Problems: Quantum Fields on the Lattice

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Sheet 6

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15 Elitzur's Theorem

Elitzur's theorem states that it is impossible to break spontaneously a local symmetry. Verify this in the following simple setup: Consider a $\mathbb{Z}_2$-gauge theory coupled to a scalar field with the action

$$S = -\beta \sum_p U_p - \kappa \sum_{\langle x,y \rangle} \phi_x U_{\langle x,y \rangle} \phi_y + h \sum_x \phi_x + V(\phi_x) \tag{1}$$

where we have included a general $\mathbb{Z}_2$-gauge invariant potential V as well as a source term parametrized by h. Prove that

$$\lim_{h \searrow 0} \langle \phi_x \rangle = 0 \tag{2}$$

uniformly in the volume and the couplings.

16 Some group integrals

1. Show that

$$\int\limits_{\mathrm{SU}(N)} \mathrm{d}U \; U = 0. \tag{3}$$

2. Let F be an $N \times N$ matrix. Prove that if $\Lambda F \Lambda^{-1} = F$ holds for all $\Lambda \in \mathrm{SU}(N)$ then $F = c\mathbb{1}$.

   Hint: Start with $N = 2$. Find two special $\mathrm{SU}(2)$ matrices which allow to show $F = c\mathbb{1}$. Embed $\mathrm{SU}(2)$ into $\mathrm{SU}(N)$ and use the $N = 2$ property to show it for all $N \in \mathbb{N}$.

3. Use the previous result to calculate

$$f_{ijkl} = \int\limits_{\mathrm{SU}(N)} \mathrm{d}U \; U_{ij} \left(U^\dagger \right)_{kl} \tag{4}$$

   and determine the constant c for this case. *(Hint: You can crosscheck parts of your result by the use of the gluing property from Problem 18.)*

17 Conjugacy slasses of SU(3)

Characterize the conjugacy classes of $\mathrm{SU}(3)$.

18 Applications of the Peter-Weyl theorem

Use the Peter-Weyl theorem to prove the following properties:

1. orthogonality:

$$(R^a_b, \chi_{R'}) = \frac{\delta_{RR'}}{d_R} \delta^a_b, \tag{5}$$

2. gluing:

$$\int \mathrm{d}\Omega \; \chi_R(U\Omega^{-1}) \chi_{R'}(\Omega V) = \frac{\delta_{RR'}}{d_R} \chi_R(UV), \tag{6}$$

3. separation:

$$\int \mathrm{d}\Omega \; \chi_R(\Omega U \Omega^{-1} V) = \frac{1}{d_R} \chi_R(U) \chi_R(V), \tag{7}$$

4. decomposition of $\mathbb{1}$:

$$\sum_R d_R \chi_R(U) = \delta(\mathbb{1}, U). \tag{8}$$