\int \frac{n(\boldsymbol{x})n(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}^3x\mathrm{d}^3y - \frac{V_{ne}}{2} = -\frac{e}{2}\int \mathrm{d}^3x\, n(\boldsymbol{x})\varphi(\boldsymbol{x}) - \frac{V_{ne}}{2}\,.$$

When solving the equation $\Delta\varphi = -4\pi e n$ for φ, for the given ansatz for $n(\boldsymbol{x})$, you arrive at the differential equation

$$\frac{1}{4\lambda^2 y^3}\left(y\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 3\frac{\mathrm{d}}{\mathrm{d}y}\right)\varphi = -4\pi e A\frac{\mathrm{e}^{-y}}{y^3}$$

The solution regular at the origin is

$$\varphi = \frac{\text{const}}{y^2}\left(1-(1+y)\mathrm{e}^{-y}\right) .$$

Check that this is a solution and fix the constant.

Abgabetermin: Wednesday 30.10.2019 in the exercise class (8:15-9:45 SR1 MWP1) in the office Abbeanum 313B between 14:00-16:00.