Problems: Quantum Fields on the Lattice

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Sheet 7

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19 Grassmannian Integration

Consider Grassmann variables $\psi_x, \bar{\psi}_x, \psi_y, \bar{\psi}_y$ on a lattice with two points x, y. The fermionic Euclidean action is given by

$$S[\psi, \bar{\psi}] = \bar{\psi}_x \psi_y + \bar{\psi}_y \psi_x + m \left(\bar{\psi}_x \psi_x + \bar{\psi}_y \psi_y \right). \tag{1}$$

Evaluate the partition function

$$Z = \int \mathscr{D}\psi \mathscr{D}\bar{\psi} \; e^{-S[\psi,\bar{\psi}]} \tag{2}$$

by the use of Grassmann integration rules. Determine the values of the 2-point functions $\left\langle \bar{\psi}_x \psi_x \right\rangle$, $\left\langle \bar{\psi}_x \psi_y \right\rangle$.

20 The Pfaffian

Let $\eta_1, \ldots, \eta_{2N}$ be an even number of anticommuting real Grassmann variables, $\{\eta_a, \eta_b\} = 0$.

1. Prove that the Gaussian integral over such variables yields the Pfaffian,
$$\int \mathrm{d}\eta_1 \ldots \eta_{2N} \; e^{\frac{1}{2}\eta^\top M \eta} = \frac{1}{2^N N!} \varepsilon_{a_1 b_1 \cdots a_N b_N} M_{a_1 b_1} \ldots M_{a_N b_N} = \mathrm{Pf}\,(M). \tag{3}$$
2. By doubling the degrees of freedom prove the important identity
$$\det M = \mathrm{Pf}\,(M). \tag{4}$$
3. Transform the Grassmann variables according to $\eta \to R\eta$ and show
$$\mathrm{Pf}\left(R^\top M R \right) = \det(R)\, \mathrm{Pf}\,(M). \tag{5}$$
4. Prove that for an antisymmetric matrix M of dimension $2N$ we have
$$\mathrm{Pf}\left(M^\top \right) = (-1)^N \, \mathrm{Pf}\,(M). \tag{6}$$
5. Show, by using the relation between the Pfaffian and determinant, that
$$\delta \ln \det(M) = \mathrm{tr}\left(M^{-1} \delta M \right) \quad \Rightarrow \quad \delta \ln \mathrm{Pf}\,(M) = \frac{1}{2} \mathrm{tr}\left(M^{-1} \delta M \right). \tag{7}$$
6. Let us assume that the antisymmetric M is a tensor product of a symmetric matrix S and an antisymmetric matrix A. By transforming both matrices into their normal forms prove that
$$\mathrm{Pf}\,(M) = (\det S)^{\dim A} \, (\mathrm{Pf} A)^{\dim S}. \tag{8}$$

21 Fermion Discretizations

When introducing fermions we need to discretize a first order derivative operator such as

$$D = \mathrm{i}\gamma^\mu \partial_\mu + \gamma^\mu A_\mu + m. \tag{9}$$

For simplicity we discard the gauge fields (that would actually be coupled via link variables) and mass ($m = 0$) and work only in a single dimension in this exercise such that we will consider operators of the form

$$D = \partial. \tag{10}$$

Consider the three discretizations

$$\left(\partial^{\text{naive}}\phi\right)_x = \frac{1}{2}\left(\phi_{x+\hat{e}} - \phi_{x-\hat{e}}\right) \tag{11}$$

$$\left(\partial^{\text{Wilson}}\phi\right)_x = \left(\partial^{\text{naive}}\phi\right)_x - \frac{r}{2}\left(\phi_{x+\hat{e}} - 2\phi_x + \phi_{x-\hat{e}}\right) \tag{12}$$

$$\left(\partial^{\text{SLAC}}\phi\right)_x = \mathcal{F}^{-1}\left[\sum_{p\in\Lambda^*} p\mathcal{F}\left[\phi\right]_p\right]_x \tag{13}$$

where $r \in [0, 1]$ is a free parameter and $\mathcal{F}$ denotes the discrete Fourier transform. Find the dispersion relations of these operators (similar to Problem 7). Taylor expand[1] your result around 0 and verify that all these operators approximate the continuum dispersion relation in a small region around 0. Sketch your findings and discuss peculiarities of the various curves.
Bonus: Compute the real space representation of ∂^{SLAC}. Is this a

- *ultra-local operator, i.e. there exists a $r > 0$ such that for all $|x - y| > r$ holds $D_{xy} = 0$?*
- *local operator, i.e. $|D_{xy}|$ decays at least as $\mathrm{e}^{-\gamma|x-y|}$ for some $\gamma > 0$?*
- *non-local operator (none of the above)?*

Could you have seen this in the dispersion relation?

22 Chemical Potential On The Lattice

The $\mathrm{U}(1)$ symmetry of the free fermionic theory is usually referred to as fermion number conservation. In the continuum its conserved current obeys

$$j^\nu = \bar{\psi}\gamma^\mu\psi, \qquad \partial_\nu j^\nu = 0 \tag{14}$$

and fermions at nonzero chemical potential are described by the Lagrangian

$$\mathcal{L} = \bar{\psi}\left(\mathrm{i}\not{\partial} + m\right)\psi + \mathrm{i}\mu j^0. \tag{15}$$

In this exercise, we will derive and discuss the corresponding expressions on the lattice.

[1] At this point, you can safely neglect the discrete nature of the lattice momentum, since for sufficiently large lattices it is almost continuous.

1. Consider the naive fermion discretization (11) and derive its conserved current and the conservation law. You will see that it comes in a point-split form inherited from the fermion discretization.

2. Since the structure of j^0 and ∂^{naive} is identical, we can couple a chemical potential according to

$$\left(\gamma^0 \partial_0 + \mu\gamma^0\right)^{\text{naive}} \psi_x = \frac{1}{2a}\gamma^0 \left(f(a\mu)\psi_{x+a\hat{0}} - g(a\mu)\psi_{x-a\hat{0}}\right) \tag{16}$$

   with explicit lattice constant a and some functional dependency f, g on μ. It is now up to you to restrict them further: Consider the limits $\mu \to 0$ and $a \to 0$ and employ time reflection invariance to find conditions that f, g have to obey.

3. In addition to the above conditions, it can be shown that $g = f^{-1}$ is needed to get non-divergent expressions in the continuum limit. Convince yourself that all restrictions are met by $f = \exp$. Find another function f to meet the restrictions. Find an additional argument why the former could be the preferred choice.

4. Assume that we are working in even dimensions. Use γ_5 to check if there is are any flavor numbers without sign problem for $\mu \neq 0$. Is there any (non-trivial) function f that would cure this for real chemical potential?

5. Finally, we broaden our concept of chemical potential a bit. Is there a sign problem for imaginary μ? Is there a sign problem for isospin chemical potential (i.e. an even number of flavors with $\mu = \mu_I$ for half of them and $\mu = -\mu_I$ for the other half)?