Problems: Quantum Fields on the Lattice

Prof. Dr. Andreas Wipf
MSc. Julian Lenz

WiSe 2019/20

Sheet 3

Problem 5: Free energy densities of scalars in 1 and 2 spatial dimensions

Calculate the free energy densities of non-interacting massive scalars in one and two spatial dimensions and discuss the limit $m \to 0$.
Hint: In the lecture we already derived a series expansion in terms of modified Bessel functions.

Problem 6: Minima of energy function

Find the configurations with minimal energy of the following spin models (assume periodic boundary conditions and, if necessary, a number of spins which is multiple of 2 or 4):

1. The Ising chain with first and second neighbor interactions

$$H = -J_1 \sum_x s_x s_{x+1} - J_2 \sum_x s_x s_{x+2}, \quad s_x \in \{-1, 1\}$$

   Consider both positive and negative values of the couplings J_1, J_2.

2. The one-dimensional clock model

$$H = -J_c \sum_x \cos\left(\frac{2\pi}{q}(n_x - n_y + \Delta)\right), \quad n_x \in \{1, 2, \dots, q\}$$

   for positive J and all values of Δ.

3. The antiferromagnetic Ising model on a triangular lattice,

$$H = J \sum_{\langle x,y \rangle} s_x s_y, \quad s_x \in \{-1, 1\}$$

   with $J > 0$.

Problem 7: Free scalar field on the lattice I

We consider a one-component real scalar field on a three dimensional (periodic) hyper-cubic lattice with Euclidean action

$$S = \sum_x \left(\frac{1}{2} \sum_\mu (\varphi_{x+\hat{\mu}} - \varphi_x)^2 + \frac{1}{2} m^2 \varphi_x^2 + g \varphi_x^4 \right)$$

or equivalently

$$S = \sum_x \Big(-\sum_\mu \varphi_x \varphi_{x+\hat\mu} + \big(d + \frac{1}{2}m^2\big)\varphi_x^2 + g\varphi_x^4 \Big)$$

In the following we set $g = 0$ and denote the corresponding action of the non-interacing scalars by S_0.

1. Other representations found in the literature are

$$S_0 = \frac{1}{2}\sum_x \Big(\sum_y D_{xy}\varphi_y \Big)^2 = \frac{1}{2}\sum_{x,y} \varphi_x M_{xy} \varphi_y \,.$$

   Find the matrix elements D_{xy} and M_{xy} and determine the eigenvalues and eigenfunctions in

$$M\varphi_p = E_p^2 \varphi_p \,.$$

   Calculate the actions $S_0[\varphi_p]$ of the eigenfunctions.

2. Determine the two-point function

$$\langle \varphi_x \varphi_y \rangle = \frac{1}{Z}\int \mathcal{D}\varphi \, \mathrm{e}^{-S_0} \varphi_x \varphi_y$$

   and apply the result to compute the expectation value of the lattice action

$$\langle S_0 \rangle = \frac{1}{Z}\int \mathcal{D}\varphi \, \mathrm{e}^{-S_0} S_0 \,.$$

Problem 8: Free scalar field on the lattice II

Use the results of the previous problem to extract the masses of the one-dimensional free theory

$$S_0 = \sum_x \Big(\frac{1}{2}(\varphi_{t+1} - \varphi_t)^2 + \frac{1}{2}m_0^2 \varphi_x^2 \Big) \,.$$

The exact analytic result in the limit $N_t \to \infty$ can be obtained from the correlator in momentum space

$$G(p) \propto (\mathcal{F}C)(p)$$

with correlator $C(t) = \langle \varphi_t \varphi_0 \rangle$. The pole p_* of the propagator yields the physical mass $m_{\text{pole}} = ip_*$.
In order to prepare the ground for MC-simulations considered later in the lecture: Determine the correlator $C(t)$ (e.g. with octave or matlab) and perform the following steps:

1. determine the mass from the (local) slope of $C(t) \propto \exp(-mt)$: the values $C(t)$ and $C(t+1)$ allow the extraction of a local mass m_t.

2. determine the mass via a cosh-fit: With N_t lattice points in time-direction the correlation function has the approximate form $C(t) \propto \cosh(m(t - N_f/2))$. Fit your data with a cosh-fit on the interval $[t, N_t - t]$ and extract a mass m_t'.

Represent the obtained masses graphically and compare the results of the two methods. Study different 'volumes' N_t and different bare masses m_0 in the action. Also compare with the bare masses and the pole masses. Investigate the limit $m_0 \to 0$. Which method would you suggest, to extract the physical mass from dynamically generated MC-data sets for the 2-point correlator.